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SINGULARLY PERTURBED STOCHASTIC SYSTEMS

СИНГУЛЯРНО ЗБУРЕНІ СТОХАСТИЧНІ СИСТЕМИ

Problems of singular perturbation of reducible invertible operators are classified and their applications to the analysis of stochastic Markov systems represented by random evolutions are considered. The phase merging, averaging, and diffusion approximation schemes are discussed for dynamical systems with rapid Markov switchings.

Розглянуто проблеми сингулярного збурення оборотних операторів та їх застосування до аналізу стохастичних марковських систем, що задаються випадковими еволюціями. Схеми фазового укрупнення, усереднення та дифузійної апроксимації застосовуються до динамічних систем зі швидкими марковськими переключеннями.

1. Preamble. Stochastic models of systems are considered in a *random medium*, i.e., the evolution of a stochastic system is developed under the influence of random factors. The specific feature of an interaction between a system and a random medium is a unilateral effect of a random medium. The local characteristics of a system change with the change of states of a random medium. This particular feature of interaction allows us to apply a unified mathematical approach based on the efficient mathematical methods of analysis in the *problems of singular perturbation* for *reducible invertible operators*.

Stochastic models of systems are determined by two processes, namely, a *switched process* describing the evolution of a system, and a *switching process* describing the changes of a random medium [1, 2].

It is assumed that the evolution of a system possesses a *semigroup property* and the random medium has an *ergodic property*.

The mentioned properties of stochastic systems select the class of systems represented by *random evolution* regarded as an operator-valued stochastic process in a Banach space [3–13].

Efficient mathematical tools of analysis are based on the problems of singular perturbation for reducible invertible operators and on the martingale characterization of Markov processes [14–19].

Stochastic systems are considered in the series scheme with a small series parameter $\varepsilon > 0$ and with two scales of time: real time for a system and rapid time for a switching process.

This approach is the main idea of various investigations of deterministic and stochastic models of systems. The systematic analysis of deterministic evolution systems with two scales of time was realized by the prominent school in nonlinear mechanics created by N. N. Bogolyubov and A. N. Krylov (see the monograph of N. N. Bogolyubov and Yu. A. Mitropol'skii [20]) and developed by Yu. A. Mitropol'skii and A. M. Samoilenko [21], I. I. Gikhman [22], E. F. Tsar'kov [23] and their students.

The stochastic model of a system determined by two processes, namely, by a switched process in real time and a switching process in rapid time, was thoroughly investigated by N. N. Krasovskii and I. Ya. Kac [24], by H. J. Kushner [15], R. Z. Khasminskii [25, 26], A. V. Skorokhod [27, 28], etc.

The random evolution as an operator-valued model of a stochastic system was realized by R. J. Griego and R. Herch [4–6] and many others (see references in [18]).

There are many methods for asymptotic analysis of stochastic systems. The most efficient of these methods is based on studying singular perturbations of reducible invertible operators.

The aim of this paper is to order the main schemes of asymptotic analysis of stochastic systems with two scales of time. In order to focus attention on the problem of singular perturbation, we deliberately simplify the considered stochastic model, which illustrates the action of the method.

Certainly, the class of stochastic systems in asymptotic analysis can be substantially extended, but the clarity of the idea of analysis would be lost (see, e.g., the analysis of a dynamical system with rapid switchings by a stationary process in [11]).

2. Dynamical system with rapid Markov switchings. Let us consider, for definiteness, a rather general and, at the same time, extremely simple model of a stochastic system given by the evolution differential equation with rapid Markov switchings

$$\frac{dU^\varepsilon(t)}{dt} = C(U^\varepsilon(t), x(t/\varepsilon)), \tag{1}$$

where $x(t)$ is an ergodic Markov process on a measurable space $(X, \mathcal{X})$ is given by the generator Q defined on the Banach space B of real-valued functions $\varphi(x)$ with the supremum norm

$$||\varphi(x)|| := \sup_{x\in X} |\varphi(x)|.$$

The velocity function $C(u, x)$, $u \in \mathrm{R}^d$, $x \in X$ is supposed to be such that the evolution equations

$$\frac{dU_x(t)}{dt} = C(U_x(t), x), \quad x \in X, \qquad U_x(0) = u \tag{2}$$

have the unique solution on every finite time interval, $t \in [0, T]$, for every fixed $x \in X$.

