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A PRESENTATION OF THE AUTOMORPHISM GROUP OF THE TWO-GENERATOR FREE METABELIAN AND NILPOTENT GROUP OF CLASS c

ЗОБРАЖЕННЯ ГРУПИ АВТОМОРФІЗМІВ ДВОПОРОДЖЕНОЇ ВІЛЬНОЇ МЕТАБЕЛЕВОЇ НІЛЬПОТЕНТНОЇ КЛАСУ c ГРУПИ

We determine the structure of $\mathrm{IA}(G)/\mathrm{Inn}(G)$ by giving a set of generators, and showing that $\mathrm{IA}(G)/\mathrm{Inn}(G)$ is a free abelian group of rank $(c-2)(c+3)/2$. Here $G = M_2$, $c = \langle x, y\rangle$, $c \geq 2$, is the free metabelian nilpotent of class c group.

Визначено структуру фактор-групи $\mathrm{IA}(G)/\mathrm{Inn}(G)$ за допомогою задання множини твірних та встановлення того факту, що $\mathrm{IA}(G)/\mathrm{Inn}(G)$ є вільною абелевою групою рангу $(c-2)(c+3)/2$. Тут $G = M_2$, $c = \langle x, y\rangle$, $c \geq 2$, є вільною метабелевою нільпотентною класу c групою.

1. Introduction. Let $F_2 = \langle \bar{x}, \bar{y}\rangle$ be the free group of rank 2. We let $V \leq [F_2, F_2]$ be a fully invariant subgroup of F_2 and $G = F_2/V$ the relatively free group generated by $\{x = \bar{x}V, y = \bar{y}V\}$. Since V is invariant, each automorphism $\bar{\alpha} \in \mathrm{Aut}(F_2)$ induces in a natural way an automorphism $\alpha \in \mathrm{Aut}(G)$. However, an automorphism $\beta \in \mathrm{Aut}(G)$ may not be induced by any automorphism $\bar{\beta} \in \mathrm{Aut}(F_2)$. For instance, the endomorphism $\alpha = [x \to x[x, y, y], y \to y]$ is an automorphism of $G = F_2/V$, $V = [F_2, F_2, F_2, F_2]$, which can not be induced by any automorphism $\bar{\alpha} \in \mathrm{Aut}(F_2)$ (Andreadakis [1]), and we say that $\alpha \in \mathrm{Aut}(F_2/V)$ is nontame (or wild); whereas, with $V = F_2''$, every automorphism $\alpha \in \mathrm{Aut}(F_2/V)$ is tame, i. e. induced by some automorphism $\bar{\alpha} \in \mathrm{Aut}(F_2)$ (Bachmuth [2]).

For $V \leq [F_2, F_2]$, an arbitrary automorphism $\alpha \in \mathrm{Aut}(F_2/V)$ may be decomposed as the product $\alpha = \alpha'\alpha''$, where $\alpha' \in \mathrm{Aut}(F_2/V)$ is induced by an automorphism of the free abelian group $F_2/[F_2, F_2]$, which in turn, is induced by some $\bar{\alpha}' \in \mathrm{Aut}(F_2)$, and hence α' is tame; and α'' is an IA-automorphism of F_2/V of the form: $\alpha'' = [x \to xd', y \to yd'']$, where $d', d'' \in [F_2/V, F_2/V]$. Thus, for $G = \mathrm{gp}\{x, y\} = F_2/V$, where $V \leq [F_2, F_2]$, we have that

$$\mathrm{Aut}(G) = \mathrm{gp}\{T(G), \mathrm{IA}(G)\}, \tag{1}$$

where $T(G) = \mathrm{sgp}\{\tau, \mu, \nu\}$ is generated by the tame automorphisms of G defined by

$$\tau = [x \to y, y \to x], \quad \mu = [x \to x^{-1}, y \to y], \quad \nu = [x \to xy, y \to y], \tag{2}$$

