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RECOGNITION OF THE GROUPS $L_5(4)$ AND $U_4(4)$ BY THE PRIME GRAPH

РОЗПІЗНАВАННЯ ГРУП $L_5(4)$ ТА $U_4(4)$ ПО ГРАФУ ПРОСТИХ ЧИСЕЛ

Let G be a finite group. The prime graph of G is the graph $\Gamma(G)$ whose vertex set is the set $\Pi(G)$ of all prime divisors of the order $|G|$ and two distinct vertices p and q of which are adjacent by an edge if G has an element of order pq. We prove that if S denotes one of the simple groups $L_5(4)$ and $U_4(4)$ and if G is a finite group with $\Gamma(G) = \Gamma(S)$, then G has a normal subgroup N such that $\Pi(N) \subseteq \{2, 3, 5\}$ and $\dfrac{G}{N} \cong S$.

Нехай G — скінченна група. Графом простих чисел групи G називають граф $\Gamma(G)$, множиною вершин якого є множина $\Pi(G)$ усіх простих дільників порядку $|G|$ і в якому дві різні вершини p та q з'єднані ребром, якщо G містить елемент порядку pq. Доведено, що, якщо S є однією з простих груп $L_5(4)$ та $U_4(4)$, а G є скінченною групою, для якої $\Gamma(G) = \Gamma(S)$, то G має нормальну підгрупу N таку, що $\Pi(N) \subseteq \{2, 3, 5\}$ та $\dfrac{G}{N} \cong S$.

1. Introduction. Let G be a finite group. The spectrum $\omega(G)$ of G is the set of orders of elements in G, where each possible order element occurs once in $\omega(G)$ regardless of how many elements of that order G has. This set is closed and partially ordered by divisibility, hence it is uniquely determined by its maximal elements. The set of maximal elements of $\omega(G)$ is denoted by $\mu(G)$. The number of isomorphic classes of finite groups H such that $\omega(G) = \omega(H)$ is denoted by $h(G)$. If $h(G) = k \geq 1$ is finite then the group G is called a k-recognizable group by spectrum. If $h(G)$ is not finite, G is called non-recognizable. A 1-recognizable group is usually called a recognizable group. The recognizability of finite groups by spectrum was first considered by W. J. Shi et al. in [16]. A list of finite simple groups which are known to be or not to be recognizable by spectrum is given in [11].

For $n \in N$, let $\Pi(n)$ denote the set of all the prime divisors of n, and for a finite group G let us set $\Pi(G) = \Pi(|G|)$. The prime graph $\Gamma(G)$ of a finite group G is a simple graph with vertex set $\Pi(G)$ in which two distinct vertices p and q are joined by an edge if and only if G has an element of order pq. It is clear that a knowledge of $\omega(G)$ determines $\Gamma(G)$ completely but not vise-versa in general. Given a finite group G, the number of non-isomorphic classes of finite groups H with $\Gamma(G) = \Gamma(H)$ is denoted by $h_\Gamma(G)$. If $h_\Gamma(G) = 1$, then G is said to be recognizable by prime graph. If $h_\Gamma(G) = k < \infty$, then G is called k-recognizable by prime graph, in case $h_\Gamma(G) = \infty$ the group G is called non-recognizable by graph. Obviously a group recognizable by spectra need not to be recognizable by prime graph, for example A_5 is recognizable by spectra but $\Gamma(A_5) = \Gamma(A_6)$.

The number of connected components of $\Gamma(G)$ is denoted by $s(G)$. As a consequence of the classification of the finite simple groups it is proved in [19] and [10], that for any finite simple group G we have $s(G) \leq 6$. Let $\Pi_i = \Pi_i(G)$, $1 \leq i \leq s$, be the connected components of G. For a group of even order we let $2 \in \Pi_1$. Recognizability of groups by prime graph was first studied in [6] where some sporadic simple groups were characterized by prime graph. As another concept we say that a non-abelian simple group G is quasi-recognizable by graph if every finite group whose prime graph is $\Gamma(G)$ has a unique non-abelian composition factor isomorphic to G.

