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ITERATION PROCESS FOR MULTIPLE ROGERS – RAMANUJAN IDENTITIES

ІТЕРАЦІЙНИЙ ПРОЦЕС ДЛЯ КРАТНИХ ТОТОЖНОСТЕЙ РОДЖЕРСА – РАМАНУДЖАНА

Replacing the monomials by an arbitrary sequence in the recursive lemma found by Bressoud (1983), we establish several general transformation formulas from unilateral multiple basic hypergeometric series to bilateral univariate ones, which are then used for the derivation of numerous multiple series identities of Rogers – Ramanujan type.

За допомогою заміни мономів довільною послідовністю в рекурентній лемі Брессо (1983) встановлено декілька загальних формул перетворення однобічних кратних основних гіпергеометричних рядів у двобічні одновимірні ряди, які потім використовуються для виведення численних тотожностей типу Роджерса – Рамануджана для кратних рядів.

1. Introduction and motivation. By means of the following well-known q-analogue of the binomial theorem

$$\frac{1}{(qa;q)_n}=\sum_{m=0}^{n}\begin{bmatrix}n\\m\end{bmatrix}\frac{q^{m^2}a^m}{(qa;q)_m},\quad\text{where}\quad\begin{bmatrix}n\\m\end{bmatrix}=\frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}. \tag{1}$$

Bressoud [19] (Lemma 2) devised ingeniously the recursive lemma on finite sums

$$\sum_{k=-n}^{n}\frac{q^{\lambda k^2}x^k}{(q;q)_{n+k}(q;q)_{n-k}}=\sum_{m=0}^{n}\frac{q^{m^2}}{(q;q)_{n-m}}\sum_{k=-m}^{m}\frac{q^{(\lambda-1)k^2}x^k}{(q;q)_{m+k}(q;q)_{m-k}}. \tag{2}$$

Iterating the last equation ℓ-times and then putting $\lambda=\ell+1/2$, he discovered the following multiple series transformation theorem [19] (Theorem):

$$\sum_{n\ge m_1\ge m_2\ge\ldots\ge m_\ell\ge 0}\frac{q^{m_1^2+m_2^2+\ldots+m_\ell^2}(x;q)_{m_\ell}(q/x;q)_{m_\ell}}{(q;q)_{n-m_1}(q;q)_{m_1-m_2}\ldots(q;q)_{m_{\ell-1}-m_\ell}(q;q)_{2m_\ell}}= \tag{3a}$$

$$=\sum_{k=-n}^{n}(-1)^k\begin{bmatrix}2n\\n+k\end{bmatrix}\frac{q^{k^2\ell+\binom{k}{2}}}{(q;q)_{2n}}x^k. \tag{3b}$$

The limiting case $n\to\infty$ of the last formula yields easy proofs and generalizations of the celebrated Rogers – Ramanujan identities (cf. Watson [59], Slater [52] (Eqs 14), and [18]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{\left[q^5,q^2,q^3;q^5\right]_\infty}{(q;q)_\infty}, \tag{4a}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{\left[q^5,q,q^4;q^5\right]_\infty}{(q;q)_\infty}. \tag{4b}$$

The similar identities in q-series are generally called identities of Rogers – Ramanujan type (RR-type), which express infinite series in terms of infinite products. The well-known Bailey lemma [13, 14] has been shown powerful in proving these identities. Slater [51, 52] exploited this technique extensively and collected 130 identities of RR-type. By iterative use of Bailey lemma, Andrews [5] showed how to embed each of classical identities of RR-type in an infinite family of multiple series identities. The typical approaches to them together with the main contributors may be sketched as follows:

Bailey lemma: Andrews and Bressoud et al. [1, 5, 6, 21, 43, 41, 54, 10, 48, 58].

Lattice path enumeration: Agarwal and Bressoud [2, 20].

Hall – Littlewood functions: Stembridge [55].

Partition bijections: Bressoud, Zeilberger [22], Garvan [35] and Lovejoy [40].

Multiple series transformations: Bressoud [17], Singh [50] and Chu [26, 27].

Observing carefully the proof of Bressoud [19], we notice that the monomials $\{x^k\}_{k\geq 0}$ appeared in (1) can be replaced by an arbitrary sequence $\{W_k\}_{k\in\mathbb{Z}}$. This suggests us to consider a more general iteration process. Following the same approach of Bressoud [19], we shall establish a remarkably useful transformation theorem involving an arbitrary sequence $\{W_k\}_{k\geq 0}$. By specifying the W-sequence, several transformation formulae from unilateral multiple series to bilateral univariate one will be derived. Their limiting cases lead us to several known and numerous new multiple series identities of RR-type.

Even though there is no technical difficulties to realize what is just described by means of Bailey lemma, we have opted to proceed along Bressoud's way. This is justified mainly by two reasons. Firstly, there is more freedom and transparency to manipulate the universal sequence $\{W_k\}_{k\geq 0}$ appearing in Lemma 1 and Theorem 1 than the two sequences $\{\alpha_k\}_{k\geq 0}$ and $\{\beta_k\}_{k\geq 0}$ that are tied by a linear relation in Bailey lemma. Secondly, in the final phase of constructing multiple series identities of RR-type, the factorization process through the identities of Jacobi's triple product and quintuple product will be facilitated by the bilateral sum displayed in (5b). This is traditionally done by incorporating two unilateral series into a bilateral one, which results often in small errors.

Following Bailey [11], Gasper, Rahman [36] and Slater [53], we shall utilize the following notations for basic hypergeometric series throughout the paper. Denote by $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ the sets of natural numbers, nonnegative integers and integers, respectively. For two indeterminate x and q, the shifted factorial of x with base q is defined by

$$(x;q)_0 = 1 \qquad \text{and} \qquad (x;q)_n = (1-x)(1-xq)\ldots(1-xq^{n-1}) \quad \text{for} \quad n\in\mathbb{N}.$$

When $|q|<1$, we have two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \qquad \text{and} \qquad (x;q)_n = (x;q)_\infty \,/\, (xq^n;q)_\infty\,.$$

With the multiparameter forms of shifted factorials being abbreviated to

$$[\,\alpha,\ \beta,\ldots,\ \gamma;q\,]_n = (\alpha;q)_n\,(\beta;q)_n\ldots(\gamma;q)_n\,,$$

$$\left[\begin{matrix}\alpha, & \beta, & \ldots, & \gamma\\ A, & B, & \ldots, & C\end{matrix}\,\middle|\,q\right]_n = \frac{(\alpha;q)_n\,(\beta;q)_n\ldots(\gamma;q)_n}{(A;q)_n\,(B;q)_n\ldots(C;q)_n};$$

we define the unilateral and bilateral basic hypergeometric series, respectively, by

$$ {}_{1+r}\phi_s\left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix}\,\Big|\, q; z\right] = \sum_{n=0}^{\infty} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} \left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ q, & b_1, & \ldots, & b_s \end{matrix}\,\Big|\, q\right]_n z^n, $$

$$ {}_{r}\psi_s\left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}\,\Big|\, q; z\right] = \sum_{n=-\infty}^{+\infty} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} \left[\begin{matrix} a_1, & a_2, & \ldots, & a_r \\ b_1, & b_2, & \ldots, & b_s \end{matrix}\,\Big|\, q\right]_n z^n, $$

where the base q will be restricted to $|q| < 1$ for nonterminating q-series.

The paper will be organized as follows. The next section will be devoted to the main theorem of this paper, which transforms a unilateral multiple series to a bilateral univariate series. Then we shall present, in the third section, its applications to multiple series transformation formulae and multiple series identities of RR-type. From the identities examined in this paper, one can see that the iteration process is powerful and simple for dealing with multiple series identities of RR-type, just like Bressoud's approach to the classical Rogers – Ramanujan identities. Finally, the paper ends with a table, putting, in evidence, the connections between the multiple series identities of RR-type and their $\ell = 1$ counterparts of single sum cases.

2. The main theorem and proof. Performing the replacements $m \to m - k$, $n \to n - k$ and $a \to q^{2k+\delta}$ with $\delta = 0, 1$, we can reformulate the equation displayed in (1) as

$$ \frac{1}{(q;q)_{n+k+\delta}} = \sum_{m=k}^{n} \begin{bmatrix} n-k \\ m-k \end{bmatrix} \frac{q^{(m-k)(m+k+\delta)}}{(q;q)_{m+k+\delta}}. $$

Let $\{W_k\}_{k\in\mathbb{Z}}$ be an arbitrary sequence. Multiplying by $W_k/(q;q)_{n-k}$ across the last equation, we may manipulate the following bilateral finite sum with respect to k over $-n-\delta \le k \le n$

$$ \sum_{k=-n-\delta}^{n} \frac{W_k}{(q;q)_{n-k}(q;q)_{n+k+\delta}} = \sum_{k=-n-\delta}^{n} \frac{W_k}{(q;q)_{n-k}} \sum_{m=k}^{n} \begin{bmatrix} n-k \\ m-k \end{bmatrix} \frac{q^{(m-k)(m+k+\delta)}}{(q;q)_{m+k+\delta}} = $$

$$ = \sum_{m=-n-\delta}^{n} \frac{q^{m^2+m\delta}}{(q;q)_{n-m}} \sum_{k=-n-\delta}^{m} \frac{q^{-k(k+\delta)}W_k}{(q;q)_{m-k}(q;q)_{m+k+\delta}} = $$

$$ = \sum_{m=0}^{n} \frac{q^{m^2+m\delta}}{(q;q)_{n-m}} \sum_{k=-m-\delta}^{m} \frac{q^{-k(k+\delta)}W_k}{(q;q)_{m-k}(q;q)_{m+k+\delta}}, $$

where the last line has been justified by the fact that the innermost summand vanishes for $m < 0$ and $k < -m - \delta$. Replacing k by $-k$ in the two extreme sums with respect to k, we find the following generalized recursive lemma.