and $\mathrm{IA}(G)$ is generated by all IA-automorphisms of the form $\phi = [x \to xd', y \to yd'']$. Let $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G. Then $\mathrm{Inn}(G)$ are tame and since $\mathrm{IA}(G) \geq \mathrm{Inn}(G) \cong G/\zeta(G)$, the central quotient of G, it follows that

$$\mathrm{Inn}(G) = \langle \psi_x, \psi_y; \Psi(V)\rangle, \tag{3}$$

where $\Psi(V) = \{v(\psi_x, \psi_y); v = v(x, y) \in V\}$ and $\psi_u = [x \to x^u, y \to y^u]$ is the inner automorphism induced by u. Note that the inner automorphism induced by $v = v(x, y)$ is $\psi_v = v(\psi_x, \psi_y)$.

For the free metabelian group $M_2 = F_2/V$, $V = [[F_2, F_2], [F_2, F_2]]$, it is a wellknown result of Bachmuth [2] that $\mathrm{IA}(M_2) = \mathrm{Inn}(M_2)$. Thus $\mathrm{IA}(M_2)$ is tame, and by (1), (2) and (3), we have

$$\mathrm{Aut}(M_2) = \mathrm{gp}\{\tau, \mu, \nu, \psi_x, \psi_y\},$$

where $\psi_x = (\mu\nu\mu\nu)^\tau$ and $\psi_y = \mu\nu\mu\nu$ define all inner automorphisms of M_2 induced by conjugation with the generators x and y respectively. Since the center of M_2 is trivial, it follows that

$$\mathrm{Inn}(M_2) = \langle \psi_x, \psi_y; \Psi'' \rangle,$$

where $\Psi'' = \Psi(F'')$.

Theorem 1 (Nielsen [3]).

(i) $\mathrm{Aut}(F_2) = \left\langle \bar\tau, \bar\mu, \bar\nu; \bar\tau^2, \bar\mu^2, (\bar\tau\bar\mu)^4, [\bar\nu, \bar\mu, \bar\nu], (\bar\tau\bar\mu\bar\tau\bar\nu)^2, (\bar\nu\bar\tau\bar\mu)^3 \right\rangle$, *where* $\bar\tau = [\bar x \to \bar y, \bar y \to \bar x]$, $\bar\mu = [\bar x \to \bar x^{-1}, \bar y \to \bar y]$ *and* $\bar\nu = [\bar x \to \bar x \bar y, \bar y \to \bar y]$.

(ii) $\mathrm{Inn}(F_2) = \langle \bar\psi_x, \bar\psi_y; \Phi \rangle$, *where* $\bar\psi_x = (\bar\mu\bar\nu\bar\mu\bar\nu)^{\bar\tau}$ *and* $\bar\psi_y = \bar\mu\bar\nu\bar\mu\bar\nu$.

From the above discussion we have the following presentation of $\mathrm{Aut}(M_2)$.

Theorem 2.

$$\mathrm{Aut}(M_2) = \left\langle \tau, \mu, \nu, \psi_x, \psi_y; \tau^2, \mu^2, (\tau\mu)^4, [\nu, \mu, \nu], (\tau\mu\tau\nu)^2, (\nu\tau\mu)^3, \Psi'' \right\rangle.$$

For an arbitrary but fixed $c \ge 2$, we now turn to the problem of finding a presentation of the group $\mathrm{Aut}(G)$, where $G = \mathrm{gp}\{x, y\} = M_{2,c}$ is the free metabelian nilpotent group of class c.

