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GROUPS WITH BOUNDED CHERNIKOV CONJUGATE CLASSES OF ELEMENTS*

ГРУПИ З ОБМЕЖЕНИМИ ЧЕРНІКОВСЬКИМИ КЛАСАМИ СПРЯЖЕНИХ ЕЛЕМЕНТІВ

We consider BCC-groups, that is groups G with Chernikov conjugacy classes in which for every element $x \in G$ the minimax rank of the divisible part of the Chernikov group $G / C_G(x^G)$ and the order of the corresponding factor-group are bounded in terms of G only. We prove that a BCC-group has a Chernikov derived subgroup. This fact extends the well-known result due to B. H. Neumann characterizing groups with bounded finite conjugacy classes (BFC-groups).

Розглянуто BCC-групи, тобто групи G з черніковськими класами спряжених елементів, у яких для кожного елемента $x \in G$ мінімаксний ранг ділимої частини черніковської групи $G / C_G(x^G)$ та порядок відповідної фактор-групи обмежені у термінах групи G. Доведено, що BCC-група має черніковський комутант, чим розширюється відомий результат Б. Неймана, який охарактеризував групи з скінченними обмеженими класами спряжених елементів (BFC-групи).

1. Introduction. The theory of groups with finiteness conditions is one of the best developed and fruitful in Infinite Group Theory. Finiteness conditions defined the natural approach to the study of infinite groups inheriting some important properties of finite groups. S. N. Chernikov was one of the main founders of this theory. He obtained first fundamental results being crucially influential on the establishment and formation of this area of algebra. His numerous principal accomplishments mainly defined the way of the further development of the theory of groups with finiteness conditions and Infinite Group Theory as a whole (see, for example, the surveys [1 – 3] and the books [4 – 6]).

The theory of FC-groups and its generalization is well-known part of the theory of groups with finiteness conditions. S. N. Chernikov was also the main founders of this theory. One of the very first interesting classes here was the class of the layer-finite groups (that is the groups G in which the layers $G[n] = \{g \in G \mid |g| = n\}$ are finite for each $n \in \mathbb{N}$). These groups has been studied completely by S. N. Chernikov in his papers [7 – 9]. He also obtained analogies of P. Hall's theorem and I. Schur – H. Zassenhaus theorem for periodic FC-groups [10]. He givs also the characterization of the FC-groups [11]. At that time B. H. Neumann [12, 13] considered the other interesting classes of the FC-groups. The FC-groups whose conjugacy classes have bounded size (that is $|G : C_G(g)| \le n$ for each element $g \in G$) have been introduced among them. They where called BFC-groups and were characterized in [12] as the groups with finite derived subgroups. A series of interesting articles dedicated to the best possible function $f(n)$ such that $|[G, G]| \le f(n)$ have been published. The latest paper of this series is the peper of D. Segal and A. Shalev [14], in which the authors obtained the following value for the function $f: f(n) = n^{A(n)}$ where $A(n) = (13 + \log_2 n) / 2$.

Considering some natural extensions of FC-groups, S. N. Chernikov introduced the approach based on the transition from the restrictions, defining by the class $\mathcal{F}$ of all finite groups to some its natural extensions, in particular, to the class C of the Chernikov groups. The first realization of this approach was the consideration by

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Ya. D. Polovicky [15] groups with Chernikov layers (that is the groups, in which every layer generates a Chernikov subgroup) (see also D. J. Robinson [16]). The next steps where the transitions from the locally finite-normal groups to the locally Chernikov-normal groups (that is the groups, in which every finite subset lie in normal Chernikov subgroup) [15], and from the FC-groups to the CC-groups (or the groups with Chernikov conjugacy classes) [17]. Recall that a *CC-group or a group with Chernikov conjugacy classes is a group G such that $G / C_G(g^G)$ is a Chernikov group for all $g \in G$*. Since a Chernikov group C is a finite extension of a characteristic subgroup $D(C)$ which is a direct product of finitely many Prüfer or quasicyclic subgroups, C has two natural invariants which depend only on C: the number $r = \mathrm{mmx}(C)$ of Prüfer subgroups occurring in the expression of $D(C)$ and $\mathrm{ord}(C) = |C/D(C)|$.

The number $\mathrm{mmx}(C)$ is called the *minimax length* of a group C (D. J. S. Robinson [5], 10.3) or a *minimax rank* of a group C [18] (Chapter IV, Section 3). A CC-group G is said to have *bounded Chernikov conjugacy classes or G is a BCC-group if there are two natural numbers $M(G)$ and $O(G)$ such that* $\mathrm{mmx}(G / C_G(g^G)) \leq M(G)$ *and* $\mathrm{ord}(G / C_G(g^G)) \leq O(G)$ *for all $g \in G$*. In this connection note that J. Otal and J. M. Peña [19] are classificied the CC-groups G with $G/\zeta(G)$ periodic and $\mathrm{ord}(G / C_G(X^G))$ bounded for every $X \subseteq G$ as central-by-Chernikov groups.

In what it concerns to the question stated in this paper, we show following theorem.

Main Theorem. *Let G be a group with bounded Chernikov conjugacy classes. Then $[G, G]$ is a Chernikov subgroup, $G / C_G([G,G])$ is finite and there are functions f and g such that* $\mathrm{mmx}([G, G]) \leq f(M(G), O(G))$ *and* $\mathrm{ord}([G, G]) \leq g(M(G), O(G))$.