Equation (2) can be considered in the equivalent operator form

$$\frac{\partial f_t(u)}{\partial t} = C(x) f_t(u), \qquad f_0(u) = f(u)$$

for evolution $f_t(u) := f(U_x(t))$, $U_x(0) = u$, with the generator of semigroup

$$C(x)f(u) := C(u, x) f'(u). \tag{3}$$

The main idea of asymptotic analysis as $\varepsilon \to 0$ of the dynamical system with rapid Markov switchings (1) is to consider the couple Markov process $U^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon)$, $t \geq 0$, determined by the generator

$$L^\varepsilon \varphi(u, x) = [\varepsilon^{-1} Q + C(x)]\varphi(u, x). \tag{4}$$

The uniformity of the ergodic property of a switching Markov process $x(t)$ means that the generator Q of the Markov process $x(t)$ possesses the reducible invertible property [17, 18], i.e., the Banach space B can be represented as the direct sum

$$B = N_Q \oplus \mathrm{R}_Q \tag{5}$$

of the null space $N_Q := \{\varphi : Q\varphi = 0\}$ and the range space $R_Q := \{\varphi : Q\psi = \varphi\}$. Decomposition (5) means that there exists the projector Π onto the null space N_Q and the potential operator R_0 defined by the relation

$$R_0 := [Q + \Pi]^{-1} - \Pi \tag{6}$$

and satisfying the following properties:

$$QR_0 = R_0Q = I - \Pi, \quad \Pi R_0 = R_0\Pi = \emptyset$$

(i.e., the potential R_0 is a reducible inverse operator to the operator Q). The general solution of the equation

$$Q\varphi = \psi$$

can be represented as follows:

$$\varphi = R_0\psi + \varphi_0, \quad \varphi_0 \in N_Q.$$

In what follows, for simplicity, we assume that the operator Q and its potential R_0 are bounded. The case where this is not true requires some additional refinement in the asymptotic analysis of singular perturbation problems [16].

3. Problems of singular perturbation. Various schemes of asymptotic analysis of stochastic systems can be reduced to the problem of singular perturbation of a reducible invertible operator, which can be formulated in the following way: For a given vector $\psi \in B$, the asymptotic solution

$$\varphi^\varepsilon = \varphi + \varepsilon\varphi_1$$

of the equation

$$[\varepsilon^{-1}Q + Q_1]\varphi^\varepsilon = \psi + \theta^\varepsilon$$

is constructed with the asymptotically negligible term θ^ε, i.e.,

$$||\theta^\varepsilon|| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Such a problem arises due to the asymptotic inversion of a singular operator:

$$[\varepsilon^{-1}Q + Q_1]^{-1} = Q^0 + \varepsilon Q^1 + \ldots .$$

There are many situations that cannot be classified (see, e.g., [16]). At the same time, it is possible to select some logically complete cases [17, 18].

The classification of problems of singular perturbation is based on the properties of a *contracted* operator $\hat{Q}_1$ defined by the relation

$$\hat{Q}_1\Pi = \Pi Q_1 \Pi. \tag{7}$$

The contracted operator $\hat{Q}_1$ acts on the contracted null space $\hat{N}_Q$.