2. Structure of $\mathrm{IA}(G)/\mathrm{Inn}(G)$. Let $G = M_{2,c} = \langle x, y \rangle$, $c \ge 2$, be the free metabelian nilpotent-of-class-c group generated by $\{x, y\}$. Since G is nilpotent, we have that every IA-endomorphism $[x \to xd', y \to yd'']$, where $d', d'' \in G'$, defines an IA-automorphism of G. Recall that

$$\mathrm{Aut}(G) = \mathrm{gp}\{T(G), \mathrm{IA}(G)\},$$

where $T(G) = \mathrm{sgp}\{\tau, \mu, \nu\}$ and $\mathrm{IA}(G) = \mathrm{sgp}\{[x \to xd', y \to yd''], d', d'' \in G'\}$. Thus, for the presentation of $\mathrm{Aut}(G)$, we need the structures of $\mathrm{Inn}(G)$ and of the quotient $\mathrm{IA}(G)/\mathrm{Inn}(G)$. We have that

$$\mathrm{Inn}(G) = G/\zeta(G) = G/\gamma_c(G) \cong M_{2,c-1}.$$

Thus, if Ψ is the subgroup of $\mathrm{Aut}(G)$ generated by ψ_x, ψ_y, then

$$\mathrm{Inn}(G) = \langle \psi_x, \psi_y; \Psi'', \gamma_c(\Psi) \rangle$$

is a free metabelian nilpotent group of class $c-1$ and, is a normal subgroup of $\mathrm{Aut}(G)$ contained in $\mathrm{IA}(G)$.

Any element $z \in G$ can be uniquely written as $z = x^a y^b [x, y]^{u(x,y)}$, where $a, b \in Z$ and $u(x, y) \in ZG$, the integral group ring of G, is of the form $u(x, y) = \sum_{0 \le i+j \le c-2} u_{ij}(x, y)$, $u_{ij}(x, y) = a_{ij}(x-1)^i (y-1)^j \in \Delta^{i+j}$, $\Delta = ZG(G-1)$, the augmentation ideal of G.

For any u, υ in ZG, we define $\alpha(u, \upsilon) = [x \to x[x, y]^u, y \to y[x, y]^\upsilon]$. Every automorphism in $\mathrm{IA}(G)/\mathrm{Inn}(G)$ is of the form $\alpha(u, \upsilon)$, where $(u, \upsilon) \in (\Delta^i, \Delta^j)$ for some $i, j \ge 0$, $(i, j) \ne (0, 0)$. For $(u, \upsilon) \in (\Delta^i, \Delta^j)$ and $(u', \upsilon') \in (\Delta^{i'}, \Delta^{j'})$, we define $(u, \upsilon) << (u', \upsilon')$ if $i + j < i' + j'$. We have the following formula:

$$\alpha(u,v)\alpha(u',v') = \alpha(u^*, v^*), \tag{4}$$

where $u^* = u' + u + u(u'(y-1) - v'(x-1))$ and $v^* = v' + v + v(u'(y-1) - v'(x-1))$.

Theorem 3. $\mathrm{IA}(G)/\mathrm{Inn}(G) = \langle \alpha(u,0), \alpha(0,v) \mid u,v \in \Delta \rangle$.

Proof. It is easily verified that

$$\alpha(u,v)\alpha(-u,0)\alpha(0,-v) =$$
$$= \alpha(-u^2(y-1) - u^2v(x-1)(y-1), -uv(y-1) + v^2(x-1) - uv^2(x-1)(y-1)).$$

Let $\hat{u} = -u^2(y-1) - u^2v(x-1)(y-1)$ and $\hat{v} = -uv(y-1) + v^2(x-1) - uv^2(x-1) \times \times (y-1)$, we have that

$$\alpha(u,v)\alpha(-u,0)\alpha(0,-v) = \alpha(\hat{u}, \hat{v})$$

and $(u,v) << (\hat{u}, \hat{v})$.

By repeated application of the above result, we conclude that, for any pair $(u,v) \in \in (\Delta^i, \Delta^j)$, $i + j > 0$, there exists a chain of pairs

$$(u,v) = (u^{(0)}, v^{(0)}) << (u^{(1)}, v^{(1)}) << (u^{(2)}, v^{(2)}) << \ldots << (u^{(t)}, v^{(t)}),$$

with $(u^{(k)}, v^{(k)}) \in (\Delta^{i(k)}, \Delta^{j(k)})$ such that

$$\alpha(u,v)\alpha(-u^{(0)},0)\alpha(0,-v^{(0)})\alpha(-u^{(1)},0)\alpha(0,-v^{(1)})\ldots\alpha(-u^{(t)},0)\alpha(0,-v^{(t)}) = 1.$$

This completes the proof.