It is proved in [20] that the simple groups $G_2(7)$ and ${}^2G_2(q)$, $q = 3^{2m+1} > 3$, are recognizable by graph, where both groups have disconnected prime graphs. A series of interesting results concerning recognition of finite simple groups were obtained by B.Khosravi et al. In particular they have stabilized quasi-recognizability of the group $L_{10}(2)$ by graph and the recognizability of $L_{16}(2)$ by graph in [8] and [9], where both groups have connected prime graphs.

Next we introduce useful notation. Let p be a prime number. The set of all non-abelian finite simple groups G such that $p \in \Pi(G) \subseteq \{2, 3, 5, \ldots, p\}$ is denoted by $\mathfrak{S}_p$. It is clear that the set of all non-abelian finite simple groups is the disjoint union of the finite sets $\mathfrak{S}_p$ for all primes p. The sets $\mathfrak{S}_p$, where p is a prime less than 1000 is given in [21].

2. Preliminary results. Let G be a finite group with disconnected prime graph. The structure of G is given in [19] which is stated as a lemma here.

Lemma 2.1. *Let G be a finite group with disconnected prime graph. Then G satisfies one of the following conditions*:

a) $s(G) = 2$, *$G = KC$ is a Frobenius group with kernel K and complement C, and the two connected components of G are $\Gamma(K)$ and $\Gamma(C)$. Moreover K is nilpotent, and here $\Gamma(K)$ is a complete graph.*

b) $s(G) = 2$ *and G is a 2-Frobeuius group, i.e., $G = ABC$ where $A, AB \trianglelefteq G$, $B \trianglelefteq BC$, and AB, BC are Frobenius groups.*

c) *There exists a non-abelian simple group P such that $P \leq \overline{G} = \dfrac{G}{N} \leq \mathrm{Aut}\,(P)$ for some nilpotent normal $\Pi_1(G)$-subgroup N of G and $\dfrac{\overline{G}}{P}$ is a $\Pi_1(G)$-group. Moreover, $\Gamma(P)$ is disconnected and $s(P) \geq s(G)$.*

If a group G satisfies condition(c) of the above lemma we may write $P = \dfrac{B}{N}$, $B \leq G$, and $\dfrac{\overline{G}}{P} = \dfrac{G}{B} = A$, hence in terms of group extensions $G = N \cdot P \cdot A$, where N is a nilpotent normal $\Pi_1(G)$-subgroup of G and A is a $\Pi_1(G)$-group.

The above structure lemma was extended to groups with connected prime graphs satisfying certain conditions [17]. Denote by $t(G)$ the maximal number of primes in $\Pi(G)$ pairwise nonadjacent in $\Gamma(G)$ and $t(2, G)$ the maximal number of primes in $\Pi(G)$ nonadjacent to 2.

Lemma 2.2. *Let G be a finite group satisfying the following conditions*:

a) *there exist three pairwise distinct primes in $\Pi(G)$ nonadjacent in $\Gamma(G)$, i.e., $t(G) \geq 3$.*

b) *there exists an odd prime in $\Pi(G)$ nonadjacent in $\Gamma(G)$ to 2, i.e., $t(2, G) \geq 2$.*

Then there is a finite non-abelian simple group S such that $S \leq \overline{G} = \dfrac{G}{K} \leq \mathrm{Aut}\,(S)$ for the maximal normal solvable subgroup K of G. Furthermore $t(S) \geq t(G) - 1$ and one of the following statements holds:

1. *$S \cong A_7$ or $L_2(q)$ for some odd q, and $t(S) = t(2, G) = 3$.*

2. *For every prime $p \in \Pi(G)$ nonadjacent to 2 in $\Gamma(G)$ a Sylow p-subgroups of G is isomorphic to a Sylow p-subgroup of S. In particular $t(2, S) \geq t(2, G)$.*

In the following we list some properties of the Frobenius group where some of its proof can be found in [15].