Lemma 1 (Recursive sums)**.**

$$ \sum_{k=-n}^{n+\delta} \frac{W_k}{(q;q)_{n+k}(q;q)_{n-k+\delta}} = \sum_{m=0}^{n} \frac{q^{m^2+m\delta}}{(q;q)_{n-m}} \sum_{k=-m}^{m+\delta} \frac{q^{k(\delta-k)}W_k}{(q;q)_{m+k}(q;q)_{m-k+\delta}}. $$

When $\delta = 0$ and $W_k \to q^{\lambda k^2} x^k$, this lemma reduces clearly to Bressoud's Lemma (2). However, with the presence of an arbitrary sequence W_k, our lemma has more flexibility in application and will be more useful.

Iterating ℓ-times the recursion in Lemma 1 leads to the following equation:

$$\sum_{k=-n}^{n+\delta} \frac{W_k}{(q;q)_{n+k}(q;q)_{n-k+\delta}} = \sum_{n\geq r_1\geq r_2\geq\ldots\geq r_\ell\geq 0} \frac{q^{r_1(r_1+\delta)+r_2(r_2+\delta)+\ldots+r_\ell(r_\ell+\delta)}}{(q;q)_{n-r_1}(q;q)_{r_1-r_2}\cdots(q;q)_{r_{\ell-1}-r_\ell}} \times$$

$$\times \sum_{k=-r_\ell}^{r_\ell+\delta} \frac{q^{k\ell(\delta-k)}W_k}{(q;q)_{r_\ell+k}(q;q)_{r_\ell-k+\delta}}.$$

In order to shorten the long expressions, we make the replacements on summation indices and fix the compact notations as follows:

$$\left.\begin{aligned} n-r_1 &\to m_0,\\ r_1-r_2 &\to m_1,\\ \ldots \quad &\quad \ldots\\ r_{\ell-1}-r_\ell &\to m_{\ell-1},\\ r_\ell &\to m_\ell, \end{aligned}\right\} \quad \text{and} \quad \begin{cases} \tilde{m}=(m_1,m_2,\ldots,m_\ell),\\ M_k=\displaystyle\sum_{\iota=k}^{\ell} m_\iota, \quad 0\leq k\leq \ell. \end{cases}$$

Further replacing W_k by $(-1)^k q^{\binom{k}{2}+k\ell(k-\delta)}W_k$ in the last finite series transformation, we may reformulate the result as the following main theorem of this paper.

Theorem 1 (Multiple series transformation). *For an arbitrary bilateral sequence $\{W_k\}_{k\in\mathbb{Z}}$, there holds the multiple series transformation*

$$\sum_{M_0=n} \frac{(q;q)_{2n+\delta}}{(q;q)_{m_0}(q;q)_{m_\ell+\delta}} \prod_{\iota=1}^{\ell} \frac{q^{M_\iota(M_\iota+\delta)}}{(q;q)_{m_\iota}} \sum_{k=-m_\ell}^{m_\ell+\delta} q^{k(m_\ell+\delta)} \frac{(q^{-m_\ell-\delta};q)_k}{(q^{m_\ell+1};q)_k} W_k = \tag{5a}$$

$$= \sum_{k=-n}^{n+\delta} (-1)^k \begin{bmatrix} 2n+\delta \\ n+k \end{bmatrix} q^{k\ell(k-\delta)+\binom{k}{2}} W_k, \tag{5b}$$

where the multiple sum on the left runs over $(m_0,m_1,\ldots,m_\ell)\in\mathbb{N}_0^{1+\ell}$ subject to the condition $M_0=m_0+m_1+\ldots+m_\ell=n$.

This theorem is remarkably useful for deriving concrete multiple transformation formulae and multiple series identities of RR-type. We first examine Bressoud's work now. More examples will be presented in the next section.

Letting $W_k=x^k$ in Theorem 1 and then evaluating the sum with respect to k displayed in (5a) by means of the bilateral q-binomial theorem (cf. Chu [29] (Eq. 5)

$$\sum_{k=-m}^{m+\delta} q^{k(m+\delta)} \frac{(q^{-m-\delta};q)_k}{(q^{m+1};q)_k} x^k = \frac{(q;q)_m(q;q)_{m+\delta}(x;q)_{m+\delta}(q/x;q)_m}{(q;q)_{2m+\delta}}$$

we derive the following variant of Bressoud's theorem stated in (3).

Theorem 2 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q;q)_{2n+\delta}(q;q)_{m_\ell}(x;q)_{m_\ell+\delta}(q/x;q)_{m_\ell}}{(q;q)_{m_0}(q;q)_{2m_\ell+\delta}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota(M_\iota+\delta)}}{(q;q)_{m_\iota}}=$$

$$=\sum_{k=-n}^{n+\delta}(-1)^k\begin{bmatrix}2n+\delta\\ n+k\end{bmatrix}q^{k\ell(k-\delta)+\binom{k}{2}}x^k.$$

For $\delta=0$, it is not hard to see that the last theorem coincides with multiple series transformation formula (3). Letting $n\to\infty$ and evaluating the last sum through Jacobi's triple product identity [39] (see [8, p. 497] for historical notes)

$$[q,x,q/x;q]_\infty=\sum_{k=-\infty}^{+\infty}(-1)^k q^{\binom{k}{2}}x^k \tag{6}$$

we obtain the following multiple nonterminating series identity.

Proposition 1 (Multiple series identity)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}(x;q)_{m_\ell+\delta}(q/x;q)_{m_\ell}}{(q;q)_{2m_\ell+\delta}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota(M_\iota+\delta)}}{(q;q)_{m_\iota}}=\frac{\left[q^{1+2\ell},q^{\ell(1-\delta)}x,q^{1+\ell(1+\delta)}/x;q^{1+2\ell}\right]_\infty}{(q;q)_\infty}.$$

This identity may be considered as a common generalization of the multiple series identities of RR-type displayed in the following corollaries.

***Corollary* 1** ($\delta=0$ and $x=1$: Andrews [4] (Eq. 2.14) and Bressoud [18] (Eq. 6.1))**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{3+2\ell},q^{1+\ell},q^{2+\ell};q^{3+2\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 2** ($\delta=1$ and $x=q$: Andrews [6] (Eq. 3.46) and Stembridge [55] (Eq. c))**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{3+2\ell},q,q^{2+2\ell};q^{3+2\ell}\right]_\infty}{(q;q)_\infty}.$$

These two corollaries are multiple series generalizations of the classical Rogers – Ramanujan identities (4a) and (4b). They have been covered extensively in literature. Additional information may further be found in [5, 10, 18, 21, 22, 34, 35, 54 – 57] for Corollary 1 and [10, 34] for Corollary 2. More comments will be made after Corollary 12.

3. Transformations and multiple series identities. Following the example illustrated in the last section, we shall derive sixteen multiple finite series transformations corresponding to different settings of W-sequence. Then their limiting cases will yield several known and numerous new multiple series identities of RR-type, that will be displayed as corollaries.

For properly chosen W_k, the corresponding finite sums displayed in (5a) will essentially be evaluated by Bailey's summation formula of very well-poised bilateral ${}_6\psi_6$-series [12] (see also [28] and [36] (II-33)) with $|qa^2/bcde|<1$

$$ {}_6\psi_6 \left[\begin{matrix} q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & qa/d, & qa/e \end{matrix} \middle| q; \frac{qa^2}{bcde} \right] = \tag{7a} $$

$$ = \left[\begin{matrix} q, qa, q/a, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de \\ qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde \end{matrix} \middle| q \right]_\infty \tag{7b} $$

together with two particular cases due to Bailey [15] (cf. Chu [25] (Eqs 3.16a-b-c))

$$ {}_4\psi_4 \left[\begin{matrix} qw, & b, & c, & d \\ w, & q/b, & q/c, & q/d \end{matrix} \middle| q; \frac{q}{bcd} \right] = \left[\begin{matrix} q, & q/bc, & q/bd, & q/cd \\ q/b, & q/c, & q/d, & q/bcd \end{matrix} \middle| q \right]_\infty, \tag{8} $$

$$ {}_5\psi_5 \left[\begin{matrix} qu, & qv, & b, & c, & d \\ u, & v, & 1/b, & 1/c, & 1/d \end{matrix} \middle| q; \frac{q^{-1}}{bcd} \right] = \frac{uv-1/q}{(1-u)(1-v)} \left[\begin{matrix} q, \ 1/bc, 1/bd, 1/cd \\ q/b, q/c, q/d, q^{-1}/bcd \end{matrix} \middle| q \right]_\infty. \tag{9} $$