We shall need the following technique result.

Lemma 1. *For any integer m, we have the following*:

(i) $\alpha(u,0)^m = \alpha\left(\sum_{i\ge1}\binom{m}{i}u^i(y-1)^{i-1}, 0\right)$;

(ii) $\alpha(0,v)^m = \alpha\left(0, \sum_{i\ge1}(-1)^{i-1}\binom{m}{i}v^i(x-1)^{i-1}\right)$,

where $\binom{m}{i} = \dfrac{\prod_{j=0}^{i-1}(m-j)}{i!}$ *is the binomial coefficient.*

Remark. Since the group G is nilpotent of class c, the above sums are finite sums.

Proof. (i) First, we prove that the result is true for positive integers. Note that if m is a positive integer and $i \ge m+1$, then $\binom{m}{i} = 0$. Thus

$$\alpha\left(\sum_{i\ge1}\binom{m}{i}u^i(y-1)^{i-1}, 0\right) = \alpha\left(\sum_{i=1}^{m}\binom{m}{i}u^i(y-1)^{i-1}, 0\right).$$

We shall use induction on m. The result is obviously true for $m = 1$. Assume that the result is true for $m-1$. That is,

$$\alpha(u,0)^{m-1} = \alpha\left(\sum_{i=1}^{m-1}\binom{m-1}{i}u^i(y-1)^{i-1}, 0\right).$$

Then

$$\alpha(u,0)^m = \alpha(u,0)\,\alpha\left(\sum_{i=1}^{m-1}\binom{m-1}{i}u^i(y-1)^{i-1}, 0\right) =$$

$$= \alpha\left(\sum_{i=1}^{m-1}\binom{m-1}{i}u^i(y-1)^{i-1} + u + u\left(\sum_{i=1}^{m-1}\binom{m-1}{i}u^i(y-1)^{i-1}\right)(y-1),\ 0\right) =$$

$$= \alpha\left(\binom{m-1}{1}u + \sum_{i=2}^{m-1}\binom{m-1}{i}u^i(y-1)^{i-1} + u + \sum_{i=2}^{m-1}\binom{m-1}{i-1}u^i(y-1)^{i-1} + \binom{m-1}{m-1}u^m(y-1)^{m-1},\ 0\right) =$$

$$= \alpha\left(\binom{m}{1}u + \sum_{i=2}^{m-1}\left(\binom{m-1}{i}+\binom{m-1}{i-1}\right)u^i(y-1)^{i-1} + \binom{m}{m}u^m(y-1)^{m-1},\ 0\right) =$$

$$= \alpha\left(\binom{m}{1}u + \sum_{i=2}^{m-1}\binom{m}{i}u^i(y-1)^{i-1} + \binom{m}{m}u^m(y-1)^{m-1},\ 0\right) =$$

$$= \alpha\left(\sum_{i=1}^{m}\binom{m}{i}u^i(y-1)^{i-1},\ 0\right).$$

Therefore the result is true for any positive integer.

Next, we shall prove that the result is also true for negative integers. That is,

$$\alpha(u,0)^{-m} = \alpha\left(\sum_{i\ge 1}\binom{-m}{i}u^i(y-1)^{i-1},\ 0\right),$$

where m is positive.