Let p be a prime. We consider two copies $A = \times_{n \in \mathbb{N}} \langle a_n \rangle$ and $B = \times_{n \in \mathbb{N}} \langle b_n \rangle$ of the direct product of cyclic groups of order $|a_n| = |b_n| = p^n$, $n \in \mathbb{N}$. If $K = \langle c_n \mid c_{n+1}^p = c_n, c_1^p = 1, n \in \mathbb{N} \rangle$ is a Prüfer p-group, the there is an action of B on $A \times K$ which satisfies $[a_n, b_n] = c_n$, $n \in \mathbb{N}$. If G is a corresponding semidirect product, then G is a periodic FC-group such that $[G, G] = K$ is a Chernikov subgroup, $K \leq \zeta(G)$ however $|G : C_G(b_n)| = p^n$ and so G is not a BCC-group. This shows that our result is the best possible.

2. Some auxiliary results. We show a couple of useful facts, the first of which is immediate.

Lemma 1. *Let G be a BCC-group, $x_1, \ldots, x_n \in G$ and define $X = \langle x_1, \ldots, x_n \rangle$. Then $G / C_G(X^G)$ is a Chernikov group and* $\mathrm{mmx}(G / C_G(X^G)) \leq nM(G)$, $\mathrm{ord}(G / C_G(X^G)) \leq O(G)^n$. *In particular, if $G / C_G(X^G)$ is finite then* $|G / C_G(X^G)| \leq O(G)^n$.

Lemma 2. *Let G be a BCC-group and H a subgroup of finite special rank r. Then $G / C_G(N^G)$ is a Chernikov group and* $\mathrm{mmx}(G / C_G(N^G)) \leq rM(G)$, $\mathrm{ord}(G / C_G(N^G)) \leq O(G)^r$. *Furthermore, if H is periodic abelian and normal in G, then $G / C_G(H)$ is finite and* $|G / C_G(N)| \leq O(G)^r$.

Proof. Since H has finite special rank r, H has an ascending chain of finitely generated subgroups

$$\langle 1 \rangle = H_0 \leq H_1 \leq \ldots \leq H_n \leq \ldots \leq \bigcup_{n \in \mathbb{N}} H_n = H$$

such that every subgroup H_n has at most r generators, $n \in \mathbb{N}$.

By Lemma 1, every factor $G / C_G(H_n^G)$ is a Chernikov group and we have that

$\mathrm{mmx}\,(G/C_G(H_n^G)) \le rM(G)$ and $\mathrm{ord}(G/C_G(H_n^G)) \le O(G)^r$. On the other hand, since

$$C_G(H_1^G) \ge C_G(H_2^G) \ge \ldots \ge C_G(H_n^G) \ge \ldots,$$

there is a number j such that $C_G(H_j^G) = C_G(H_{j+n}^G)$ for all $n \in \mathbb{N}$. Thus $C_G(H_j^G) = C_G(H^G)$ and so $\mathrm{mmx}\,(G/C_G(H^G)) \le rM(G)$, $\mathrm{ord}(G/C_G(N^G)) \le O(G)^r$.

If H is a normal periodic abelian subgroup, every subgroup H_n^G is bounded and, being abelian of finite special rank r, it is finite. By Lemma 1, $|G/C_G(H_n^G)| \le O(G)^r$. Consequently, $G/C_G(H^G)$ is finite and $|G/C_G(H^G)| \le O(G)^r$.

Let G be a CC-group. Define $D(G)$ to be the subgroup of G generated by all the Chernikov divisible normal subgroups of G.

It is easy to see that $D(G)$ is a periodic divisible abelian normal subgroup of G, moreover $D(G)$ is the largest periodic divisible abelian subgroup of G. The factor-group $G/D(G)$ does not include the normal Chernikov divisible subgroups, in particular, $G/D(G)$ is an FC-group.

Lemma 3. *Let G be a BCC-group. Then $Q = G/D(G)$ is a BFC-group. Moreover, for every element $x \in Q$ we have $|Q/C_Q(x^Q)| \le O(G)$.*

Indeed, for every element $g \in G$ the factor-group $G/C_G(g^G)$ is Cernikov and we have that

$$C_G(g^G)D(G)/D(G) \le C_{G/D(G)}\big((gD(G))^{G/D(G)}\big).$$

If G is a BFC-group then there is a number $b(G)$ such that $|G/C_G(g)| \le b(G)$ for every $g \in G$. By B. Neumann's theorem the derived subgroup $[G, G]$ is finite. Moreover, there exists a function f_1 such that $|[G, G]| \le f_1(b(G))$. For example, by [14] $f_1(n) = b^{A(n)}$ where $A(n) = (13 + \log_2 n)/2$. If G is central-by-finite group then a derived subgroup $[G, G]$ is finite by Schur's theorem (see, for example, [4], theorem 4.12). Moreover, there is a function f_2 such that $|[G, G]| \le f_2(c)$ where $c = |G/\zeta(G)|$. In fact, we may take $f_2(c) = c^{W(c)}$ where $W(c) = (\log_p c - 1)/2$ where p is the least prime dividing c (see, for example, [4, p. 103]).

Lemma 4. *Let G be a BCC-group and A a normal abelian subgroup such that $|G/C_G(A)| = n$ is finite. Then $[A, G]$ is a Chernikov subgroup and* $\mathrm{mmx}\,([A, G]) \le nM(G)$, $\mathrm{ord}([A, G]) \le O(G)^n$.

Proof. Put $C = C_G(A)$ and let $\{x_1, \ldots, x_n\}$ be a transversal to C in G. For each j, $1 \le j \le n$, the mapping $\phi_j : a \mapsto [a, x_j]$, $a \in A$, is an endomorphism of A with $\mathrm{Im}\phi_j = [A, x_j]$, $\mathrm{Ker}\phi_j = C_A(x_j)$. Since $C_G(x_j^G) \cap A \le C_A(x_j)A/C_A(x_j)$ is a Chernikov group and besides $\mathrm{mmx}\,(A/C_A(x_j)) \le M(G)$, $\mathrm{ord}(A/C_A(x_j)) \le O(G)$, $1 \le j \le \le n$. But $A/C_A(x_j) \cong [A, x_j]$, so that $[A, x_j]$ is a Chernikov subgroup such that $\mathrm{mmx}\,([A, x_j]) \le M(G)$, $\mathrm{ord}([A, x_j]) \le O(G)$, $1 \le j \le n$.