The limiting cases of the multiple series transformations will be simplified through Jacobi's triple product identity (6) and its variant, in view of the parity of summation index, which is originally due to Bailey [16] (Eq. 4.1)

$$ \left[q^2, \, qy, \, q/y; q^2\right]_\infty = \sum_{n=-\infty}^{+\infty} \left\{1 - yq^{1+4n}\right\} q^{4n^2} \, y^{2n} \tag{10} $$

as well as the quintuple product identity [23, 30, 31, 33, 60]

$$ [q, z, q/z; q]_\infty \times \left[qz^2, q/z^2; q^2\right]_\infty = \sum_{n=-\infty}^{+\infty} \left\{1 - zq^n\right\} q^{3\binom{n}{2}} \left(qz^3\right)^n. \tag{11} $$

Considering that the computations from multiple series transformations to multiple series identities of RR-type are entirely routine, we shall not reproduce them in details. Instead, the specific parameter settings will briefly be indicated in the headers of corollaries. Throughout this section, the Gaussian binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ will be denoted by $\begin{bmatrix} n \\ k \end{bmatrix}_{q^m}$ under the base change $q \to q^m$ for $m \in \mathbb{N}$. In addition, we shall also fix $\varepsilon = \pm 1$ and $\delta_{m,n}$, the usual Kronecker symbol.

3.1. Letting $\delta = 0$ and

$$ W_k = \frac{1 - q^k w}{1 - w} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bd}\right)^k $$

we may evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (8) as

$$ \sum_{k=-m}^{m} q^{km} \frac{(q^{-m}; q)_k}{(q^{m+1}; q)_k} \frac{1 - q^k w}{1 - w} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q \right]_k \left(\frac{q}{bd}\right)^k = \left[\begin{matrix} q, & q/bd \\ q/b, & q/d \end{matrix} \middle| q \right]_m. $$

In view of Theorem 1, this leads to the following transformation formula.

Theorem 3 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q;q)_{2n}}{(q;q)_{m_0}}\left[\begin{matrix}q/bd\\ q/b,q/d\end{matrix}\Big|q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}}:=$$

$$:=\sum_{k=-n}^{n}\left(\frac{-1}{bd}\right)^k\frac{1-q^kw}{1-w}\left[\begin{matrix}2n\\ n+k\end{matrix}\right]\left[\begin{matrix}b, & d\\ q/b, & q/d\end{matrix}\Big|q\right]_k q^{\ell k^2+\binom{k+1}{2}}.$$

Letting $n\to\infty$, we establish the nonterminating multiple transformation.
Proposition 2 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\left[\begin{matrix}q/bd\\ q/b,q/d\end{matrix}\Big|q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}}=$$

$$=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\left(\frac{-1}{bd}\right)^k\frac{1-q^kw}{1-w}\left[\begin{matrix}b, & d\\ q/b, & q/d\end{matrix}\Big|q\right]_k q^{\ell k^2+\binom{k+1}{2}}.$$

Five multiple series identities of RR-type are derived from this proposition.
Corollary **3** ($w=b=-1,\ d\to0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-1)^{m_\ell}}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{2\ell},q^\ell,q^\ell;q^{2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **4** ($b,\ d\to0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}(-1)^{m_\ell}q^{-\binom{1+m_\ell}{2}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{2\ell-1},q^{\ell-1},q^\ell;q^{2\ell-1}\right]_\infty}{(q;q)_\infty}.$$

Corollary **5** ($w=b=-1,\ d\to\infty$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{2+2\ell},q^{1+\ell},q^{1+\ell};q^{2+2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **6** ($b\to0,\ d=q^{1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{q^{-m_\ell}}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{4\ell},-q^{2\ell-1},-q^{1+2\ell};q^{4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

Corollary **7** ($b\to\infty,\ d=q^{1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{4+4\ell},-q^{1+2\ell},-q^{3+2\ell};q^{4+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

The identities displayed in Corollaries 5 and 7 were found, through iterative use of Bailey lemma, by Paule [43] (Eqs 44 and 54), where the first one contains printing errors. For different proofs, refer to Warnaar [57, 56] for the first and Andrews [9] (Eq. 7.26), Bressoud [21] and Chu [27] (Example 16) for the second.

3.2. For $\delta = 0$, first take

$$W_k = \frac{1-q^{-k}w}{1-w}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\Big| q\right]_k \left(\frac{q}{bd}\right)^k$$

in Theorem 1. Then evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (9) as

$$\sum_{k=-m}^{m} q^{km}\frac{(q^{-m};q)_k}{(q^{m+1};q)_k}\frac{1-q^{-k}w}{1-w}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\Big| q\right]_k \left(\frac{q}{bd}\right)^k = \frac{1-q^m w}{1-w}\left[\begin{matrix} q, & 1/bd \\ q/b, & q/d \end{matrix}\Big| q\right]_m.$$

Therefore we derive the following terminating series transformation formula.

Theorem 4 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{1-q^{m_\ell}w}{1-w}\frac{(q;q)_{2n}}{(q;q)_{m_0}}\left[\begin{matrix} 1/bd \\ q/b, q/d \end{matrix}\Big| q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}} =$$

$$= \sum_{k=-n}^{n}\left(\frac{-1}{bd}\right)^k\frac{1-q^{-k}w}{1-w}\begin{bmatrix} 2n \\ n+k \end{bmatrix}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\Big| q\right]_k q^{\ell k^2+\binom{k+1}{2}}.$$

The limiting case $n \to \infty$ leads to the nonterminating series transformation.

Proposition 3 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1-q^{m_\ell}w}{1-w}\left[\begin{matrix} 1/bd \\ q/b, q/d \end{matrix}\Big| q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}} =$$

$$= \frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\left(\frac{-1}{bd}\right)^k\frac{1-q^{-k}w}{1-w}\left[\begin{matrix} b, & d \\ 1/b, & 1/d \end{matrix}\Big| q\right]_k q^{\ell k^2+\binom{k+1}{2}}.$$

Five multiple series identities of RR-type are derived from this proposition.

Corollary **8** ($w = 0$, $b = -1$, $d \to 0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-q)^{-m_\ell}}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}} = \frac{\left[q^{2\ell}, q^{\ell-1}, q^{1+\ell}; q^{2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **9** ($w = 0$, $b, d \to 0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}(-1)^{m_\ell}q^{-\binom{m_\ell}{2}-2m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}} = \frac{\left[q^{2\ell-1}, q^{\ell-2}, q^{1+\ell}; q^{2\ell-1}\right]_\infty}{(q;q)_\infty}.$$

Corollary **10** ($w, d \to \infty$, $b = -1$: Stanton [54, p. 63])**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{q^{m_\ell}}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}} = \frac{\left[q^{2+2\ell}, q^{\ell}, q^{2+\ell}; q^{2+2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **11** ($b=-1$, $w=-d=q^{1/2} \mid q\to q^2$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{1-q^{2m_\ell+1}}{1-q}\frac{(q^{-1};q^2)_{m_\ell}}{(-q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{2+4\ell},q^{2\ell-1},q^{3+2\ell};q^{2+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

Corollary **12** ($b,d,w\to\infty$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}q^{m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{3+2\ell},q^{\ell},q^{3+\ell};q^{3+2\ell}\right]_\infty}{(q;q)_\infty}.$$

There is a more general multiple series identity due to Andrews [3] and Gordon [37]

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}q^{\sum_{\iota=n}^{\ell}m_\iota}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{3+2\ell},q^{n},q^{3-n+2\ell};q^{3+2\ell}\right]_\infty}{(q;q)_\infty}$$

which results in a common extension of Corollaries 1, 2 and 12. This important identity has been studied extensively in literature. Different proofs may be found in [27, 1, 2, 7, 10, 17, 18, 20, 21, 44, 34, 35, 43, 24, 41], just for examples.

3.3. For $\delta=1$ and

$$W_k=\frac{(1-q^ku)(1-q^kv)}{uv-q^{-1}}\left[\begin{matrix}b, & d\\ 1/b, & 1/d\end{matrix}\middle|q\right]_k\left(\frac{q^{-1}}{bd}\right)^k$$

evaluate the sum displayed in (5a) through the bilateral ${}_5\psi_5$-series identity (9) as

$$\sum_{k=-m}^{m+1}q^{km}\frac{(q^{-m-1};q)_k}{(q^{m+1};q)_k}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(bd)^k}\left[\begin{matrix}b, & d\\ 1/b, & 1/d\end{matrix}\middle|q\right]_k=\frac{(q;q)_{m+1}(1/bd;q)_m}{(q/b;q)_m(q/d;q)_m}.$$

According to Theorem 1, we have the following transformation formula.

Theorem 5 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}}{(q;q)_{m_0}}\left[\begin{matrix}1/bd\\ q/b,q/d\end{matrix}\middle|q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}}=$$

$$=\sum_{k=-n}^{n+1}(-1)^k\begin{bmatrix}2n+1\\ n+k\end{bmatrix}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(qbd)^k}\left[\begin{matrix}b, & d\\ 1/b, & 1/d\end{matrix}\middle|q\right]_k q^{(2\ell+1)\binom{k}{2}}.$$

Letting $n\to\infty$ in this theorem leads to the nonterminating series transformation.