For $m = 1$, we have that $\binom{-1}{i} = \dfrac{-1(-1-1)\ldots(-1-i+1)}{i!} = (-1)^i$. It is easily verified that $\alpha(u,0)^{-1} = \alpha\left(\sum_{i\ge 1}(-1)^i u^i(y-1)^{i-1}, 0\right) = \alpha\left(\sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1}, 0\right)$. Thus the result is true for -1. Assume that the result is true for $m-1$, then

$$\alpha(u,0)^{-(m-1)} = \alpha\left(\sum_{i\ge 1}\binom{-(m-1)}{i}u^i(y-1)^{i-1}, 0\right).$$

Now we have

$$\alpha(u,0)^{-m} = \alpha(u,0)^{-1}\alpha(u,0)^{-(m-1)} =$$

$$= \alpha\left(\sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1}, 0\right)\alpha\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^j(y-1)^{j-1}, 0\right) =$$

$$= \alpha\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^j(y-1)^{j-1} + \sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1} + \right.$$

$$\left. + \sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1}\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^j(y-1)^{j-1}\right)(y-1), 0\right) =$$

$$= \alpha\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^j(y-1)^{j-1} + \sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1} + \right.$$

$$\left. + \sum_{i\ge 1}\sum_{j\ge 1}\binom{-1}{i}\binom{-(m-1)}{j}u^{i+j}(y-1)^{i+j-1}, 0\right) =$$

$$= \alpha\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^j(y-1)^{j-1} + \sum_{i\ge 1}\binom{-1}{i}u^i(y-1)^{i-1} + \right.$$

$$+ \sum_{t\ge 2}\left(\sum_{i+j=t}\binom{-1}{i}\binom{-(m-1)}{j}\right)u^{i+j}(y-1)^{i+j-1}, 0 \ =$$

$$= \ \alpha\left(\sum_{j\ge 1}\binom{-(m-1)}{j}u^{j}(y-1)^{j-1} + \sum_{i\ge 1}\binom{-1}{i}u^{i}(y-1)^{i-1} + \right.$$

$$\left. + \sum_{t\ge 2}\left(\sum_{j=1}^{t-1}\binom{-1}{t-j}\binom{-(m-1)}{j}\right)u^{t}(y-1)^{t-1}, 0\right) \ =$$

$$= \ \alpha\left(\sum_{i\ge 1}\binom{-(m-1)}{i}u^{i}(y-1)^{i-1} + \sum_{i\ge 1}\binom{-1}{i}u^{i}(y-1)^{i-1} + \right.$$

$$\left. + \sum_{i\ge 2}\left(\sum_{j=1}^{i-1}\binom{-1}{i-j}\binom{-(m-1)}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-(m-1)}{1}u + \binom{-1}{i}u + \sum_{i\ge 2}\binom{-(m-1)}{i}u^{i}(y-1)^{i-1} + \right.$$

$$\left. + \sum_{i\ge 2}\binom{-1}{i}u^{i}(y-1)^{i-1} + \sum_{i\ge 2}\left(\sum_{j=1}^{i-1}\binom{-1}{i-j}\binom{-(m-1)}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-(m-1)}{1}u + \binom{-1}{1}u + \sum_{i\ge 2}\left(\binom{-(m-1)}{i} + \binom{-1}{i} + \right.\right.$$

$$\left.\left. + \sum_{j=1}^{i-1}\binom{-1}{i-j}\binom{-(m-1)}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-m}{1}u + \sum_{i\ge 2}\left(\binom{-1}{0}\binom{-(m-1)}{i} + \binom{-1}{i}\binom{-(m-1)}{0} + \right.\right.$$

$$\left.\left. + \sum_{j=1}^{i-1}\binom{-1}{i-j}\binom{-(m-1)}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-m}{1}u + \sum_{i\ge 2}\left(\sum_{j=0}^{i}\binom{-1}{i-j}\binom{-(m-1)}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-m}{1}u + \sum_{i\ge 2}\left(\sum_{j=0}^{i}(-1)^{i-j}\binom{-m+1}{j}\right)u^{i}(y-1)^{i-1}, 0\right) \ =$$

$$= \ \alpha\left(\binom{-m}{1}u + \sum_{i\ge 2}\binom{-m}{i}u^{i}(y-1)^{i-1}, 0\right) \ = \ \alpha\left(\sum_{i\ge 1}\binom{-m}{i}u^{i}(y-1)^{i-1}, 0\right).$$

This completes the proof.