Example [16]. Let Q be a generator of a Markov ergodic process with finitely many ergodic classes, let $X = \cup_{k=1}^N X_k$, and let $\pi_k(dx)$, $1 \le k \le N$, be stationary distributions on X_k, $1 \le k \le N$. The projector Π onto the null space N_Q acts as follows:

$$\Pi\varphi(x) = \sum_{k=1}^N \hat{\varphi}_k I_k(x), \quad \hat{\varphi}_k := \int_{X_k} \varphi(x)\pi_k(dx);$$

here,

$$I_k(x) := \begin{cases} 1, & x \in X_k, \\ 0, & x \notin X_k. \end{cases}$$

The contracted null space $\hat{N}_Q$ is the N-dimensional Euclidean space of vectors $\hat{\varphi} = (\hat{\varphi}_k,\ 1 \le k \le N)$.

Assume that a perturbation operator Q_1 acts as follows:

$$Q_1\varphi(x) = \int_X Q_1(x, dy)\varphi(y), \quad x \in X.$$

Then the contracted operator $\hat{Q}_1$ on $\hat{N}_Q$ is defined according to relation (7) by the matrix

$$\hat{Q}_1 = [q_{kr} :\ 1 \le k, r \le N],$$

where

$$q_{kr} := \int_{X_k} \pi_k(dx) Q_1(x, X_r)$$

and

$$\hat{Q}_1\hat{\varphi} := \left(\sum_{r=1}^{N} q_{kr}\hat{\varphi}_r, 1 \le k \le N \right).$$

There are three logically complete possibilities:

(i) $\hat{Q}_1$ is invertible, i.e., there exists $\hat{Q}_1^{-1}$;

(ii) $\hat{Q}_1$ is the zero operator, i.e., $\hat{Q}_1\hat{\varphi} = 0$ for all $\hat{\varphi} \in \hat{N}_Q$;

(iii) $\hat{Q}_1$ is reducible invertible, i.e., there exists a null space $\hat{N}_{\hat{Q}_1} \subset \hat{N}_Q$ such that

$$\hat{N}_Q = \hat{N}_{\hat{Q}_1} \oplus \hat{\mathrm{R}}_{\hat{Q}_1}.$$

There also exists the potential operator $\hat{\mathrm{R}}_0 := [\hat{Q}_1 + \hat{\Pi}]^{-1} - \hat{\Pi}$, where $\hat{\Pi}$ is the projector onto $\hat{N}_{\hat{Q}_1}$ defined by relation

$$\hat{\Pi}\hat{\varphi} = \hat{\varphi}\hat{1}, \quad \hat{\varphi} \in \hat{N}_{\hat{Q}_1}.$$

Here, $\hat{1}$ is the unit vector in $\hat{N}_{\hat{Q}_1}$.

The solutions of singular perturbation problems in these three cases are given by the following three propositions (see [16–19]):

Proposition 1. *Let the contracted operator $\hat{Q}_1$ be invertible, i.e., $\exists\, \hat{Q}_1^{-1}$. Then the asymptotic representation*

$$[\varepsilon^{-1}Q + Q_1](\varphi + \varepsilon\varphi_1) = \psi + \theta^\varepsilon$$

can be realized by the following relations:

$$\hat{Q}_1\hat{\varphi} = \hat{\psi},$$
$$\varphi_1 = \mathrm{R}_0(\psi - Q_1\varphi),$$
$$\theta^\varepsilon = \varepsilon Q_1 \mathrm{R}_0(\psi - Q_1\varphi).$$

Proposition 2. *Let the contracted operator $\hat{Q}_1$ be the zero operator, i.e., $\hat{Q}_1\hat{\varphi} = 0$ $\forall\hat{\varphi} \in \hat{N}_Q$. Also assume that, after contraction to the space $\hat{N}_Q$, the operator $Q_0 = Q_2 - Q_1\mathrm{R}_0Q_1$ has the inverse operator $\hat{Q}_0^{-1}$.*