Proposition 4 (Nonterminating series transformation)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\left[\begin{matrix}1/bd\\ q/b,q/d\end{matrix}\middle|q\right]_{m_\ell}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}}=$$

$$=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(-qbd)^k}\left[\begin{matrix}b,\ d\\ 1/b,1/d\end{matrix}\middle|q\right]_k q^{(2\ell+1)\binom{k}{2}}.$$

Five multiple series identities of RR-type are derived from this proposition.

***Corollary* 13** ($b=-1,\ d\to 0$).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-q)^{-m_\ell}}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{2\ell},q,q^{2\ell-1};q^{2\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 14** ($b,\ d\to 0$).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}(-1)^{m_\ell}q^{-\binom{m_\ell}{2}-2m_\ell}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{2\ell-1},q,q^{2\ell-2};q^{2\ell-1}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 15** ($u=-v=b=-d=q^{-1/2}$: Chu [27] (Example 2)).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-q;q)_{m_\ell}}{(q;q^2)_{m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{1+2\ell},-q^{1+2\ell},-q^{1+2\ell};q^{1+2\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 16** ($d\to 0,\ b=u=q^{-1/2}\mid q\to q^2$).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{q^{-m_\ell}}{(q;q^2)_{m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{4\ell},-q,-q^{4\ell-1};q^{4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

***Corollary* 17** ($d\to\infty,\ b=u=q^{-1/2}\mid q\to q^2$: Paule [43] (Eq. 53)).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1}{(q;q^2)_{m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{4+4\ell},-q,-q^{3+4\ell};q^{4+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

3.4. Letting $\delta=0$ and

$$W_{2k+1}=0\qquad\text{and}\qquad W_{2k}=\frac{1-q^{2k}w}{1-w}\,\frac{(c;q^2)_k}{(q^2/c;q^2)_k}\left(\frac{q}{c}\right)^k$$

in Theorem 1, then evaluating the sum with respect to k displayed in (5a) by means of the bilateral ${}_4\psi_4$-series identity (8) as

$$\sum_k q^{2km}\frac{(q^{-m};q)_{2k}}{(q^{m+1};q)_{2k}}\,\frac{1-q^{2k}w}{1-w}\,\frac{(c;q^2)_k}{(q^2/c;q^2)_k}\left(\frac{q}{c}\right)^k=\frac{(q;q)_m(q/c;q^2)_m}{(q/c;q)_m(q;q^2)_m}$$

we establish the following terminating series transformation formula.

Theorem 6 (Terminating series transformation).

$$\sum_{M_0=n}\frac{(q;q)_{2n}(q/c;q^2)_{m_\ell}}{(q;q)_{m_0}(q;q^2)_{m_\ell}(q/c;q)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}}=$$

$$=\sum_k\frac{1-q^{2k}w}{1-w}\begin{bmatrix}2n\\n+2k\end{bmatrix}\frac{(c;q^2)_k}{(q^2/c;q^2)_k}\,\frac{q^{(4\ell+2)k^2}}{c^k}.$$

Its limiting case $n\to\infty$ yields the nonterminating multiple series transformation.

Proposition 5 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q/c;q^2)_{m_\ell}}{(q;q^2)_{m_\ell}(q/c;q)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2}}{(q;q)_{m_\iota}}=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\frac{1-q^{2k}w}{1-w}\frac{(c;q^2)_k}{(q^2/c;q^2)_k}\frac{q^{(4\ell+2)k^2}}{c^k}.$$

Four multiple series identities of RR-type are derived from this proposition.

Corollary **18** ($c\to 0$: Agarwal, Bressoud [2] (Eq. 1.8) and [20] (Eq. 1.8))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{q^{\binom{m_\ell}{2}}}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{2+8\ell},q^{4\ell},q^{2+4\ell};q^{2+8\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **19** ($w=c=-1$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-q;q^2)_{m_\ell}}{(q;q^2)_{m_\ell}(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{4+8\ell},q^{2+4\ell},q^{2+4\ell};q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **20** ($c=\varepsilon q$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(\varepsilon;q^2)_{m_\ell}}{(q;q^2)_{m_\ell}(\varepsilon;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{4+8\ell},-\varepsilon q^{1+4\ell},-\varepsilon q^{3+4\ell};q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **21** ($c\to\infty$: Agarwal, Bressoud [2, 20] and Warnaar [58])**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1}{(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{6+8\ell},q^{2+4\ell},q^{4+4\ell};q^{6+8\ell}\right]_\infty}{(q;q)_\infty}.$$

We remark that when $\ell=1$, the last corollary reduces to a classical identity of RR-type due to Rogers [45] (see also Slater [52] (Eq. 61)).

3.5. Let $\delta=1$ and

$$W_{2k+1}=0\qquad\text{and}\qquad W_{2k}=\frac{1-q^{-2k}w}{1-w}\frac{(c;q^2)_k}{(1/c;q^2)_k}\left(\frac{q}{c}\right)^k$$

in Theorem 1. Then evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_5\psi_5$-series identity (9) as

$$\sum_k q^{2km}\frac{(q^{-m-1};q)_{2k}}{(q^{m+1};q)_{2k}}\frac{1-q^{-2k}w}{1-w}\frac{(c;q^2)_k}{(1/c;q^2)_k}\left(\frac{q^3}{c}\right)^k=\frac{1-q^m w}{1-w}\frac{(q;q)_{m+1}(q/c;q^2)_m}{(q;q^2)_{m+1}(q/c;q)_m}.$$

We therefore find the following terminating series transformation formula.

Theorem 7 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{1-q^{m_\ell}w}{1-w}\frac{(q;q)_{2n+1}(q/c;q^2)_{m_\ell}}{(q;q)_{m_0}(q;q^2)_{m_\ell+1}(q/c;q)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}}=$$

$$=\sum_k\frac{1-q^{-2k}w}{1-w}\begin{bmatrix}2n+1\\n+2k\end{bmatrix}\frac{(c;q^2)_k}{(1/c;q^2)_k}\frac{q^{(2\ell+1)\binom{2k}{2}+k}}{c^k}.$$

Letting $n\to\infty$ in this theorem gives the nonterminating series transformation.

Proposition 6 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{1-q^{m_\ell}w}{1-w} \frac{(q/c;q^2)_{m_\ell}}{(q;q^2)_{m_\ell+1}(q/c;q)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}} =$$

$$= \frac{1}{(q;q)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1-q^{-2k}w}{1-w} \frac{(c;q^2)_k}{(1/c;q^2)_k} \frac{q^{(2\ell+1)\binom{2k}{2}+k}}{c^k}.$$

Five multiple series identities of RR-type are derived from this proposition.

***Corollary* 22** ($w=0$, $c\to 0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{q^{\binom{m_\ell}{2}}}{(q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{2+8\ell}, q^{2+2\ell}, q^{6\ell}; q^{2+8\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 23** ($c=-1, w\to\infty$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} q^{m_\ell} \frac{(-q;q^2)_{m_\ell}}{(q;q^2)_{m_\ell+1}(-q;q)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{4+8\ell}, q^{2\ell}, q^{4+6\ell}; q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 24** ($c=\varepsilon q^{-1}$, $w=\varepsilon q$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(\varepsilon q^2;q^2)_{m_\ell}}{(q;q^2)_{m_\ell+1}(\varepsilon q;q)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{4+8\ell}, -\varepsilon q^{1+2\ell}, -\varepsilon q^{3+6\ell}; q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 25** ($w=0$, $c=-1$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q;q^2)_{m_\ell}}{(q;q^2)_{m_\ell+1}(-q;q)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{4+8\ell}, q^{2+2\ell}, q^{2+6\ell}; q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 26** ($c, w\to\infty$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{q^{m_\ell}}{(q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{6+8\ell}, q^{2\ell}, q^{6+6\ell}; q^{6+8\ell}\right]_\infty}{(q;q)_\infty}.$$

3.6. Letting $\delta=1$ and

$$W_{2k+1}=0 \qquad \text{and} \qquad W_{2k} = \frac{q^{4k}-q}{(qbd)^k} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q^2\right]_k$$

we may evaluate the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (7a), (7b) as

$$\sum_k q^{2k(m+1)} \frac{(q^{-m-1};q)_{2k}}{(q^{m+1};q)_{2k}} \left[\begin{matrix} b, & d \\ q/b, & q/d \end{matrix} \middle| q^2\right]_k \frac{q^{4k}-q}{(qbd)^k} = \frac{(q;q)_{m+1}(q/bd;q^2)_m}{(-q;q)_m[q/b,q/d;q]_m}.$$

In view of Theorem 1, this leads to the following transformation formula.

Theorem 8 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}(q/bd;q^2)_{m_\ell}}{(q;q)_{m_0}(-q;q)_{m_\ell}[q/b,q/d;q]_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}}=$$

$$=\sum_k\frac{q^{4k}-q}{(qbd)^k}\begin{bmatrix}2n+1\\n+2k\end{bmatrix}\left[\begin{matrix}b, & d\\ q/b, & q/d\end{matrix}\,\middle|\,q^2\right]_k q^{(2\ell+1)\binom{2k}{2}}.$$

Letting $n\to\infty$, we obtain the nonterminating multiple series transformation.