Lemma 2.3. *Let G be a Frobenius group with kernel K and complement H. Then*:

a) *K is nilpotent and $|H| \mid (|K| - 1)$.*

b) *The connected components of G are $\Gamma(K)$ and $\Gamma(H)$.*

c) $|\mu(K)| = 1$ *and* $\Gamma(K)$ *is a complete graph.*

d) *If* $|H|$ *is even, then* K *is abelian.*

e) *Every subgroup of* H *of order* pq, p *and* q *not necessary distinct primes, is cyclic. In particular if* H *is abelian, then it would be cyclic.*

f) *If* H *is non-solvable, then there is a normal subgroup* H_0 *of* H *such that* $[H\colon H_0] \leq 2$ *and* $H_0 \cong SL_2(5) \times Z$, *where every Sylow subgroup of* Z *is cyclic and* $|Z|$ *is prime to* $2, 3$ *and* 5.

A Frobenius group with cyclic kernel of order m and cyclic complement of order n is denoted by $m\colon n$.

The following result is also used in this paper whose proof is included in [4].

Lemma 2.4. *Every* 2*-Frobenius group is solvable.*

Lemma 2.5 [7]. *Let* G *be a finite solvable group all of whose elements are of prime power order. Then the order of* G *is divisible by at most two distinct primes.*

Lemma 2.6 [12]. *Let* G *be a finite group,* $K \trianglelefteq G$, *and let* $\dfrac{G}{K}$ *be a Frobenius group with kernel* F *and cyclic complement* C. *If* $(|F|, |K|) = 1$ *and* F *does not lie in* $\dfrac{K \cdot C_G(K)}{K}$, *then* $r \cdot |C| \in w(G)$ *for some prime divisor* r *of* $|K|$.

Lemma 2.7 [18]. (1) *If there exists a primitive prime divisor* r *of* $q^n - 1$, *then* $L_n(q)$ *has a Frobenius subgroup with kernel of order* r *and cyclic complement of order* n.

(2) $L_n(q)$ *contains a Frobenius subgroup with kernel of order* q^{n-1} *and cyclic complement of order* $\dfrac{q^{n-1} - 1}{(n, q-1)}$.

Using [3] we can find $\mu(L_5(4))$ and using [13] we can find $\mu(U_4(4))$.

Lemma 2.8. *For the groups* $L_5(4)$ *and* $U_4(4)$ *we have*

$$\mu(L_5(4)) = \{8, 60, 126, 255, 315, 341\},$$

$$\mu(U_4(4)) = \{51, 65, 30, 20\}.$$

Using Lemma 2.8 we can draw the prime graphs of the groups $L_5(4)$ and $U_4(4)$ (see Figures 1 and 2).

Our main results are the following:

Theorem 2.1. *If* G *is a finite group such that* $\Gamma(G) = \Gamma(L_5(4))$, *then* G *has a normal subgroup* N *such that* $\Pi(N) \subseteq \{2, 3, 5\}$ *and* $\dfrac{G}{N} \cong L_5(4)$.

Theorem 2.2. *If* G *is a finite group such that* $\Gamma(G) = \Gamma(U_4(4))$, *then* G *has a normal subgroup* N *such that* $\Pi(N) \subseteq \{2, 3, 5\}$ *and* $\dfrac{G}{N} \cong U_4(4)$.

3. Proof of Theorem 2.1. First we prove Theorem 2.1 in series of steps. Therefore we assume G is a group with $\Gamma(G) = \Gamma(PSL_5(4))$. By Fig. 1 we have $s(G) = 2$, hence G has disconnected prime graph and we can use the structure theorem for G which is denoted by Lemma 2.1 here:

a) G is non-solvable.

If G is solvable, then consider a $\{7, 11, 17\}$-Hall subgroup of G and call it H. By Fig. 1, H dose not contain elements of order $7 \cdot 11$, $7 \cdot 17$, $11 \cdot 17$, and since it is solvable, by [7] we deduce $\Pi(H) \leq 2$, a contradiction.

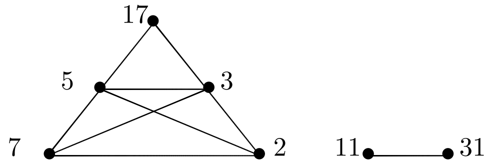

Fig. 1. The prime graph of $L_5(4)$.

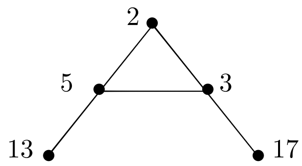

Fig. 2. The prime graph of $U_4(4)$.

b) G is neither a Frobenius nor a 2-Frobenius group.