Given $g \in G$, there exist some j such that $g \in x_j C$, that is $g = x_j c$ for some $c \in C$. If $a \in A$, then $[a, g] = [a, x_j c] = [a, c][a, x_j]^c = [a, x_j]$. Thus $[A, G] = [A, x_1] \ldots [A, x_n]$ is a Chernikov subgroup group and $\mathrm{mmx}\,([A, G]) \le nM(G)$, $\mathrm{ord}([A, G]) \le O(G)^n$.

***Corollary* 1.** *Let G be a BCC-group and A a normal abelian subgroup such that $|G/A| = n$ is finite. Then $[G, G]$ is a Chernikov subgroup and* $\mathrm{mmx}\,([G, G]) \le nM(G)$, $\mathrm{ord}([G, G]) \le f_2(n)O(G)^n$.

Proof. In fact, $[A, G]$ is a Chernikov subgroup by Lemma 4. Thus, $A/[A, G] \leq \leq \zeta(G/[A, G])$, in particular, $|(G/[A, G])/\zeta(G/[A, G])| \leq n$. By Schur's theorem $|[G/[A, G], G/[A, G]]| \leq f_2(n)$.

Proposition 1. *Let G be a BCC-group. Then G has a series of normal subgroups $C \leq A \leq G$, where C is a Chernikov subgroup, A/C is a periodic divisible abelian subgroup (more precisely, $A/C = D(G)C/C$), G/A is abelian. Moreover,*

$$\mathrm{mmx}(C) \leq f_1(O(G))M(G) \quad \textit{and} \quad \mathrm{ord}(C) \leq f_1(O(G))^U f_2(f_1(O(G)))$$

where $U = O(G)f_1(O(G))O(G)^{f_1(O(G))}$.

Proof. Put $D = D(G)$. By Lemma 3 G/D is a BFC-group and $b(G/D) \leq \leq O(G)$. We consider $A/D = [G/D, G/D]$ so that A/D is finite and $|A/D| \leq \leq f_1(O(G))$. Put $B = [A, A]$. By Corollary 1 B is Chernikov subgroup, moreover, $\mathrm{mmx}(B) \leq M(G)f_1(O(G))$, $\mathrm{ord}(B) \leq f_2(f_1(O(G)))O(G)^{f_1(O(G))}$.

On the other hand, A/B is periodic abelian normal subgroup and $A/B = L/B \times \times DB/B$ for some subgroup L (see, for example, [20], theorem 21.2). Since L/B is finite abelian, we have $L/B = \langle x_1B\rangle \times \ldots \times \langle x_nB\rangle$ for suitable x_j, $1 \leq j \leq n$ and $n \leq f_1(O(G))$. Every element x_jB has in G/B at most $O(G)$ conjugates and $|x_jB| \leq f_1(O(G))$, therefore $|(L/B)^{G/B}| \leq f_1(O(G))^{O(G)f_1(O(G))}$.

Put $C/B = (L/B)^{G/B}$ so that C is a normal Chernikov subgroup of G and $A/C = DC/C$. If follows that $\mathrm{mmx}(C) \leq \mathrm{mmx}(B) \leq M(G)f_1(O(G))$ and $\mathrm{ord}(C) = = \mathrm{ord}(B)|C/B|$.

Lemma 5. *Let G be a BCC-group, A a normal abelian subgroup. Define $A_1 = A \cap \zeta(G)$ and assume that A/A_1 is divisible Chernikov group and $A/A_1 \leq \leq \zeta(G/A_1)$. Then $[A, G]$ is a Chernikov subgroup and $\mathrm{mmx}([A, G]) \leq \leq M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([A, G]) \leq O(G)^{\mathrm{mmx}(A/A_1)}$.*

Proof. Since A/A_1 is divisible Chernikov subgroup, there exists an ascending chain

$$A_1 = H_1 \leq H_2 \leq \ldots \leq H_n \leq \ldots \leq \bigcup_{n\in\mathbb{N}} H_n = A$$

such that

$$H_n/A_1 = \langle h_{n1}A_1\rangle \times \ldots \times \langle h_{nk}A_1\rangle, \qquad n \in \mathbb{N},$$

where $k = \mathrm{mmx}(A/A_1)$. Since $A_1 \leq \zeta(G)$ and $A/A_1 \leq \zeta(G/A_1)$,

$$C_G\left(\langle h_{nj}\rangle^G\right) = C_G(\langle h_{nj}\rangle A_1), \qquad 1 \leq j \leq k.$$

We can suppose that $\langle h_{1j}A_1\rangle \leq \langle h_{2j}A_1\rangle \leq \ldots \leq \langle h_{nj}A_1\rangle \leq \ldots$ for every j, $1 \leq j \leq k$. Then $C_G(\langle h_{1j}\rangle A_1) \geq C_G(\langle h_{2j}\rangle A_1) \geq \ldots \geq C_G(\langle h_{nj}\rangle A_1) \geq \ldots$, that is $C_G\left(\langle h_{1j}\rangle^G\right) \geq \geq C_G\left(\langle h_{2j}\rangle^G\right) \geq \ldots \geq C_G\left(\langle h_{nj}\rangle^G\right) \geq \ldots$.