Proposition 7 (Nonterminating series transformation)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{(q/bd;q^2)_{m_\ell}}{(-q;q)_{m_\ell}[q/b,q/d;q]_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}}=$$

$$=\frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\frac{q^{4k}-q}{(qbd)^k}\left[\begin{matrix}b, & d\\ q/b, & q/d\end{matrix}\,\middle|\,q^2\right]_k q^{(2\ell+1)\binom{2k}{2}}.$$

Five multiple series identities of RR-type are derived from this proposition.

Corollary **27** ($b=-d=q^{1/2}$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{(-1;q^2)_{m_\ell}}{(-q;q)_{m_\ell}(q;q^2)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[-q^{1+2\ell},q,-q^{2\ell};-q^{1+2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **28** ($b=-d=q^{-1/2}$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{(-q^2;q^2)_{m_\ell}}{(-q;q)_{m_\ell}(q;q^2)_{m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{4+8\ell},q^{1+2\ell},q^{3+6\ell};q^{4+8\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **29** ($b,\ d\to\infty$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{1}{(-q;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=\frac{\left[q^{2+2\ell},q,q^{1+2\ell};q^{2+2\ell}\right]_\infty}{(q;q)_\infty}.$$

Corollary **30** ($d\to 0,\ b=-q^{1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}(-1)^{m_\ell}\frac{q^{m_\ell^2-2m_\ell}}{(-q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{1+4\ell},q^2,q^{4\ell-1};q^{1+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

Corollary **31** ($d\to\infty,\ b=-q^{1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde m\in\mathbb{N}_0^\ell}\frac{1}{(-q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\left[q^{3+4\ell},q^2,q^{1+4\ell};q^{3+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

The univariate cases of the last two identities can be found in Rogers [45], where the latter is usually called the second Rogers, Selberg identity [47].

3.7. For $\delta = 0$ and

$$W_{3k+1} = W_{3k+2} = 0 \qquad \text{and} \qquad W_{3k} = \frac{1 - q^{3k}w}{1 - w}$$

in Theorem 1, evaluate the sum displayed in (5a) by means of (8) as

$$\sum_k q^{3km} \frac{(q^{-m};q)_{3k}}{(q^{m+1};q)_{3k}} \frac{1 - q^{3k}w}{1 - w} = \frac{(q;q)_m (q^3;q^3)_{m-1+\delta_{0,m}}}{(q;q)_{2m-1+\delta_{0,m}}}.$$

We derive consequently the following multiple series transformation formula.

Theorem 9 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q;q)_{2n}(q^3;q^3)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q)_{m_0}(q;q)_{2m_\ell-1+\delta_{0,m_\ell}}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q;q)_{m_k}} = \sum_k (-1)^k \frac{1-q^{3k}w}{1-w} \begin{bmatrix} 2n \\ n+3k \end{bmatrix} q^{9\ell k^2 + \binom{3k}{2}}.$$

Letting $n \to \infty$ in this theorem leads us to the multiple series identity of RR-type.

***Corollary* 32.**

$$\sum_{\tilde{m} \in \mathbb{N}_0^\ell} \frac{(q^3;q^3)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q)_{2m_\ell-1+\delta_{0,m_\ell}}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q;q)_{m_k}} = \frac{\left[q^{9+18\ell}, q^{3+9\ell}, q^{6+9\ell}; q^{9+18\ell}\right]_\infty}{(q;q)_\infty}.$$

The last corollary may be considered as a multiple series generalization of a classical identity of RR-type due to Bailey, Dyson [13] (Eq. B4).

3.8. Letting $\delta = 1$ and

$$W_{3k+1} = W_{3k+2} = 0 \qquad \text{and} \qquad W_{3k} = 1$$

we may evaluate the sum displayed in (5a) through (9) as

$$\sum_k q^{3k(m+1)} \frac{(q^{-m-1};q)_{3k}}{(q^{m+1};q)_{3k}} = \frac{(q;q)_{m+1}(q^3;q^3)_m}{(q;q)_{2m+1}}.$$

In view of Theorem 1, we derive the following transformation formula.

Theorem 10 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q;q)_{2n+1}(q^3;q^3)_{m_\ell}}{(q;q)_{m_0}(q;q)_{2m_\ell+1}} \prod_{\iota=1}^{\ell} \frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}} = \sum_k (-1)^k \begin{bmatrix} 2n+1 \\ n+3k \end{bmatrix} q^{(2\ell+1)\binom{3k}{2}}.$$

For $n \to \infty$, this theorem becomes the multiple series identity of RR-type.

***Corollary* 33.**

$$\sum_{\tilde{m} \in \mathbb{N}_0^\ell} \frac{(q^3;q^3)_{m_\ell}}{(q;q)_{2m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{9+18\ell}, q^{3+6\ell}, q^{6+12\ell}; q^{9+18\ell}\right]_\infty}{(q;q)_\infty}.$$

This corollary extends another identity due to Bailey, Dyson [13] (Eq. B3).

3.9. Taking $\delta = 1$ and

$$W_{3k+1} = W_{3k+2} = 0 \qquad \text{and} \qquad W_{3k} = \frac{(c;q^3)_k}{(q^2/c;q^3)_k}\frac{q^{6k}-q}{(q^2c)^k}$$

in Theorem 1 and then evaluating the sum with respect to k displayed in (5a) by means of the bilateral ${}_6\psi_6$-series identity (7a), (7b) as

$$\sum_k q^{3k(m+1)}\frac{(q^{-m-1};q)_{3k}}{(q^{m+1};q)_{3k}}\frac{(c;q^3)_k}{(q^2/c;q^3)_k}\frac{q^{6k}-q}{(q^2c)^k} = \frac{(q;q)_m(q;q)_{m+1}(q/c;q^3)_m}{(q;q)_{2m}(q/c;q)_m}$$

we establish the following multiple series transformation formula.

Theorem 11 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q;q)_{2n+1}(q;q)_{m_\ell}(q/c;q^3)_{m_\ell}}{(q;q)_{m_0}(q;q)_{2m_\ell}(q/c;q)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}} =$$

$$= \sum_k \frac{q^{6k}-q}{(-q^2c)^k}\begin{bmatrix}2n+1\\ n+3k\end{bmatrix}\frac{(c;q^3)_k}{(q^2/c;q^3)_k}q^{(2\ell+1)\binom{3k}{2}}.$$

The limiting case $n\to\infty$ leads us to the nonterminating series transformation.

Proposition 8 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}(q/c;q^3)_{m_\ell}}{(q;q)_{2m_\ell}(q/c;q)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}} = \frac{1}{(q;q)_\infty}\sum_{k=-\infty}^{+\infty}\frac{q^{6k}-q}{(-q^2c)^k}\frac{(c;q^3)_k}{(q^2/c;q^3)_k}q^{(2\ell+1)\binom{3k}{2}}.$$

Four multiple series identities of RR-type are derived from this proposition.

Corollary **34** ($c\to 0$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} q^{2\binom{m_\ell}{2}}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{2+6\ell},q,q^{1+6\ell};q^{2+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{6\ell},q^{4+6\ell};q^{4+12\ell}\right]_\infty.$$

Corollary **35** ($c=-q$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}(-1;q^3)_{m_\ell}}{(q;q)_{2m_\ell}(-1;q)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{3+6\ell},q,q^{2+6\ell};q^{3+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{1+6\ell},q^{5+6\ell};q^{6+12\ell}\right]_\infty.$$

Corollary **36** ($c\to\infty$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{4+6\ell},q,q^{3+6\ell};q^{4+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{2+6\ell},q^{6+6\ell};q^{8+12\ell}\right]_\infty.$$

Corollary **37** ($c = -q^{-1/2} \mid q \to q^2$).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q^2;q^2)_{m_\ell}(-q^3;q^6)_{m_\ell}}{(q^2;q^2)_{2m_\ell}(-q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}} =$$

$$= \frac{\left[q^{6+12\ell}, q, q^{5+12\ell}; q^{6+12\ell}\right]_\infty}{(q^2;q^2)_\infty} \left[q^{4+12\ell}, q^{8+12\ell}; q^{12+24\ell}\right]_\infty.$$

The $\ell = 1$ cases of Corollaries 34 and 36 have been discovered by Jackson [38] (1928) and Rogers [45] (1894) respectively. Instead, the $\ell = 1$ case of Corollary 37 has recently been found respectively by Chu, Zhang [32] (No. 197) and McLaughlin, Sills [42] (Eq. 4.17).

3.10. For $\delta = 0$, first replace q by q^2 in Theorem 1. Then put

$$W_k = \frac{1-q^k w}{1-w} \frac{(c;q)_k}{(q/c;q)_k} \Big(-\frac{q}{c}\Big)^k.$$

The corresponding finite sum displayed in (5a) may be evaluated by means of the bilateral ${}_4\psi_4$-series identity (8) as

$$\sum_{k=-m}^{m} q^{2km} \frac{(q^{-2m};q^2)_k}{(q^{2m+2};q^2)_k} \frac{1-q^k w}{1-w} \frac{(c;q)_k}{(q/c;q)_k} \Big(-\frac{q}{c}\Big)^k = \frac{(-q/c;q)_{2m}(q^2;q^2)_m}{(-q;q)_{2m}(q^2/c^2;q^2)_m}.$$

We therefore derive the following transformation formula.