Similarly, we can prove (ii).

Next, for each pair (i, j), $1 \le i + j \le c - 2$, we define IA-automorphisms ξ_{ij} and η_{ij} as

$$\xi_{ij} \ = \ [x \to x[x, y]^{(x-1)^{i}(y-1)^{j}},\ y \to y]$$

and

$$\eta_{ij} = [x \to x,\ y \to y[x,y]^{(x-1)^i(y-1)^j}].$$

By Lemma 1, we have the following corollary.

Corollary 1. *For any positive integers i, j and any integer m, we have the following*:

(i) $\xi_{ij}^m = \alpha\left(\sum_{t\ge1}\binom{m}{t}(x-1)^{ti}(y-1)^{tj+t-1}, 0\right)$;

(ii) $\eta_{ij}^m = \alpha\left(0, \sum_{t\ge1}(-1)^{t-1}\binom{m}{t}(x-1)^{ti+t-1}(y-1)^{tj}\right)$,

where $\binom{m}{t} = \dfrac{\prod_{j=0}^{t-1}(m-j)}{t!}$ *is the binomial coefficient.*

Proof. (i) By Lemma 1, we have that

$$\xi_{ij}^m = \alpha\left(\sum_{t\ge1}\binom{m}{t}\left((x-1)^i(y-1)^j\right)^t(y-1)^{t-1}, 0\right) =$$

$$= \alpha\left(\sum_{t\ge1}\binom{m}{t}(x-1)^{ti}(y-1)^{tj+t-1}, 0\right).$$

Similarly, we can prove (ii).

The coefficients of a polynomial can be positive as well as negative. To deal with the situation where the coefficients of a polynomial are negative, we shall need the following result. Again, we omit the proof.

Lemma 2. (i) $\alpha(-(x-1)^i(y-1)^j, 0) = \prod_{m\ge0}\xi^{-1}_{2^m i, 2^m j+2^m-1}$.

(ii) $\alpha(0, -(x-1)^k) = \chi_k^{-1}\chi_{2k+1}$, *where* $\chi_k = [x \to x,\ y \to y[x,y]^{(x-1)^k}]$.

For any $u \in \Delta$, we can write $u = u_s + u_{s+1} + \ldots$, where $u_k = \sum_{i+j=k} a_{ij}(x-1)^i \times (y-1)^j$ for $k \ge s$, and $a_{ij} \ne 0$ for some $i+j=s$. If $u' = u'_t + u'_{t+1} + \ldots$, where $u'_k = \sum_{i+j=k} a'_{ij}(x-1)^i(y-1)^j$ for $k \ge t$, and $a'_{ij} \ne 0$ for some $i+j=t$. We say that $u << u'$ if $s < t$. We have the following lemma.

Lemma 3. (i) *Let* $u = u_s + \sum_{k>s} u_k$, *where* $u_k = \sum_{i+j=k} a_{ij}(x-1)^i(y-1)^j$. *Then there exists* u' *such that* $u << u'$ *and* $\alpha(u,0)\prod_{i+j=s}\xi_{ij}^{-a_{ij}} = \alpha(u',0)$.

(ii) *Let* $\upsilon = \upsilon_t + \sum_{k>t}\upsilon_k$, *where* $\upsilon_k = \sum_{i+j=k} b_{ij}(x-1)^i(y-1)^j$. *Then there exists* υ' *such that* $\upsilon << \upsilon'$ *and* $\alpha(0,\upsilon)\prod_{i+j=t}\eta_{ij}^{-b_{ij}} = \alpha(0,\upsilon')$.