Since $h_{nj} \in \zeta_2(G)$, the mapping $g \mapsto [h_{nj}, g]$, $g \in G$, is an endomorphism of G and it follows that $[h_{nj}, G] \cong G/C_G(h_{nj}) = G/C_G\left(\langle h_{nj}\rangle^G\right)$. On the other hand, there is a number t_n such that $(h_{nj})^{t}_{n} \in A_1 \leq \zeta(G)$. It follows that $1 = [(h_{nj})^{t}_{n}, g] = = [h_{nj}, g]^{t}_{n}$. In particular, $[h_{nj}, G]$ is bounded and so is $G/C_G(h_{nj})$. This means that $G/C_G(h_{nj})$ is finite. In this case $|G/C_G(h_{nj})| \leq O(G)$ for all $n \in \mathbb{N}$, $1 \leq j \leq k$. Therefore $|G/C_G(H_n)| \leq O(G)^k$, $n \in \mathbb{N}$.

However, $C_G(H_1) \geq C_G(H_2) \geq \ldots \geq C_G(H_n) \geq \ldots$. Hence there is a number $d \in \mathbb{N}$ such that $C_G(H_d) = C_G(H_{d+n})$ for all $n \in \mathbb{N}$. Put $C = C_G(H_d)$. Then $H_n \leq \zeta(C)$ for every $n \in \mathbb{N}$, so that $A = \bigcup_{n\in\mathbb{N}} H_n \leq \zeta(C)$. Consequently $G/C_G(A)$ is finite and $|G/C_G(A)| \leq O(G)^k$. By Lemma 4 $[A, G]$ is Chernikov subgroup and $\mathrm{mmx}([A, G]) \leq M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([A, G]) \leq O(G)^{\mathrm{mmx}(A/A_1)}$.

Corollary 1. *Let G be a BBC-group, A a normal abelian subgroup of G, $A_1 = A \cap \zeta(G)$. Suppose that A/A_1 is Chernikov divisible. Then and $[A, G]$ is a Chernikov subgroup and* $\mathrm{mmx}([A, G]) \leq 2M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([A, G]) \leq O(G)^R$ *where* $R = \mathrm{mmx}(A/A_1) + O(G)^{\mathrm{mmx}(A/A_1)}$.

Proof. Since A/A_1 is Chernikov, it has a finite special rank. By Lemma 2 $|G/C_G(A/A_1)| \leq O(G)^{\mathrm{mmx}(A/A_1)}$. Put $C = C_G(A/A_1)$. By Lemma 5 $[A, C]$ is a Chernikov subgroup and $\mathrm{mmx}([A, C]) \leq M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([A, C]) \leq O(G)^{\mathrm{mmx}(A/A_1)}$. If $Q = G/[A, C]$ and $U = A/[A, C]$, then we have that $|Q/C_Q(U)| \leq O(G)^{\mathrm{mmx}(A/A_1)}$. On the other hand, by Lemma 4 $[U, Q]$ is Chernikov, $\mathrm{mmx}([U, Q]) \leq M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([U, Q]) \leq O(G)^T$ where $T = O(G)^{\mathrm{mmx}(A/A_1)}$. Then it follows that $[A, C]$ is a Chernikov subgroup, $\mathrm{mmx}([A, G]) \leq 2M(G)O(G)^{\mathrm{mmx}(A/A_1)}$, $\mathrm{ord}([A, G]) \leq O(G)^R$ where $R = \mathrm{mmx}(A/A_1) + O(G)^{\mathrm{mmx}(A/A_1)}$.

Corollary 2. *Let G be a BBC-group, H a normal subgroup of G such that the index $|G : H| = n$ is finite. Suppose that A is a G-invariant abelian subgroup of H such that $A/(A \cap \zeta(H))$ is a Chernikov group. Then $[A, G]$ is a Chernikov subgroup and there are the functions f_3, f_4 such that*

$$\mathrm{mmx}([A, G]) \leq f_3(M(G), O(G), \mathrm{mmx}(A/(A \cap \zeta(H))), \mathrm{ord}(A/(A \cap \zeta(H))), n),$$

$$\mathrm{ord}([A, G]) \leq f_4(M(G), O(G), \mathrm{mmx}(A/(A \cap \zeta(H))), \mathrm{ord}(A/(A \cap \zeta(H))), n).$$

Proof. Let $D/(A \cap \zeta(H))$ be the divisible part of $A/(A \cap \zeta(H))$. By Corollary 1 $[D, H]$ is a Chernikov subgroup and there are integers

$$M_1 = M_1(M(G), O(G), \mathrm{mmx}(A/(A \cap \zeta(H))),$$

$$O_1 = O_1(M(G), O(G), \mathrm{mmx}(A/(A \cap \zeta(H)))$$

such that $\mathrm{mmx}([D, H]) \leq M_1$, $\mathrm{ord}([D, H]) \leq O_1$. Put $Q = G/[D, H]$, $V = D/[D, H]$, $U = A/[D, H]$ and $Y = H/[D, H]$. Then $|U/V| \leq \mathrm{ord}(A/(A \cap \zeta(H))$. If $Z = C_Y(U/V)$ then $|Y/Z| \leq (\mathrm{ord}(A/(A \cap \zeta(H)))!$. Let $\{u_1, \ldots, u_s\}$ be a transversal to V in U and pick $z \in Z$, $u \in U$. Then $u \in u_jV$ for some j, that is $u \in u_jv$ for suitable $v \in V$. Thus we have $[z, u] = [z, u_jv] = [z, u_j]$. It follows that $[Z, U] \leq [Z, u_1] \ldots [Z, u_s]$. Since $(u_j)^{k_j} \in V$ for some $k_j \in \mathbb{N}$, it follows that each subgroup $[Z, u_j]$ is bounded. By isomorphism $[Z, u_j] \cong Z/C_Z(u_j)$ and inclusion $C_Z(u_j) \geq C_Q(u_j^Q) \cap Z$ we obtain that $[Z, u_j]$ is also Chernikov and hence this is finite. Actually $|[Z, u_j]| \leq O(G)$, for every j. Therefore $[Z, U]$ is finite and $|[Z, U]| \leq O(G)^{\mathrm{ord}(A/(A\cap\zeta(H)))}$.