Theorem 12 (Terminating series transformation).

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n}(-q/c;q)_{2m_\ell}}{(q^2;q^2)_{m_0}(-q;q)_{2m_\ell}(q^2/c^2;q^2)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2}}{(q^2;q^2)_{m_\iota}} =$$

$$= \sum_{k=-n}^{n} \frac{1-q^k w}{1-w} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \frac{(c;q)_k}{(q/c;q)_k} \frac{q^{(2\ell+1)k^2}}{c^k}.$$

Letting $n \to \infty$, we have the nonterminating multiple series transformation.

Proposition 9 (Nonterminating series transformation).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q/c;q)_{2m_\ell}}{(-q;q)_{2m_\ell}(q^2/c^2;q^2)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2}}{(q^2;q^2)_{m_\iota}} =$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1-q^k w}{1-w} \frac{(c;q)_k}{(q/c;q)_k} \frac{q^{(2\ell+1)k^2}}{c^k}.$$

Two multiple series identities of RR-type are derived from this proposition.

Corollary **38** ($c \to 0$).

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-1)^{m_\ell} q^{m_\ell^2}}{(-q;q)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}} = \frac{\left[q^{1+4\ell}, q^{2\ell}, q^{1+2\ell}; q^{1+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

***Corollary* 39** ($c \to \infty$: Warnaar [58] (Eq. 5.14))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{1}{(-q;q)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}} = \frac{\left[q^{3+4\ell}, q^{1+2\ell}, q^{2+2\ell}; q^{3+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

The last corollary generalizes the first Rogers, Selberg identity [45, 47].

3.11. Let $\delta = 0$ and $q \to q^2$ in Theorem 1. Then for

$$W_k = (1+q^{2k}) \left[\begin{matrix} b, & d \\ -q/b, & -q/d \end{matrix}\Big| q\right]_k \Big(-\frac{q}{bd}\Big)^k$$

the corresponding sum displayed in (5a) can be evaluated through (7a), (7b) as

$$\sum_{k=-m}^{m} q^{k(2m+1)} \frac{(q^{-2m};q^2)_k}{(q^{2m+2};q^2)_k} \left[\begin{matrix} b, & d \\ -q/b, & -q/d \end{matrix}\Big| q\right]_k \frac{1+q^{2k}}{(-bd)^k} = \frac{2(q^2;q^2)_m(-q/bd;q)_{2m}}{(q;q^2)_m[q^2/b^2, q^2/d^2;q^2]_m}.$$

We establish consequently the following transformation formula.

Theorem 13 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n}(-q/bd;q)_{2m_\ell}}{(q^2;q^2)_{m_0}(q;q^2)_{m_\ell}[q^2/b^2, q^2/d^2;q^2]_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2}}{(q^2;q^2)_{m_\iota}} =$$

$$= \sum_{k=-n}^{n} \frac{1+q^{2k}}{2(bd)^k} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^2} \left[\begin{matrix} b, & d \\ -q/b, & -q/d \end{matrix}\Big| q\right]_k q^{(2\ell+1)k^2}.$$

Letting $n \to \infty$ leads us to the nonterminating multiple series transformation.

Proposition 10 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q/bd;q)_{2m_\ell}}{(q;q^2)_{m_\ell}[q^2/b^2, q^2/d^2;q^2]_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2}}{(q^2;q^2)_{m_\iota}} =$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{k=-\infty}^{+\infty} \frac{1+q^{2k}}{2(bd)^k} \left[\begin{matrix} b, & d \\ -q/b, & -q/d \end{matrix}\Big| q\right]_k q^{(2\ell+1)k^2}.$$

Three multiple series identities of RR-type are derived from this proposition.

***Corollary* 40** ($b = -d = \sqrt{-q}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-1;q)_{m_\ell}}{(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2}}{(q;q)_{m_k}} = \frac{\left[q^{1+2\ell}, -q^{\ell}, -q^{1+\ell}; q^{1+2\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 41** ($b = -d = \sqrt{-1}$: Chu [27] (Example 15))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q;q^2)_{m_\ell}}{(-q;q)_{2m_\ell}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2}}{(q^2;q^2)_{m_k}} = \frac{\left[q^{2+4\ell}, q^{1+2\ell}, q^{1+2\ell}; q^{2+4\ell}\right]_\infty}{(q^2;q^2)_\infty}.$$

Corollary **42** ($b=\sqrt{-1}$, $d=\sqrt{-q}\mid q\to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q^2)_{2m_\ell}}{(q^2;q^4)_{m_\ell}(-q^2;q^2)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{4M_k^2}}{(q^4;q^4)_{m_k}}=\frac{\big[q^{4+8\ell},q^{1+4\ell},q^{3+4\ell};q^{4+8\ell}\big]_\infty}{(q^4;q^4)_\infty}.$$

For the last identity, its univariate version has been found by Slater [52] (Eq. 53).

3.12. In Theorem 1, let $\delta=1$ and $q\to q^2$. Then for

$$W_k=\frac{(1-q^ku)(1-q^kv)}{uv-q^{-1}}\frac{(c;q)_k}{(1/c;q)_k}\Big(-\frac{q^{-1}}{c}\Big)^k$$

we may compute the corresponding sum displayed in (5a) via (9) as

$$\sum_{k=-m}^{m+1}q^{2km+k}\frac{(q^{-2m-2};q^2)_k}{(q^{2m+2};q^2)_k}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(-c)^k}\frac{(c;q)_k}{(1/c;q)_k}=\frac{(q^2;q^2)_{m+1}(-q/c;q)_{2m}}{(q^2/c^2;q^2)_m(-q;q)_{2m+1}}.$$

This establishes the following terminating series transformation formula.

Theorem 14 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q^2;q^2)_{2n+1}(-q/c;q)_{2m_\ell}}{(q^2;q^2)_{m_0}(-q;q)_{2m_\ell+1}(q^2/c^2;q^2)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{2M_\iota^2+2M_\iota}}{(q^2;q^2)_{m_\iota}}=$$

$$=\sum_{k=-n}^{n+1}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(qc)^k}\begin{bmatrix}2n+1\\n+k\end{bmatrix}_{q^2}\frac{(c;q)_k}{(1/c;q)_k}q^{(4\ell+2)\binom{k}{2}}.$$

Letting $n\to\infty$, we get the nonterminating multiple series transformation.

Proposition 11 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-q/c;q)_{2m_\ell}}{(-q;q)_{2m_\ell+1}(q^2/c^2;q^2)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{2M_\iota^2+2M_\iota}}{(q^2;q^2)_{m_\iota}}=$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{+\infty}\frac{(1-q^ku)(1-q^kv)}{(uv-q^{-1})(qc)^k}\frac{(c;q)_k}{(1/c;q)_k}q^{(4\ell+2)\binom{k}{2}}.$$

Three multiple series identities of RR-type are derived from this proposition.

Corollary **43** ($c\to 0$: Paule [43] (Eq. 58))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(-1)^{m_\ell}q^{m_\ell^2}}{(-q;q)_{2m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\big[q^{1+4\ell},q,q^{4\ell};q^{1+4\ell}\big]_\infty}{(q^2;q^2)_\infty}.$$

Corollary **44** ($c\to\infty$: Paule [43] (Eq. 46))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{1}{(-q;q)_{2m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}}=\frac{\big[q^{3+4\ell},q,q^{2+4\ell};q^{3+4\ell}\big]_\infty}{(q^2;q^2)_\infty}.$$

***Corollary* 45** ($c = u = q^{-1/2} \mid q \to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q;q^2)_{2m_\ell+1}}{(-q^2;q^2)_{2m_\ell+1}(q^2;q^4)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{4M_k^2+4M_k}}{(q^4;q^4)_{m_k}} = \frac{\left[q^{4+8\ell}, q, q^{3+8\ell}; q^{4+8\ell}\right]_\infty}{(q^4;q^4)_\infty}.$$

When $\ell = 1$, the last two corollaries reduce respectively to the third Rogers, Selberg identity [46, 47] and an identity due to Slater [52] (Eq. 55).

3.13. First let $\delta = 1$ and $q \to q^2$ in Theorem 1. Then for

$$W_k = (q + q^{2k}) \left[\begin{matrix} b, & d \\ -1/b, & -1/d \end{matrix} \middle| q\right]_k \left(-\frac{q^{-1}}{bd}\right)^k$$

the corresponding finite sum displayed in (5a) can be evaluated by means of the bilateral ${}_6\psi_6$-series identity (7a), (7b) as

$$\sum_{k=-m}^{m+1} q^{k(2m+1)} \frac{(q^{-2m-2};q^2)_k}{(q^{2m+2};q^2)_k} \left[\begin{matrix} b, & d \\ -1/b, & -1/d \end{matrix} \middle| q\right]_k \frac{q+q^{2k}}{(-bd)^k} =$$

$$= \frac{2(1+bd)(q^2;q^2)_{m+1}(-q/bd;q)_{2m}}{(1+b)(1+d)(q;q^2)_{m+1}[q^2/b^2, q^2/d^2; q^2]_m}.$$

We therefore derive the following transformation formula.