Proof. (i) By Corollary 1 and formula (4), for any i, j, i', j' with $i+j=k$ and $i'+j'=k$, we have that

$$\xi_{ij}^{-a_{ij}}\xi_{i'j'}^{-a_{i'j'}} = \alpha\left(-a_{ij}(x-1)^i(y-1)^j + \sum_{t\ge2}\binom{-a_{ij}}{t}(x-1)^{ti}(y-1)^{tj+t-1}, 0\right) \times$$

$$\times\, \alpha\left(-a_{i'j'}(x-1)^{i'}(y-1)^{j'} + \sum_{t\ge2}\binom{-a_{i'j'}}{t}(x-1)^{ti'}(y-1)^{tj'+t-1}, 0\right) =$$

$$= \alpha\left(-a_{ij}(x-1)^i(y-1)^j - a_{i'j'}(x-1)^{i'}(y-1)^{j'} + w', 0\right),$$

where $(x-1)^i(y-1)^j << w'$. By repeated use of formula (4), we have

$$\prod_{i+j=s} \xi_{ij}^{-a_{ij}} = \alpha\left(-\sum_{i+j=k} a_{ij}(x-1)^i(y-1)^j + w'', 0 \right),$$

where $(x-1)^i(y-1)^j << w''$. Thus,

$$\alpha(u,0) \prod_{i+j=s} \xi_{ij}^{-a_{ij}} = \alpha(u',0),$$

where $u << u'$.

Similarly, we can prove (ii).

Lemma 4. $\mathrm{IA}(G)/\mathrm{Inn}(G) = \langle \xi_{ij}, \chi_k \mid i+j \geq 1,\ k \geq 1 \rangle$, *where* $\chi_k = [x \to x, y \to y[x,y]^{(x-1)^k}]$.

Proof. Using Lemma 3, we can prove that $\alpha(u,0) \in \mathrm{sgp}\{\xi_{ij}\}$ and $\alpha(0,v) \in \mathrm{sgp}\{\eta_{ij}\}$. We shall now show that, for $t > 0$, η_{st} can be written as a product of the ξ_{ij}'s and their inverses.

Let $w = [x,y]^{(x-1)^s(y-1)^{t-1}}$ and $I_w = [x \to x^w, y \to y^w]$ be the inner automorphism induced by w. We have that $[x,y]^{(x-1)^s(y-1)^t} = [[x,y]^{(x-1)^s(y-1)^{t-1}}, y] = [y, w^{-1}]$. Thus

$$\eta_{st} = [x \to x, y \to y[x,y]^{(x-1)^s(y-1)^t}] = [x \to x, y \to y[y, w^{-1}]] = [x \to x, y \to y^{w^{-1}}].$$

Hence,

$$\eta_{st} I_w = [x \to x, y \to y^{w^{-1}}][x \to x^w, y \to y^w] =$$
$$= [x \to x^w, y \to y] = [x \to x[x,y]^{-(x-1)^{s+1}(y-1)^{t-1}}, y \to y] =$$
$$= \alpha(-(x-1)^{s+1}(y-1)^{t-1}, 0),$$

which is a product of the ξ_{ij}'s and their inverses by Lemma 2. This completes the proof.

We shall now state and prove our main result.

Theorem 4. $\mathrm{IA}(G)/\mathrm{Inn}(G)$ *is a free abelian group of rank* $(c-2)(c+3)/2$ *and freely generated by* ξ_{ij} *and* χ_k, *where* $0 \leq i \leq c-2$, $0 \leq j \leq c-2$, $1 \leq i+j \leq c-2$, *and* $1 \leq k \leq c-2$.