Finally, we have that $CQ/[Z, U](U/[Z, U]) \geq Z/[Z, U]$ and so this centralizers has finite index at most $n\ (\mathrm{ord}(A/(A \cap \zeta(H)))!$. Thus, it suffices to apply Lemma 4.

Lemma 6. *Let G be a BCC-group, A a normal abelian subgroup of G*

such that $G/G_G(A)$ *is locally cyclic. Then* $[A, G]$ *is a Chernikov subgroup and* $\mathrm{mmx}([A, G]) \le M(G)$, $\mathrm{ord}([A, G]) \le O(G)$.

Proof. Put $C = G_G(A)$, so that G/C has an ascending series of cyclic subgroups

$$\langle g_1C\rangle \le \langle g_2C\rangle \le \ldots \le \langle g_nC\rangle \le \ldots$$

such that $G/C = \bigcup_{n\in\mathbb{N}} \langle g_nC\rangle$. For every element $g \in G$ the mapping $\phi_g\colon a \mapsto [a, g]$, $a \in A$, is a $\mathbb{Z}G$-endomorphism of A and then $\mathrm{Im}\,\phi_g = [A, g]$, $\mathrm{Ker}\,\phi_g = C_A(g)$ are the G-invariant subgroups of A such that $[A, g] \cong A/C_A(g)$. Since $C_A(g) \ge A \cap C_G(g^G)$, $[A, g]$ is a Chernikov subgroup, $\mathrm{mmx}([A, G]) \le M(G)$, $\mathrm{ord}([A, G]) \le O(G)$. Furthermore, $[A, g] = [A, \langle g\rangle] = [A, \langle g\rangle C]$. It follows that

$$[A, g_1] \le [A, g_2] \le \ldots \le [A, g_n] \le \ldots .$$

Since $\mathrm{mmx}([A, g_n]) \le M(G)$, $\mathrm{ord}([A, g_n]) \le O(G)$ for each $n \in \mathbb{N}$, there is an integer d such that $[A, g_d] = [A, g_{d+n}]$ for every $n \in \mathbb{N}$. It follows that $[A, G] = \bigcup_{n\in\mathbb{N}}[A, g_{d+n}] = [A, g_d]$ so that $[A, G]$ is a Chernikov subgroup, $\mathrm{mmx}([A, G]) \le M(G)$, $\mathrm{ord}([A, G]) \le O(G)$.

Corollary 1. *Let* G *be a* BCC*-group,* A *a normal abelian subgroup of* G *such that* $G/G_G(A)$ *is abelian group of finite special rank* r. *Then* $[A, G]$ *is a Chernikov subgroup and* $\mathrm{mmx}([A, G]) \le rM(G)$, $\mathrm{ord}([A, G]) \le O(G)^r$.

Corollary 2. *Let* G *be a* BCC*-group,* A *a normal abelian subgroup of* G *such that* $Q = G/G_G(A)$ *is a Chernikov group. Then* $[A, G]$ *is a Chernikov subgroup and* $\mathrm{mmx}([A, G]) \le (\mathrm{mmx}(Q) + \mathrm{ord}(Q))M(G)$, $\mathrm{ord}([A, G]) \le O(G)^{\mathrm{mmx}(Q)+\mathrm{ord}(Q)}$.

Indeed, we can apply Corollary 1 and Lemma 4.

Corollary 3. *Let* G *be a* BCC*-group,* H *a normal subgroup of* G *such that* G/H *is a Chernikov group. Suppose that* A *is a* G*-invariant abelian subgroup of* H *such that* A/A_1 *is Chernikov where* $A_1 = A \cap \zeta(H)$. *Then* $[A, G]$ *is a Chernikov subgroup and there are functions* f_5, f_6 *such that* $\mathrm{mmx}([A, G]) \le f_5(M(G), O(G), \mathrm{mmx}(A/A_1), \mathrm{ord}(A/A_1), \mathrm{mmx}(G/H), \mathrm{ord}(G/H))$, $\mathrm{ord}([A, G]) \le f_6(M(G), O(G), \mathrm{mmx}(A/A_1), \mathrm{ord}(A/A_1), \mathrm{mmx}(G/H), \mathrm{ord}(G/H))$.

Corollary 4. *Let* G *be a* BCC*-group,* H *a normal subgroup of* G *such that* G/H *is a Chernikov group,* A *is a normal abelian subgroup of* G, $A_1 = A \cap H$, $A_2 = A \cap \zeta(H)$. *Suppose that* A_1/A_2 *is Chernikov. Then* $[A, G]$ *is a Chernikov subgroup and there are functions* f_7, f_8 *such that* $\mathrm{mmx}([A, G]) \le f_7(M(G), O(G), \mathrm{mmx}(A_1/A_2), \mathrm{ord}(A_1/A_2), \mathrm{mmx}(G/H), \mathrm{ord}(G/H))$, $\mathrm{ord}([A, G]) \le f_8(M(G), O(G), \mathrm{mmx}(A_1/A_2), \mathrm{ord}(A_1/A_2), \mathrm{mmx}(G/H), \mathrm{ord}(G/H))$.

Indeed, by above Corollary 3 $[A_1, G]$ is a Chernikov subgroup. Apply then Corollary 1 of Lemma 5 to a factor-group $G/[A_1, G]$.

Proposition 2. *Let* G *be a* BCC*-group,* $D = D(G)$. *Then* G *has a series of normal subgroups* $C \le A \le G$ *such that* C *is a Chernikov subgroup,* $A = DC$ *(in particular,* A/C *is a periodic divisible abelian subgroup),* G/A *is abelian and* $[A, C] \le C$. *Moreover, there are the functions* f_9, f_{10} *such that* $\mathrm{mmx}(C) \le f_9(M(G), O(G))$, $\mathrm{ord}(C) \le f_{10}(M(G), O(G))$.