Theorem 15 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q^2;q^2)_{2n+1}(-q/bd;q)_{2m_\ell}}{(q^2;q^2)_{m_0}(q;q^2)_{m_\ell+1}[q^2/b^2, q^2/d^2; q^2]_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2+2M_\iota}}{(q^2;q^2)_{m_\iota}} =$$

$$= \frac{(1+b)(1+d)}{2(1+bd)} \sum_{k=-n}^{n+1} \frac{q+q^{2k}}{(qbd)^k} \begin{bmatrix} 2n+1 \\ n+k \end{bmatrix}_{q^2} \left[\begin{matrix} b, & d \\ -1/b, & -1/d \end{matrix} \middle| q\right]_k q^{(4\ell+2)\binom{k}{2}}.$$

Letting $n \to \infty$ yields the nonterminating multiple series transformation.

Proposition 12 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q/bd;q)_{2m_\ell}}{(q;q^2)_{m_\ell+1}[q^2/b^2, q^2/d^2; q^2]_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{2M_\iota^2+2M_\iota}}{(q^2;q^2)_{m_\iota}} =$$

$$= \frac{(1+b)(1+d)}{2(1+bd)(q^2;q^2)_\infty} \sum_{k=-\infty}^{+\infty} \frac{q+q^{2k}}{(qbd)^k} \left[\begin{matrix} b, & d \\ -1/b, & -1/d \end{matrix} \middle| q\right]_k q^{(4\ell+2)\binom{k}{2}}.$$

Four multiple series identities of RR-type are derived from this proposition.

***Corollary* 46** ($b = -d = \sqrt{-q^{-1}} \mid q \to q^{1/2}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q;q)_{m_\ell}}{(q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\left[q^{1+2\ell}, -q^{1+2\ell}, -q^{1+2\ell}; q^{1+2\ell}\right]_\infty}{(q;q)_\infty}.$$

***Corollary* 47** ($b = -d = \sqrt{-1}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q;q^2)_{m_\ell}}{(-q;q)_{2m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{2M_k^2+2M_k}}{(q^2;q^2)_{m_k}} = \frac{\big[q^{2+4\ell}, q, q^{1+4\ell}; q^{2+4\ell}\big]_\infty}{(q^2;q^2)_\infty}.$$

***Corollary* 48** ($b = \sqrt{-q^{-1}},\ d \to 0 \mid q \to q^{1/2}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{q^{\binom{1+m_\ell}{2}}}{(q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\big[q^{2+8\ell}, q^{2\ell}, q^{2+6\ell}; q^{2+8\ell}\big]_\infty}{(q;q)_\infty}.$$

***Corollary* 49** ($b = \sqrt{-q^{-1}},\ d \to \infty \mid q \to q^{1/2}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{1}{(q;q^2)_{m_\ell+1}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} = \frac{\big[q^{6+8\ell}, q^{2+2\ell}, q^{4+6\ell}; q^{6+8\ell}\big]_\infty}{(q;q)_\infty}.$$

The univariate case of Corollary 49 is due to Rogers [46] (1917) (cf. Slater [52] (Eq. 60).

3.14. For $\delta = 0$, replace q by q^3 in Theorem 1. Let further

$$W_k = \frac{1-q^{2k}\omega}{1-\omega}\frac{(\omega^2 c;q)_k}{(q\omega^2/c;q)_k}\Big(\frac{q}{c}\Big)^k$$

where $\omega \neq 1$ is a cubic root of unity. Then we can evaluate the corresponding sum with respect to k displayed in (5a) by means of (7a), (7b) as

$$\sum_{k=-m}^{m} q^{3km}\frac{(q^{-3m};q^3)_k}{(q^{3m+3};q^3)_k}\frac{1-q^{2k}\omega}{1-\omega}\frac{(\omega^2 c;q)_k}{(q\omega^2/c;q)_k}\Big(\frac{q}{c}\Big)^k = \frac{(q^3;q^3)_m^2 (q/c;q)_{3m}}{(q^3;q^3)_{2m}(q^3/c^3;q^3)_m}.$$

This leads us to the following transformation formula.

Theorem 16 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q^3;q^3)_{2n}(q^3;q^3)_{m_\ell}(q/c;q)_{3m_\ell}}{(q^3;q^3)_{m_0}(q^3;q^3)_{2m_\ell}(q^3/c^3;q^3)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{3M_\iota^2}}{(q^3;q^3)_{m_\iota}} =$$

$$= \sum_{k=-n}^{n} \Big(\frac{-1}{c}\Big)^k \frac{1-q^{2k}\omega}{1-\omega} \begin{bmatrix} 2n \\ n+k \end{bmatrix}_{q^3} \frac{(\omega^2 c;q)_k}{(q\omega^2/c;q)_k} q^{3\ell k^2+3\binom{k}{2}+k}.$$

Letting $n \to \infty$, we have the nonterminating multiple series transformation.

Proposition 13 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q^3;q^3)_{m_\ell}(q/c;q)_{3m_\ell}}{(q^3;q^3)_{2m_\ell}(q^3/c^3;q^3)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{3M_\iota^2}}{(q^3;q^3)_{m_\iota}} =$$

$$= \frac{1}{(q^3;q^3)_\infty} \sum_{k=-\infty}^{+\infty} \Big(\frac{-1}{c}\Big)^k \frac{1-q^{2k}\omega}{1-\omega} \frac{(\omega^2 c;q)_k}{(q\omega^2/c;q)_k} q^{3\ell k^2+3\binom{k}{2}+k}.$$

Four multiple series identities of RR-type are derived from this proposition.

***Corollary* 50** ($c=\varepsilon$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q^3;q^3)_{m_\ell}(\varepsilon q;q)_{3m_\ell}}{(q^3;q^3)_{2m_\ell}(\varepsilon q^3;q^3)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{3M_k^2}}{(q^3;q^3)_{m_k}}=\frac{\left[q^{3+6\ell},\varepsilon q^{1+3\ell},\varepsilon q^{2+3\ell};q^{3+6\ell}\right]_\infty}{(q^3;q^3)_\infty}.$$

***Corollary* 51** ($c=q^{1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q^6;q^6)_{m_\ell}(q;q^2)_{3m_\ell}}{(q^6;q^6)_{2m_\ell}(q^3;q^6)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{6M_k^2}}{(q^6;q^6)_{m_k}}=\frac{\left[q^{6+12\ell},q^{1+6\ell},q^{5+6\ell};q^{6+12\ell}\right]_\infty}{(q^6;q^6)_\infty}.$$

***Corollary* 52** ($c\to0\mid q\to q^{1/3}$: McLaughlin, Sills [41] (Eq. 5.8))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}q^{m_\ell^2}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{2+6\ell},q^{\ell},q^{2+5\ell};q^{2+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{2+4\ell},q^{2+8\ell};q^{4+12\ell}\right]_\infty.$$

***Corollary* 53** ($c\to\infty\mid q\to q^{1/3}$: Andrews [5] (Eq. 3.4) and [41] (Eq. 5.10))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell}}\prod_{k=1}^{\ell}\frac{q^{M_k^2}}{(q;q)_{m_k}}=\frac{\left[q^{4+6\ell},q^{1+\ell},q^{3+5\ell};q^{4+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{2+4\ell},q^{6+8\ell};q^{8+12\ell}\right]_\infty.$$

When $\varepsilon=1$, the univariate case of Corollary 50 has been discovered by Bailey [13] (Eq. 1.6) (cf. Slater [52] (Eq. 42)). However, when $\varepsilon=-1$, it seems new even for $\ell=1$.

3.15. Similarly, for $\delta=1$, replace q by q^3 in Theorem 1. Then for

$$W_k=\frac{1-q^{2k-1}\omega}{1-q^{-1}\omega}\frac{(\omega^2c;q)_k}{(\omega^2/c;q)_k}\Big(\frac{q^{-1}}{c}\Big)^k$$

we can evaluate the corresponding sum displayed in (5a) through (7a), (7b) as

$$\sum_{k=-m}^{m+1}q^{3k(m+1)}\frac{(q^{-3m-3};q^3)_k}{(q^{3m+3};q^3)_k}\frac{1-q^{2k-1}\omega}{1-q^{-1}\omega}\frac{(\omega^2c;q)_k}{(\omega^2/c;q)_k}\Big(\frac{q^{-1}}{c}\Big)^k=$$

$$=\frac{(1-\omega)(q^3;q^3)_m(q^3;q^3)_{m+1}(q/c;q)_{3m+1}}{(1-\omega^2/c)(1-q\omega^2)(q^3;q^3)_{2m+1}(q^3/c^3;q^3)_m}$$

which yields consequently the following transformation formula.