By (a) and Lemma 2.4, G is not a 2-Frobenius group. If G is a Frobenius group, then by Lemma 2.1, $G = KC$ with Frobenius kernel K and Frobenius complement C with connected components $\Gamma(K)$ and $\Gamma(C)$. Obviously $\Gamma(K)$ is a graph with vertices $\{11, 31\}$ and $\Gamma(C)$ with vertex set $\{2, 3, 5, 7, 17\}$. Since G is non-solvable, by Lemma 2.3(a) C must be non-solvable. Therefore, by Lemma 2.3(f) C has a subgroup isomorphic to H_0 and $[C : H_0] \leq 2$, where $H_0 \cong SL_2(5) \times Z$ with Z cyclic of order prime to $2, 3, 5$. But $\mu(SL_2(5)) = \{4, 6, 10\}$ from which we can observe that H_0 has no element of order 15. This implies that C has no element of order 15, contradicting Fig. 1.

(a) and (b) imply that case (c) of Lemma 2.1 holds for G. Hence there is a non-abelian simple group P such that $P \leq \overline{G} = \dfrac{G}{N} \leq \mathrm{Aut}\,(P)$ where N is a nilpotent normal $\Pi_1(G)$-subgroup of G and $\dfrac{\overline{G}}{P}$ is a $\Pi_1(G)$-group and $s(P) \geq 2$. We have $\Pi_1(G) = \{2, 3, 5, 7, 17\}$ and $\Pi(G) = \{2, 3, 5, 7, 11, 17, 31\}$. Therefore P is a simple group with $\Pi(P) \subseteq \{2, 3, 5, 7, 11, 17, 31\}$, i.e., $P \in \mathfrak{S}_p$ where p is a prime number satisfying $p \leq 31, p \neq 13, 19, 23, 29$. Using [21] we list the possibilities for P in Table 1.

c) $\{11, 31\} \subseteq \Pi(P)$.

By Table 1, $|\mathrm{Out}\,(P)|$ is a number of the form $2^{\alpha} \cdot 3^{\beta} \cdot 5^{\gamma}$, therefore if $\dfrac{G}{N} = P \cdot S$ where $S \leq \mathrm{Out}\,(P)$, then $|P|_p = \left|\dfrac{G}{N}\right|_p / |S|_p$ for all $p \in \Pi(G)$, where n_p denotes the p-part of the integer $n \in N$. Hence $|N|_p = \dfrac{|G|_p}{|P|_p . |S|_p}$, from which the claim follows because $\Pi(N) \subseteq \{2, 3, 5, 7, 17\}$.

Therefore, only the following possibilities arise for P: $L_2(32)$, $L_5(4)$, $O_{12}^{+}(2)$, $S_{10}(2)$.

d) $P \cong L_5(4)$.

By [14] the group $L_2(32)$ has three prime graph components as follows $\Pi_1 = \{2\}$, $\Pi_2 = \{31\}$ and $\Pi_3 = \{3, 11\}$. Both groups $S_{10}(2)$ and $O_{12}^{+}(2)$ have two prime graph components with the

Table 1. Simple groups in $\mathfrak{S}_p$, $p \leq 31, p \neq 13, 19, 23, 29$.

P	$\|P\|$	$\|\text{out}\,(P)\|$	P	$\|P\|$	$\|\text{out}\,(P)\|$
A_5	$2^2 \cdot 3 \cdot 5$	2	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	2
$S_4(3)$	$2^6 \cdot 3^4 \cdot 5$	2	$U_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2
$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4	$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2
$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6	$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$O_{10}^-(2)$	$2^{20} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	2
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2	$L_3(5)$	$2^5 \cdot 3 \cdot 5^3 \cdot 31$	2
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2	$L_2(32)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	5
$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8	$L_2(5^3)$	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	6
$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2	$G_2(5)$	$2^6 \cdot 3^3 \cdot 5^6 \cdot 7 \cdot 31$	1
$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1	$L_5(2)$	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	2
$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6	$L_6(2)$	$2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$	2
$L_2(11)$	$2^2 \cdot 3 \cdot 5 \cdot 11$	2	$O_{10}^+(2)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$	2
M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	1	$L_5(4)$	$2^{20} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	
M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$	2		31	4
$U_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$	2	$S_{10}(2)$	$2^{25} \cdot 3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 17$	
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	2		31	1
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	2	$O_{12}^+(2)$	$2^{30} \cdot 3^8 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 17$	
M^cL	$2^6 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$	2		31	2

second component $\Pi_2 = \{31\}$. In any case the above facts violates the prime graph of $L_5(4)$ in Fig. 1, and this completes our claim.