Proof. By Proposition 1 G includes a normal Chernikov subgroup C_1 such that

G/DC_1 is abelian. Further, there are the numbers $M_2 = M_2(M(G), O(G))$ and $O_2 = O_2(M(G), O(G))$ such that $\mathrm{mmx}(C_1) \le M_2$ and $\mathrm{ord}(C_1) \le O_2$.

Clearly we may assume that $C_1 = \langle 1 \rangle$. In other words, we may suppose that G/D is abelian. In this case for each element $g \in G$ the mapping $\phi_g: d \mapsto [d, g]$, $d \in D$, is a $\mathbb{Z}G$-endomorphism of D and then $\mathrm{Im}\phi_g = [D, g]$, $\mathrm{Ker}\phi_g = C_D(g)$ are the G-invariant subgroups of D such that $[D, g] \cong D/C_D(g)$. Since D is periodic divisible subgroup and $C_D(g) \ge D \cap C_G(g^G)$, $D/C_D(g)$ is a divisible Chernikov group such that $\mathrm{mmx}(D/C_D(g)) \le M(G)$. Therefore $\mathrm{mmx}([D, g]) \le M(G)$.

Suppose that $\zeta(G)$ does not include D. Then there is an element $x_1 \in G$ such that $B_1 = [D, x_1] \ne \langle 1 \rangle$. Put $D_1 = C_D(x_1)$, then D/D_1 is a divisible Chernikov group and $\mathrm{mmx}(D/D_1) \le M(G)$. If $\mathrm{Soc}(D/D_1) = \langle a_1 D_1 \rangle \times \ldots \times \langle a_n D_1 \rangle$ is the socle, define S as $S = \langle a_1, \ldots, a_n \rangle$. Since $S \cap D_1 \ge \mathrm{Fratt}(S)$, S is a product of n cyclic subgroups. Let E_1 be a divisible envelope of S in D. Then $\mathrm{mmx}(E_1) = n = \mathrm{mmx}(D/D_1)$. Clearly, $E_1 \cap D_1 = S \cap D_1$, in particular, $E_1 \cap D_1$ is finite. Hence $E_1 D_1 / D_1 \cong E_1/(E_1 \cap D_1)$ is a divisible Chernikov group and

$$\mathrm{mmx}(E_1 D_1 / D_1) = \mathrm{mmx}(E_1/(E_1 \cap D_1)) = \mathrm{mmx}(E_1) = \mathrm{mmx}(D/D_1)$$

what means that $E_1 D_1 = D$.

Let $e \in E_1$. Since D is a periodic abelian divisible subgroup, $\langle e \rangle^G$ is bounded. On other hand, $\langle e \rangle^G$ is a Chernikov subgroup [17]. This means that $|G/C_G(\langle e \rangle^G)| \le O(G)$. By Lemma 2, $G/C_G(E_1^G)$ is also finite, moreover, $|G/C_G(E_1^G)| \le O(G)^{M(G)}$. Put $G_1 = C_G(E_1^G) \cap C_G(x_1^G)$, so that G/G_1 is a Chernikov group such that $\mathrm{mmx}(G/G_1) \le M(G)$, $\mathrm{ord}(G/G_1) \le O(G)^{M(G)+1}$.

Consider now $[D_1, x]$ where $x \in G_1$. If $[D_1, x] \le B_1$ for each $x \in G_1$ then and $[D_1, G_1] \le B_1$ and we it suffices to apply Corollary 4 of Lemma 6 to a factor-group G/B_1. Therefore we suppose that there is an element $x_2 \in G_1$ such that B_1 does not include $[D_1, x_2]$. Since B_1 is divisible, we have $[D_1, x_2]B_1 = B_1 \times U_1$ for some nonidentity subgroup U_1 (see, for example, [20], theorem 21.2). There is some $u_1 \ne 1$ such that $u_1 \in B_1 U_1$ and $\langle u_1 \rangle \cap B_1 = \langle 1 \rangle$. Then

$u_1 = [e_1, x_1][d_2, x_2]$ where $e_1 \in E_1$, $d_2 \in D_1$. If $e \in E_1$, then we have

$$[e, x_1 x_2] = [e, x_2]x_2^{-1}[e, x_1]x_2 = [x_2^{-1} e x_2, x_2^{-1} x_1 x_2] = [e, x_1],$$

so that

$$B_1 = [D, x_1] = [E_1, x_1] = [E_1, x_1 x_2]$$

and

$$[d_2, x_1 x_2] = [d_2, x_2]x_2^{-1}[d_2, x_1]x_2 = [d_2, x_2].$$

It follows that $\langle u_1, B_1 \rangle \le [D, x_1 x_2]$, in particular, $[D, x_1 x_2]$ is a divisible Chernikov subgroup such that $\mathrm{mmx}([D, x_1 x_2]) > \mathrm{mmx}([D, x_1])$.

Put $D_2 = C_D(x_1 x_2)$ and $B_2 = [D, x_1 x_2]$. One again there exists a divisible subgroup E_2 such that $D = E_2 D_2$ and $\mathrm{mmx}(E_2) = \mathrm{mmx}(D/D_2)$. Moreover, the index $|G : C_G(E_2^G)|$ is finite and $|G/C_G(E_2^G)| \le O(G)^{M(G)}$. Put

$$G_2 = C_G(E_2^G) \cap C_G((x_1 x_2)^G),$$

so that G/G_2 is a Chernikov group, $\mathrm{mmx}(G/G_2) \le M(G)$, $\mathrm{ord}(G/G_2) \le$ $\le O(G)^{M(G)+1}$.