Then the asymptotic representation

$$[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = \psi + \theta^\varepsilon$$

can be realized by the following relations:

$$\hat{Q}_0\hat{\varphi} = \hat{\psi},$$
$$\varphi_1 = -\mathrm{R}_0 Q_1 \varphi,$$
$$\varphi_2 = \mathrm{R}_0(\psi - Q_0\varphi),$$
$$\theta^\varepsilon = \varepsilon[Q_2\varphi_1 + [Q_1 + \varepsilon Q_2]\varphi_2].$$

Proposition 3. *Let the contracted operator $\hat{Q}_1$ be reducible invertible with the null space $\hat{N}_{\hat{Q}_1} \subset \hat{N}_Q$ defined by the projector $\hat{\Pi}$. Let the twice contracted operator $\hat{\hat{Q}}_2$ on $\hat{N}_{\hat{Q}_1}$ defined by the relation*

$$\hat{\hat{Q}}_2\hat{\Pi} = \hat{\Pi}\hat{Q}_2\hat{\Pi}, \qquad \hat{Q}_2\Pi = \Pi Q_2 \Pi,$$

be invertible, i.e., $\exists\, \hat{\hat{Q}}_2^{-1}$.

Then the asymptotic representation

$$[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + Q_2](\varphi + \varepsilon\varphi_1 + \varepsilon^2\varphi_2) = \psi + \theta^\varepsilon$$

can be realized by the following relations:

$$\hat{\hat{Q}}_2\hat{\varphi} = \hat{\psi},$$
$$\hat{\varphi}_1 = \hat{R}_0(\hat{\psi} - \hat{Q}_2\hat{\varphi}),$$
$$\varphi_2 = R_0(\psi - Q_2\varphi - Q_1\varphi_1),$$
$$\theta^\varepsilon = \varepsilon\big[Q_2\varphi_1 + [Q_1 + \varepsilon Q_2]\varphi_2\big].$$

The remarkable fact is that every solution of a singular perturbation problem has a corresponding interpretation in the analysis of stochastic systems.

4. Analysis of stochastic systems. Let us start with a rather simple example.

4.1. *Sojourn time in a subset of states* [17]. Let x_n^ε, $n \geq 0$, be a Markov chain on a measurable space $(X, \mathcal{X})$ with unique absorbing state 0 such that $X = X_0 + \{0\}$. The transition probabilities are represented as follows:

$$P_\varepsilon(x, B) = P(x, B) - \varepsilon P_1(x, B),$$

where the stochastic kernel $P(x, B)$ is the transition probabilities of the support ergodic Markov chain x_n, $n \geq 0$, on the subset X_0 with stationary distribution $\pi(dx)$. The perturbing kernel $P_1(x, B)$ provides the absorption of the initial Markov chain x_n^ε, $n \geq 0$, with probabilities

$$P_\varepsilon(x, \{0\}) = \varepsilon P_1(x, X_0) =: \varepsilon\psi(x).$$

Introduce the sojourn time in the subset of states

$$\tau_x^\varepsilon := \min\{n : x_n^\varepsilon = 0 / x_n^\varepsilon = x \in X_0\}.$$

The generating function $\varphi_\varepsilon(x, s) := Ee^{-\varepsilon s \tau_x^\varepsilon}$ is determined by the solution of a singular perturbed problem of the following form [16]:

$$[\varepsilon^{-1}Q + Q_1 + \theta^\varepsilon]\varphi_\varepsilon = \psi,$$

where $Q := I - P$, $Q_1 := sI + P_1$, and the operator θ^ε satisfies the negligibility condition

$$||\theta^\varepsilon|| \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

By assumption, the contracted null space $\hat{N}_Q$ is a one-dimensional Euclidean space, i.e., the real line. The contracted operator $\hat{Q}_1$ acts as follows:

$$\hat{Q}_1 \hat{\varphi} = (s + \hat{\psi})\hat{\varphi}, \quad \hat{\psi} := \int_{X_0} \pi(dx)\psi(x)$$