e) $\dfrac{G}{N} \cong L_5(4)$. So far we proved that $P \leq \dfrac{G}{N} \leq \text{Aut}\,(P)$ where $P \cong L_5(4)$. But $\text{Aut}\,(L_5(4)) = L_5(4)\colon A$ where A is a four group. If σ_2 denotes the field automorphism and Θ the graph automorphism of $L_5(4)$, then $A = \langle \sigma_2, \Theta \rangle$ and we have the following possibilities for $\dfrac{G}{N}$:

$$\frac{G}{N} \cong L_5(4), \quad \frac{G}{N} \cong L_5(4)\colon \langle \sigma_2 \rangle, \quad \frac{G}{N} \cong L_5(4)\colon \langle \Theta \rangle,$$

$$\frac{G}{N} \cong L_5(4)\colon \langle \sigma_2 \cdot \Theta \rangle \quad \text{or} \quad \frac{G}{N} \cong L_5(4)\colon \langle \sigma_2, \Theta \rangle.$$

It is shown in [5] that all the above possibilities except $\frac{G}{N} \cong L_5(4)$ violates the structure of the prime graph of G in Fig. 1, therefore our claim is proved.

f) $\Pi(N) \subseteq \{2, 3, 5\}$.

We know that N is a nilpotent normal $\{2, 3, 5, 7, 17\}$-subgroup of G. Regarding Fig. 1 we obtain:

If $2 \mid |N|$, then $\Pi(N) \subseteq \{2, 3, 5, 7\}$.

If $17 \mid |N|$, then $\Pi(N) \subseteq \{3, 5, 17\}$.

If $7 \mid |N|$, then $\Pi(N) \subseteq \{2, 3, 5, 7\}$.

If $7 \mid |N|$ we may assume M is the characteristic $7'$-subgroup of N such that $\frac{H}{K} \cong L_5(4)$, where $H = \frac{G}{M}$ and $K = \frac{N}{M}$ is a non-trivial 7-group. By Lemma 2.7(1) $L_5(4)$ has a Frobenius group of the shape $4^4 : 255$, where 4^4 denotes Z_4^4 and is the Frobenius kernel and 255 is the cyclic group of order $5 \cdot 3 \cdot 17$ and is the Frobenius complement. Now by Lemma 2.6, H would have an element of order $7 \cdot 17$ violating Fig. 1. Also $L_5(4)$ has a Frobenius group of the shape $11 : 2$, then, if $17 \mid |N|$. Therefore by Lemma 2.6, H would have an element of order $2 \cdot 17$ violating Fig. 1. Therefore, the only possibility is $\Pi(N) \subseteq \{2, 3, 5\}$.

Theorem 2.1 is proved.

Proof of Theorem 2.2. Therefore we will assume that G is a group with $\Gamma(G) = \Gamma(U_4(4))$. By Fig. 2 we have $s(G) = 1$, i.e. the prime graph of G is connected. In this case Lemma 2.2 is applicable for the structure of G, because $\{2, 13, 17\}$ is an independent set as well as a 2-independent set for G, hence $t(G) = 3$ and $t(2, G) = 3$. Therefore there is a finite non-abelian simple group S such that $S \leq \overline{G} = \frac{G}{K} \leq \mathrm{Aut}\,(S)$ for the maximal normal solvable subgroup K of G.

Before we continue our investigation, we need a table similar to Table 1 for simple groups G with $13 \in \Pi(G) \subseteq \{2, 3, 5, \ldots, 13\}$ but $7 \nmid |G|$, $11 \nmid |G|$. Using [21] we obtain Table 2.