In this way, consider now $[D_2, x]$ for all $x \in G_2$. If $[D_2, x] \le B_2$ for every $x \in$ $\in G_2$, then we argue as above using G/B_2. Similarly, if B_2 does not include $[D_2, x_3]$ for some $x_3 \in G_2$, we show that $\mathrm{mmx}([D, x_1x_2x_3]) > \mathrm{mmx}([D, x_1x_2])$.

Since $\mathrm{mmx}([D, g]) \le M(G)$ for every element $g \in G$, then after $t \le M(G)$ steps we obtain that $[D_t, G_t]$ is a Chernikov subgroup and complete the proof applying again Corollary 4 of Lemma 6 to a factor-group $G/[D_t, G_t]$.

For a group G put $t(G) = \{x \in G / x$ has finite order$\}$.

If G is a CC-group (in particular, BCC-group) then $t(G)$ is a subgroup of G, moreover, $[G, G] \le t(G)$ [16].

Lemma 7. *Let G be a BCC-group, $D = D(G)$, $T = t(G)$. Suppose that $D \le \zeta(G)$ and G/D is abelian. Then $[T, G]$ is a Chernikov subgroup. Moreover, there are the functions f_{11}, f_{12} such that* $\mathrm{mmx}([T, G]) \le f_{11}(M(G), O(G))$, $\mathrm{ord}([T, G]) \le f_{12}(M(G), O(G))$.

Proof. Since $G/\zeta(G)$ is abelian, the mapping $\vartheta_g : x \mapsto [x, g]$, $x \in G$, is an endomorphism of G for each $g \in G$. If $g \in T$ then there is a number $n \in \mathbb{N}$ such that $g^n \in \zeta(G)$. It follows that $1 = [x, g^n] = [x, g]^n$ for any $x \in G$. In other words, $\mathrm{Im}\,\vartheta_g = [G, g]$ is a bounded subgroup. By isomorphism $[G, g] = \mathrm{Im}\,\vartheta_g \cong$ $\cong G/\mathrm{Ker}\,\vartheta_g = G/C_G(g)$ we obtain that $[G, g]$ is a Chernikov subgroup. Thus $[G, g]$ is finite for all $g \in T$, moreover, $\mathrm{ord}([G, g]) \le O(G)$. It follows that T is a BFC-group, hence $[T, T]$ is finite and $|[T, T]| \le f_1(O(G))$. Therefore we may assume that T is an abelian subgroup.

Suppose that $\zeta(G)$ does not include T. Choose an element $y_1 \in T \setminus \zeta(G)$. If $B_1 = [G, y_1]$, then B_1 is finite and $|B_1| \le O(G)$. A factor-group $G/C_G(y_1)$ is also finite, so that $G/C_G(y_1) = \langle u_1C_G(y_1)\rangle \times \ldots \times \langle u_kC_G(y_1)\rangle$ for some elements $u_1, \ldots, u_k$ where $k \le O(G)$. Put $U_1 = \langle u_1, \ldots, u_k\rangle$, then by Lemma 1 $G/C_G(U_1)$ is Chernikov and $\mathrm{mmx}(G/C_G(U_1)) \le M(G)O(G)$, $\mathrm{ord}(G/C_G(U_1)) \le O(G)^{O(G)}$. Put $G_1 =$ $= C_G(y_1) \cap C_G(U_1)$, then G/G_1 is a Chernikov group such that $\mathrm{mmx}(G/G_1) \le$ $\le M(G)O(G)$, $\mathrm{ord}(G/G_1) \le O(G)^{O(G)+1}$. Put $T_1 = T \cap G_1$. If $[g, y] \in B_1$ for any $g \in G_1$, $y \in T_1$ then and $[G_1, T_1] \le B_1$. For this case we can use a Corollary 4 of Lemma 6 to G/B_1. Therefore we may suppose that there are the elements $y_2 \in T_1$, $g_1 \in G_1$ such that $[g_1, y_2] \notin B_1$.

If $u \in U_1$, we will have $[u, y_1y_2] = [u, y_1][u, y_2] = [u, y_1]$, so that $B_1 = [U_1, y_1] = [U_1, y_1y_2]$. On the other hand, $[g_1, y_1y_2] = [g_1, y_1][g_1, y_2] = [g_1, y_2]$. It follows that $[G, y_1y_2] \ge \langle B_1, [g_1, y_2]\rangle$, in particular, $\mathrm{ord}([G, y_1]) < \mathrm{ord}([G, y_1y_2])$.

Put $B_1 = [G, y_1y_2]$. Similarly, there is a finite subgroup U_2 such that $G =$ $= U_2C_G(y_1y_2)$ and $G/G_G(U_2)$ is a Chernikov group, furthermore, $\mathrm{mmx}(G/C_G(U_2)) \le M(G)O(G)$, $\mathrm{ord}(G/C_G(U_2)) \le O(G)^{O(G)}$. If $G_2 =$ $= C_G(y_1y_2) \cap C_G(U_2)$ then G/G_2 is a Chernikov group with $\mathrm{mmx}(G/G_2) \le$ $\le M(G)O(G)$, $\mathrm{ord}(G/G_2) \le O(G)^{O(G)+1}$. If $[G_2, G_2 \cap T] \le B_2$ then we must consider G/B_2 and use Corollary 4 of Lemma 6. If there are the elements $y_3 \in G_2 \cap$ T, $g_2 \in G_2$ such that $[g_2, y_2] \notin B_2$, then repeating the above arguments we prove that $\mathrm{ord}([G, y_1y_2y_3]) > \mathrm{ord}([G, y_1y_2])$.

Since $\mathrm{ord}([G, y]) \le M(G)$ for every element $y \in T$, after $t \le O(G)$ steps we obtain that $[G_t, T \cap G_t]$ is a Chernikov subgroup and complete the proof applying again Corollary 4 of Lemma 6 to $G/[G_t, T \cap G_t]$.