By Proposition 1, we obtain

$$\varphi_\varepsilon(x, s) \to \varphi(s) = \hat{\psi}/(s + \hat{\psi}) \quad \text{as} \quad \varepsilon \to 0.$$

Thus,

$$\mathcal{P}\{\varepsilon\tau_x^\varepsilon > t\} \to e^{-\hat{\psi}t} \quad \text{as} \quad \varepsilon \to 0.$$

4.2. *Phase merging scheme* [17]. Let $x_\varepsilon(t)$, $t \geq 0$, be a Markov jump process on a measurable split space $(X, \mathcal{X})$,

$$X = \cup_{v \in V} X_v, \quad X_v \cap X_{v'} = \emptyset, \quad v \neq v', \tag{8}$$

given by the generator

$$Q^\varepsilon \varphi(x) = q(x) \int_X P_\varepsilon(x, dy) [\varphi(y) - \varphi(x)]. \tag{9}$$

The stochastic kernel $P_\varepsilon(x, dy)$ is represented as follows:

$$P_\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B), \tag{10}$$

where the stochastic kernel $P(x, B)$ is coordinated with the splitting of the phase space (8) as follows:

$$P(x, X_v) = \begin{cases} 1, & x \in X_v, \\ 0, & x \notin X_v. \end{cases}$$

We also assume that the support Markov chain x_n, $n \geq 0$, with the transition probabilities $P(x, B)$ is uniformly ergodic in every class X_v, $v \in V$, with stationary distribution $\pi_v(dx)$, $v \in V$. For sufficiently small enough $\varepsilon > 0$, the Markov process $x_\varepsilon(t)$, $t \geq 0$, stays for a long time in every class X_v, $v \in V$, and, provided that

$$p_v = \int_{X_v} \pi_v(dx) P_1(x, X_v) > 0, \quad v \in V,$$

sooner or later leaves each class X_v of ergodicity of the support Markov process $x(t)$, $t \geq 0$, defined by the generator

$$Q\varphi(x) = q(x) \int_X P(x, dy) [\varphi(y) - \varphi(x)]. \tag{11}$$

The asymptotic behavior of the Markov process $x_\varepsilon(t)$, $t \geq 0$, as $\varepsilon \to 0$ can be investigated by using the martingale characterization of the Markov process $x^\varepsilon(t) := x_\varepsilon(t/\varepsilon)$ in the following form [16]:

$$\mu_t^\varepsilon = \varphi(x^\varepsilon(t)) - \int_0^t L^\varepsilon \varphi(X^\varepsilon(s)) ds,$$
$$L^\varepsilon \varphi(x) = [\varepsilon^{-1} Q + Q_1] \varphi(x),$$

where

$$Q_1 \varphi(x) := q(x) \int_X P_1(x, dy) \varphi(y). \tag{12}$$

The phase merging effect is realized by Proposition 1 for the test functions

$$\varphi^\varepsilon(x) = \varphi(v(x)) + \varepsilon\varphi_1(x),$$

where $v(x) = v$, $x \in X_v$, is the merging function on X corresponding to splitting (8).

The martingale characterization for the limit Markov process

$$\hat{x}(t) = P\text{-}\lim_{\varepsilon\to 0} v(x^\varepsilon(t))$$

on the merged phase space $(V, \mathcal{V})$ is given by the relation

$$\mu_t = \hat{\varphi}(\hat{x}(t)) - \int_0^t \hat{Q}_1\hat{\varphi}(\hat{x}(s))ds$$

with the generator

$$\hat{Q}_1\hat{\varphi}(v) := \hat{q}(v)\int_V \hat{P}(v, dv')\left[\hat{\varphi}(v') - \hat{\varphi}(v)\right], \tag{13}$$

where

$$\hat{P}(v,\Gamma) := \int_{X_v} \pi_v(dx)P_1(x, X_\Gamma), \quad X_\Gamma := \cup_{v\in\Gamma}X_v,$$