Now suppose G satisfies condition (a) of Lemma 2.2. We have $S \ncong A_7$ because $7 \nmid |G|$. If $S \cong L_2(q)$, q odd, then by Tables 1 and 2 we obtain $S \cong L_2(5), L_2(9), L_2(17)$ or $L_2(25)$. Regarding the order of outer automorphism of the groups S listed above we obtain the following facts:

If $S \cong PSL_2(5)$ or $PSL_2(9)$, then $\{13, 17\} \subseteq \Pi(K)$.

If $S \cong PSL_2(17)$, then $\{13\} \subseteq \Pi(K)$.

If $S \cong PSL_2(25)$, then $\{17\} \subseteq \Pi(K)$.

Table 2. Simple groups G with $13 \in \Pi(G) \subseteq \{2, 3, \ldots, 13, 17\}$ but $7, 11 \nmid |G|$.

S	$\|S\|$	$\|\mathrm{out}\,(S)\|$	S	$\|S\|$	$\|\mathrm{out}\,(S)\|$
$A_5 \cong L_2(5)$	$2^2 \cdot 3 \cdot 5$	2	$S_4(5)$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 13$	2
$A_6 \cong L_2(9)$	$2^3 \cdot 3^2 \cdot 5$	4	$L_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 13$	4
$L_3(3)$	$2^4 \cdot 3^3 \cdot 13$	2	${}^2F_4(2)'$	$2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$	2
$L_2(25)$	$2^3 \cdot 3 \cdot 5^2 \cdot 13$	4	$U_4(4)$	$2^{12} \cdot 3^2 \cdot 5^3 \cdot 13 \cdot 17$	4
$U_3(4)$	$2^6 \cdot 3 \cdot 5^2 \cdot 13$	4			

Now by Lemma 2.7(2), $PSL_2(q)$ has a Frobenius group of the shape $q\colon \dfrac{q-1}{2}$. Since $\dfrac{q-1}{2}$ for $q = 5, 9, 17, 25$ is even, Lemma 2.6 implies that G has an element of order $2 \cdot 13$ or $2 \cdot 17$, both contradicting Fig. 2.

Therefore, G must satisfy condition (b) of Lemma 2.2. The primes non-adjacent to 2 are 13 and 17, hence $\{13, 17\} \subseteq \Pi(S)$, and regarding Tables 1 and 2 the only simple group whose order is divisible by 13 and 17 is $U_4(4)$. Therefore we obtain $U_4(4) \leq \dfrac{G}{K} \leq \mathrm{Aut}\,(U_4(4))$.

Now we observe that the group $U_4(4)$ contains Frobenius subgroups of types $17\colon 4$ and $13\colon 3$. We may assume K is elementary abelian p-group for $p \in \{2, 3, 5, 13, 17\}$. Therefore by Lemma 2.6 and Fig. 2 the orders of K can not be divisible by 13. By Lemma 2.7 in [14] we have $17 \nmid |K|$. Therefore $\Pi(K) \subseteq \{2, 3, 5\}$.

By [2] the outer automorphism group of $U_4(4)$ is a cyclic group isomorphic to Z_4, hence we have the following lemma:

Lemma 4.1. *If G is an almost simple group related to $L = U_4(4)$, then G is isomorphic to one of the following groups*: L, $L\colon 2$ *or* $L\colon 4$.

If $U_4(4) \leq \dfrac{G}{K} \leq U_4(4)\colon 4$, then by above lemma, we have $\dfrac{G}{K} = U_4(4)$, $U_4(4)\colon 2$ or $U_4(4)\colon 4$.

If $\dfrac{G}{K} = U_4(4)\colon 2$, then let t denote the outer automorphism of order 2, by [1] we have $C^{(t)}_{U_4(4)} = S_4(4)$ implying that t centralizes an element of order 17 violating Fig. 2.

If $\dfrac{G}{K} = U_4(4)\colon 4$, then, similar to the above case, let t denote the outer automorphism of order 4, by [1] we have $C^{(t)}_{U_4(4)} = S_4(4)$ implying that t centralizes an element of order 17 violating Fig. 2.

Therefore, the only possibility is $\dfrac{G}{K} \cong U_4(4)$.

Theorem 2.2 is proved.

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