Corollary. *Let G be a BCC-group, $T = t(G)$. Then G includes a normal Chernikov subgroup B such that $[G, T] \le B$ and there are the functions f_{13}, f_{14} such that* $\mathrm{mmx}(B) \le f_{13}(M(G), O(G))$, $\mathrm{ord}(B) \le f_{14}(M(G), O(G))$.

In fact, we may use Proposition 2 and then Lemma 7.

Lemma 8. *Let G be a BCC-group, A a normal abelian subgroup of G. If G/A is a Chernikov group then $[G, G]$ is also Chernikov and there are the functions f_{15}, f_{16} such that* $\mathrm{mmx}([G, G]) \le f_{15}(M(G), O(G))$, $\mathrm{ord}([G, G]) \le$ $\le f_{16}(M(G), O(G))$.

Proof. Since $G/C_G(A)$ is a Chernikov group then we may apply Corollary 2 of Lemma 6. In other words, we may assume that $A \le \zeta(G)$. Let D/A be a divisible part of G/A. Theorem 2.1 of [19] yields that D is abelian and we complete the proof applying Corollary 1 of Lemma 4.

Proposition 3. *Let G be a BCC-group. If $[G, G] \le \zeta(G)$, then $[G, G]$ is a Chernikov subgroup and there exist the functions f_{17}, f_{18} such that* $\mathrm{mmx}([G, G]) \le f_{17}(M(G), O(G))$, $\mathrm{ord}([G, G]) \le f_{18}(M(G), O(G))$.

Proof. Since $[G, G] \le \zeta(G)$, $C_G(g) = C_G(g^G)$ and $G/C_G(g) \cong [g, G]$ for all elements $g \in G$, moreover, $\mathrm{mmx}([g, G]) \le M(G)$, $\mathrm{ord}([g, G]) \le O(G)$.

Put $k = M(G)$ and proceed by induction on k. If $k = 0$ then $|G/C_G(g)| \le$ $\le O(G)$ for every $g \in G$, i.e. G is a BFC-group and all is done.

Suppose that $k > 0$. If $\mathrm{mmx}(G/C_G(g)) < k$ for every $g \in G$, then result follows by induction. Hence we assume that there is an element $g_1 \in G$ such that $\mathrm{mmx}(G/C_G(g_1)) = k$. Put $H = C_G(g_1)$, $B = [g_1, G]$. Given $g \in G$ we consider $HC_G(g)/C_G(g)$. If this group is finite, then so is $H/(C_G(g) \cap H) = H/C_G(g)$ and $\mathrm{mmx}(H/C_H(g)) = 0 < k$. Let now $HC_G(g)/C_G(g)$ is infinite. Then it includes a Prüfer p-subgroup for some prime p, that is H includes a subset $\{x_n \mid n \in \mathbb{N}\}$ with the following properties: $y_2^p = y_1c_1$, $y_{n+1}^p = y_nc_n$ for some elements $c_n \in C_G(g)$, $n \in \mathbb{N}$. Thus

$$[g, y_2]^p = [g, y_2^p] = [g, (y_2^p)(c_1^{-1})] = [g, y_1],$$

$$[g, y_{n+1}]^p = [g, y_{n+1}^p] = [g, (y_{n+1}^p)(c_n^{-1})] = [g, y_n], \qquad n \in \mathbb{N}.$$

This means that $\langle [g, y_n] \mid n \in \mathbb{N} \rangle$ is likewise a Prüfer p-subgroup. As $[g, y] = [g, y][g_1, y] = [gg_1, y]$, for every $y \in H$, we have that $[g, y_n] = [gg_1, y_n]$. It follows that $[g, G] \cap [gg_1, G]$ is infinite for any $n \in \mathbb{N}$.

Put $C = \langle g, g_1 \rangle = \langle g, gg_1 \rangle$ and $K = [C, G]$. Since

$$[g^q g_1^r, x] = [g^q, x][g_1^r, x] = [g, x]^q [g_1, x]^r,$$

we obtain $K = [g, G][g_1, G]$. By the same reasons $K = [g, G][gg_1, G]$. Since $[g, G] \cap [gg_1, G]$ is infinite, $\mathrm{mmx}(K) \le \mathrm{mmx}([g, G]) + \mathrm{mmx}([gg_1, G]) - 1 \le 2k - 1$. Let $g \in H$ be. Then

$$[gB, H/B] = [g, H]B/B \le [g, G]B/B.$$

If $HC_G(g)/C_G(g)$ is finite then so is $[g, G]$, so that and $[gB, H/B]$ is finite, in particular, $\mathrm{mmx}([gB, H/B]) < k$. If $HC_G(g)/C_G(g)$ is infinite then

$$[gB, H/B] = [g, H]B/B \le [g, G][g_1, G]/[g_1, G].$$

Hence

$$\mathrm{mmx}([gB, H/B]) \le \mathrm{mmx}([g, G][g_1, G]/[g_1, G]) \le 2k - 1 - k = k - 1.$$

By induction hypothesis $[H/B, H/B]$ is a Chernikov subgroup and there are the functions f_{19}, f_{20} such that $\mathrm{mmx}([H/B, H/B]) \le f_{19}(M(G), O(G))$, $\mathrm{ord}([H/B, H/B]) \le f_{20}(M(G), O(G))$. It follows that $[H, H]$ is Chernikov, because B is Chernikov. Now it suffices to apply Lemma 8 to $G/[H, H]$.

3. The proof of main theorem. By Proposition 2 G includes a normal subgroups $A \ge C$ such that C is Chernikov subgroup, A/C is abelian and $[G, A] \le C$. Now we may apply Proposition 3 to G/C. A second assertion follows from Lemma 2.

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