Theorem 17 (Terminating series transformation)**.**

$$\sum_{M_0=n}\frac{(q^3;q^3)_{2n+1}(q^3;q^3)_{m_\ell}(q/c;q)_{3m_\ell+1}}{(q^3;q^3)_{m_0}(q^3;q^3)_{2m_\ell+1}(q^3/c^3;q^3)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{3M_\iota^2+3M_\iota}}{(q^3;q^3)_{m_\iota}}=$$

$$=(q\omega/c-q\omega^2)\sum_{k=-n}^{n+1}\Big(\frac{-1}{qc}\Big)^k\frac{1-q^{2k-1}\omega}{1-\omega}\begin{bmatrix}2n+1\\n+k\end{bmatrix}_{q^3}\frac{(\omega^2c;q)_k}{(\omega^2/c;q)_k}q^{(6\ell+3)\binom{k}{2}}.$$

Letting $n\to\infty$, we obtain the nonterminating multiple series transformation.

Proposition 14 (Nonterminating series transformation)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q^3;q^3)_{m_\ell}(q/c;q)_{3m_\ell+1}}{(q^3;q^3)_{2m_\ell+1}(q^3/c^3;q^3)_{m_\ell}}\prod_{\iota=1}^{\ell}\frac{q^{3M_\iota^2+3M_\iota}}{(q^3;q^3)_{m_\iota}}=$$

$$=\frac{q\omega/c-q\omega^2}{(q^3;q^3)_\infty}\sum_{k=-\infty}^{+\infty}\left(\frac{-1}{qc}\right)^k\frac{1-q^{2k-1}\omega}{1-\omega}\frac{(\omega^2c;q)_k}{(\omega^2/c;q)_k}q^{(6\ell+3)\binom{k}{2}}.$$

Four multiple series identities of RR-type are derived from this proposition.

Corollary **54** ($c=\varepsilon$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q^3;q^3)_{m_\ell}(\varepsilon q;q)_{3m_\ell+1}}{(q^3;q^3)_{2m_\ell+1}(\varepsilon q^3;q^3)_{m_\ell}}\prod_{k=1}^{\ell}\frac{q^{3M_k^2+3M_k}}{(q^3;q^3)_{m_k}}=$$

$$=\frac{\left[q^{3+6\ell},\varepsilon q,\varepsilon q^{2+6\ell};q^{3+6\ell}\right]_\infty}{(q^3;q^3)_\infty}.$$

Corollary **55** ($c=q^{-1/2}\mid q\to q^2$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q^6;q^6)_{m_\ell}(q;q^2)_{3m_\ell+2}}{(q^6;q^6)_{2m_\ell+1}(q^3;q^6)_{m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{6M_k^2+6M_k}}{(q^6;q^6)_{m_k}}=$$

$$=\frac{\left[q^{6+12\ell},q,q^{5+12\ell};q^{6+12\ell}\right]_\infty}{(q^6;q^6)_\infty}.$$

Corollary **56** ($c\to 0\mid q\to q^{1/3}$)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}q^{m_\ell^2+m_\ell}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=$$

$$=\frac{\left[q^{2+6\ell},q^{1+2\ell},q^{1+4\ell};q^{2+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{2\ell},q^{4+10\ell};q^{4+12\ell}\right]_\infty.$$

Corollary **57** ($c\to\infty\mid q\to q^{1/3}$: Andrews [5] (Eq. 3.13))**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell}\frac{(q;q)_{m_\ell}}{(q;q)_{2m_\ell+1}}\prod_{k=1}^{\ell}\frac{q^{M_k^2+M_k}}{(q;q)_{m_k}}=$$

$$=\frac{\left[q^{4+6\ell},q^{1+2\ell},q^{3+4\ell};q^{4+6\ell}\right]_\infty}{(q;q)_\infty}\left[q^{2+2\ell},q^{6+10\ell};q^{8+12\ell}\right]_\infty.$$

For $\varepsilon=1$, the univariate case of Corollary 54 has been discovered by Bailey [13] (Eq. 1.7) (cf. Slater [52] (Eq. 40)). However, Corollary 55 seems new, whose $\ell=1$ case has recently been discovered by Chu Zhang [32] (No. 134).

3.16. In Theorem 1, let $\delta = 1$ and

$$W_{4k+1} = W_{4k+2} = W_{4k+3} = 0 \qquad \text{and} \qquad W_{4k} = q^{4k} - q^{1-4k}.$$

Evaluating the corresponding sum displayed in (5a) by means of Bailey's bilateral ${}_6\psi_6$-series identity (7a), (7b) as

$$\sum_k q^{4k(m+1)} \frac{(q^{-m-1};q)_{4k}}{(q^{m+1};q)_{4k}} (q^{4k} - q^{1-4k}) = \frac{(q;q)_{m+1}(-q^2;q^2)_{m-1+\delta_{0,m}}}{(-q;q)_m (q;q^2)_m}$$

we therefore establish the following transformation formula.

Theorem 18 (Terminating series transformation)**.**

$$\sum_{M_0=n} \frac{(q;q)_{2n+1}(-q^2;q^2)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q)_{m_0}(-q;q)_{m_\ell}(q;q^2)_{m_\ell}} \prod_{\iota=1}^{\ell} \frac{q^{M_\iota^2+M_\iota}}{(q;q)_{m_\iota}} =$$

$$= \sum_k (q^{4k} - q^{1-4k}) \begin{bmatrix} 2n+1 \\ n+4k \end{bmatrix} q^{(2\ell+1)\binom{4k}{2}}.$$

Its limiting case $n \to \infty$ yields the nonterminating multiple series identity.

Corollary **58** (Difference of infinite products)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(-q^2;q^2)_{m_\ell-1+\delta_{0,m_\ell}}}{(-q;q)_{m_\ell}(q;q^2)_{m_\ell}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} =$$

$$= \frac{1}{(q;q)_\infty} \left\{ \begin{array}{c} \left[q^{16+32\ell}, -q^{10+12\ell}, -q^{6+20\ell}; q^{16+32\ell}\right]_\infty \\ -q\left[q^{16+32\ell}, -q^{2+12\ell}, -q^{14+20\ell}; q^{16+32\ell}\right]_\infty \end{array} \right\}.$$

There exist other multiple series identities with the right members being differences of infinite products. For example, letting $c = q$ in Proposition 13 leads us to the following identity.

Corollary **59** (Difference of infinite products)**.**

$$\sum_{\tilde{m}\in\mathbb{N}_0^\ell} \frac{(q;q)_{m_\ell}(q^3;q^3)_{m_\ell-1+\delta_{0,m_\ell}}}{(q;q)_{2m_\ell}(q;q)_{m_\ell-1+\delta_{0,m_\ell}}} \prod_{k=1}^{\ell} \frac{q^{M_k^2+M_k}}{(q;q)_{m_k}} =$$

$$= \frac{1}{(q;q)_\infty} \left\{ \begin{array}{c} \left[q^{9+18\ell}, q^{6+6\ell}, q^{3+12\ell}; q^{9+18\ell}\right]_\infty \\ -q\left[q^{9+18\ell}, q^{6\ell}, q^{9+12\ell}; q^{9+18\ell}\right]_\infty \end{array} \right\}.$$

However, we shall not pursue further along this direction due to the space limitation.

Concluding comments. Most of the multiple series identities displayed in this paper generalize classical single sum identities. The comparisons are summarized in the following table, where most of the references to the reduced $\ell = 1$ cases of the corollaries are positioned in Slater's list [52].

Corollary	Case $\ell = 1$	Corollary	Case $\ell = 1$
1	Slater [52], (18)	35	McLaughlin, Sills [41], (1.3)
2	Slater [52], (14)	36	Slater [52], (99)
6	Slater [52], (85)	37	McLaughlin, Sills [42], (4.17)
7	Slater [52], (39)	38	Slater [52], (19)
11	Stanton [54], (D3)(S1)	39	Slater [52], (136)
15	Bailey[11], §8.6(10)	40	Slater [52], (6)
16	Slater [52], (9, 84)	41	Sills [49], (4.6)
17	Slater [52], (38)	42	Slater [52], (53)
18	Slater [52], (46)	43	Slater [52], (15)
19	Slater [52], (29)	44	Slater [52], (31)
20	Slater [52], (54)	45	Slater [52], (55, 57)
21	Slater [52], (61)	46	Bailey[11], §8.6(10)
22	Agarwal, Bressoud [2], (2.3)	47	Slater [52], (27)
23	Slater [52], (50)	48	Slater [52], (44)
24	Slater [52], (28)	49	Slater [52], (60)
25	Slater [52], (11, 51)	50	Slater [52], (42)
26	Slater [52], (59)	51	Chu, Zhang [32], No.135
27	Slater [52], (48)	52	Slater [52], (83)
28	Slater [52], (28)	53	Slater [52], (98)
29	Slater [52], (7)	54	Slater [52], (40)
30	Slater [52], (19)	55	Chu, Zhang [32], No.134
31	Slater [52], (32)	56	Slater [52], (86)
32	Slater [52], (93)	57	Slater [52], (94)
33	Slater [52], (92)	58	Slater [52], (120)
34	Slater [52], (83)	59	Slater [52], (89)

Given a known identity of Rogers – Ramanujan type, there may exist several multiple counterparts. For example, we have ten multiple series identities displayed in Corollaries 3 – 5, 8 – 10, 12 – 14 and 29 that reduce, when $\ell = 1$, to special cases of Euler's second q-exponential function.

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