$$\hat{q}(v) := q_v \cdot p_v,$$

$$q_v^{-1} := \int_{X_v} \pi_v(dx)/q(x). \tag{14}$$

4.3. *Phase averaging scheme* [18]. Now let us consider a dynamical system with rapid Markov switching (see Section 2) The corresponding problem of singular perturbation is formulated for the martingale characterization of the coupled Markov process $U^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon)$, $t \geq 0$,

$$\mu_t^\varepsilon = \varphi^\varepsilon(U^\varepsilon(t), x^\varepsilon(t)) - \int_0^t L^\varepsilon\varphi^\varepsilon\big(U^\varepsilon(s), x^\varepsilon(s)\big)ds \tag{15}$$

on the test functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x).$$

According to (3) and (4), the generator

$$L^\varepsilon = \varepsilon^{-1}Q + C(x), \tag{16}$$

where

$$C(x)\varphi(u) := C(u,x)\varphi'(u).$$

Under the conditions of uniform ergodicity of the switching Markov process $x(t)$ with stationary distribution $\pi(dx)$, the phase averaging effect is realized by Proposition 1 for martingale (15) with generator (16) in the following form:

$$\mu_t^\varepsilon = \varphi(U^\varepsilon(t)) - \int_0^t \hat{C}\varphi(U^\varepsilon(s))ds + \theta_t^\varepsilon, \tag{17}$$

where the averaged operator

$$\hat{C}\varphi(u) := \hat{C}(u)\varphi'(u), \quad \hat{C}(u) := \int_X \pi(dx)C(u,x)$$

and the negligible term

$$E|\theta_t^\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{18}$$

The limit average system is determined by the equation

$$\varphi(\hat{U}(t)) - \int_0^t \hat{C}\varphi(\hat{U}(s))\,ds = 0$$

or, in the equivalent form for the averaged dynamical system,

$$\frac{d\hat{U}(t)}{dt} = \hat{C}(\hat{U}(t)).$$

4.4. *Diffusion approximation* [18]. Consider a dynamical system with accelerated Markov switching

$$\frac{dU^\varepsilon(t)}{dt} = C^\varepsilon(U^\varepsilon(t), x(t/\varepsilon^2))$$

and the velocity depending on the series parameter ε as follows:

$$C^\varepsilon(u,x) = \varepsilon^{-1}C(u,x) + C_1(u,x).$$

Now consider the problem of singular perturbation for the martingale

$$\mu_t^\varepsilon = \varphi^\varepsilon(U^\varepsilon(t), x^\varepsilon(t)) - \int_0^t L^\varepsilon\varphi^\varepsilon(U^\varepsilon(s), x^\varepsilon(s))\,ds \tag{19}$$

on the test functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x).$$

The generator of the coupled Markov process $U^\varepsilon(t)$, $x^\varepsilon(t) := x(t/\varepsilon^2)$, $t \geq 0$, is given by the following relation:

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}C(x) + C_1(x)\right]\varphi(u,x),$$

where, as usual,

$$C(x)\varphi(u) := C(u,x)\varphi'(u).$$

Under the condition of uniform ergodicity of the switching Markov process $x(t)$ with stationary distribution $\pi(dx)$ and the additional balance condition on the velocity function $C(u,x)$

$$\hat{C}(u) = \int_X \pi(dx)C(u,x) \equiv 0,$$

which guarantees the existence of the limit diffusion, by Proposition 2, martingale (19) is reduced to the form

$$\mu_t^\varepsilon = \varphi(U^\varepsilon(t)) - \int_0^t L^0\varphi(U^\varepsilon(s))ds + \theta_t^\varepsilon$$

with negligible term θ_t^ε satisfying condition (18). The limit generator L^0 is determined by the formulas (see Proposition 2)

$$L^0 := \hat{Q}_0, \quad \hat{Q}_0\Pi = \Pi Q_0\Pi, \quad Q_0 := C_1 - C\mathrm{R}_0C.$$

Note that

$$C\mathrm{R}_0C\varphi(u) = C(u,x)\mathrm{R}_0C(u,x)\varphi''(u) + C(u,x)\mathrm{R}_0C_u'(u,x)\varphi'(u).$$

It is now easy to verify that

$$L^0\varphi(u) = \sigma^2(u)\varphi''(u) + a(u)\varphi'(u),$$

where

$$\sigma^2(u) := \int_X C(u,x)\mathrm{R}_0C(u,x)\pi(dx)$$

and

$$a(u) := \int_X \left[C(u,x)\mathrm{R}_0 C_u'(u,x) + C_1(u,x)\right]\pi(dx).$$

The limit diffusion process $U(t)$, $t \geq 0$, is determined by the martingale characterization

$$\mu_t = \varphi(U(t)) - \int_0^t L^0\varphi(U(s))ds.$$

4.5. *Double averaging scheme* [17]. Combining the phase merging and averaging schemes, we consider a dynamical system with accelerated Markov switchings in a splittable phase space

$$\frac{dU^\varepsilon(t)}{dt} = C\big(U^\varepsilon(t), x_\varepsilon(t/\varepsilon^2)\big),$$

where the Markov process $x_\varepsilon(t)$ is considered, as in Sec. 4.2, on a split phase space $X = \cup_{v\in V} X_v$ with generator (9)–(10). The generator of the coupled Markov process $U^\varepsilon(t)$, $x^\varepsilon(t) := x_\varepsilon(t/\varepsilon^2)$, $t \geq 0$, is determined by the relation

$$L^\varepsilon\varphi(u,x) = \left[\varepsilon^{-2}Q + \varepsilon^{-1}Q_1 + C(x)\right]\varphi(u,x), \tag{20}$$

where Q and Q_1 are given by relations (11) and (12).

Now consider the problem of singular perturbation for the martingale

$$\mu_t^\varepsilon = \varphi^\varepsilon\big(U^\varepsilon(t), x^\varepsilon(t)\big) - \int_0^t L^\varepsilon\varphi^\varepsilon(U^\varepsilon(s), x^\varepsilon(s))ds \tag{21}$$

on the test functions

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon\varphi_1(u, v(x)) + \varepsilon^2\varphi_2(u,x),$$

where $v(x) = x$, $x \in X_v$, is a merging function on X, corresponding to the split of the phase space (8).

The singularity of the perturbed operator (20) is guaranteed by the reducible invertible operator $\hat{Q}_1$ determined by (13)–(14). The double averaging effect is realized by Proposition 3 under the additional condition of the uniform ergodicity of the merged limit Markov process $\hat{x}(t)$, $t \geq 0$, on the merged phase space $(V, \mathcal{V})$ defined by the generator $\hat{Q}_1$ (see Sec. 4.2). According to Proposition 3, martingale (21) is reduced to the form

$$\mu_t^\varepsilon = \varphi(U^\varepsilon(t)) - \int_0^t \hat{\hat{C}}\varphi(U^\varepsilon(s))ds + \theta_t^\varepsilon$$

with negligible term θ_t^ε. The double averaged operator $\hat{\hat{C}}$ is determined by the following formulas:

$$\hat{\hat{C}}\varphi(u) = \hat{C}(u)\varphi'(u),$$

$$\hat{\hat{C}}(u) := \int_V \hat{C}(u,v)\pi(dv),$$

$$\hat{C}(u,v) := \int_{X_v} C(u,x)\pi(dx).$$

Here, $\hat{\pi}(dv)$ is the stationary distribution of the merged limit Markov process $\hat{x}(t)$.

The double averaged limit dynamical system is defined by the equation

$$\frac{d\hat{U}(t)}{dt} = \hat{\hat{C}}(\hat{U}(t)).$$

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