Proof. By Lemma 4, $\mathrm{IA}(G)/\mathrm{Inn}(G)$ is generated by ξ_{ij} and χ_k, where $0 \leq i \leq c-2$, $0 \leq j \leq c-2$, $1 \leq i+j \leq c-2$, and $1 \leq k \leq c-2$. It is clear that

$$[\xi_{ij}, \xi_{kl}] = 1 \quad \text{for all } i, j, k, l$$

and

$$[\chi_k, \chi_l] = 1 \quad \text{for all } k, l.$$

Thus it suffices to prove that

$$[\xi_{ij}, \chi_k] = 1 \quad \text{for all } i, j \text{ and } k.$$

We shall now prove that the above relation is true. First, we have that

$$\chi_k^{-1}\xi_{ij}: \ x \to x \to x[x,y]^{(x-1)^i(y-1)^j},$$

$$y \to y[x,y]^{-\sum_{m\geq 1}(x-1)^{mk+m-1}} \to y\left[x[x,y]^{(x-1)^i(y-1)^j}, y\right]^{-\sum_{m\geq 1}(x-1)^{mk+m-1}} =$$

$$= y\Big([x,y]\Big[[x,y]^{(x-1)^i(y-1)^j}, y\Big]\Big)^{-\sum_{m\geq 1}(x-1)^{mk+m-1}} =$$

$$= y\Big([x,y][x,y]^{(x-1)^i(y-1)^{j+1}}\Big)^{-\sum_{m\geq 1}(x-1)^{mk+m-1}} =$$

$$= y[x,y]^{-\sum_{m\geq 1}(x-1)^{mk+m-1}}[x,y]^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}},$$

$$\chi_k^{-1}\xi_{ij}\chi_k: \; x \to x[x,y]^{(x-1)^i(y-1)^j} \to x\Big[x, y[x,y]^{(x-1)^k}\Big]^{(x-1)^i(y-1)^j} =$$

$$= x\Big([x,y][x,y]^{-(x-1)^{k+1}}\Big)^{(x-1)^i(y-1)^j} = x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j},$$

$$y \to \Big(y[x,y]^{-\sum_{m\geq 1}(x-1)^{mk+m-1}}\Big)[x,y]^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}} \to$$

$$\to y\Big[x, y[x,y]^{(x-1)^k}\Big]^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}} =$$

$$= y\Big([x,y][x,y]^{-(x-1)^{k+1}}\Big)^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}} =$$

$$= y[x,y]^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}+(x-1)^{k+1}(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}} =$$

$$= y[x,y]^{-(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+mk+m-1}+(y-1)^{j+1}\sum_{m\geq 1}(x-1)^{i+(m+1)k+(m+1)-1}} =$$

$$= y[x,y]^{-(y-1)^{j+1}\left(-\sum_{m\geq 1}(x-1)^{i+mk+m-1}+\sum_{m\geq 2}(x-1)^{i+mk+m-1}\right)} = y[x,y]^{-(x-1)^{i+k}(y-}$$

nd

$$\xi_{ij}^{-1}\chi_k^{-1}\xi_{ij}\chi_k: \; x \to x[x,y]^{\sum_{m\geq 1}(-1)^m(x-1)^{mi}(y-1)^{m(j+1)-1}} \to$$

$$\to x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j}\Big[x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j},$$

$$y[x,y]^{-(x-1)^{i+k}(y-1)^{j+1}}\Big]^{\sum_{m\geq 1}(-1)^m(x-1)^{mi}(y-1)^{m(j+1)-1}} =$$

$$= x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j}\Big([x,y][x,y]^{(x-1)^{i+k+1}(y-1)^{j+1}} \times$$

$$\times [x,y]^{(x-1)^i(y-1)^{j+1}-(x-1)^{i+k+1}(y-1)^{j+1}}\Big)^{\sum_{m\geq 1}(-1)^m(x-1)^{mi}(y-1)^{m(j+1)-1}} =$$

$$= x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j} \times$$

$$\times [x,y]^{\sum_{m\geq 1}(-1)^m(x-1)^{mi}(y-1)^{m(j+1)-1}+\sum_{m\geq 1}(-1)^m(x-1)^{(m+1)i}(y-1)^{(m+1)(j+1)-1}} =$$

$$= x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j} \times$$

$$\times [x,y]^{\sum_{m\geq 1}(-1)^m(x-1)^{mi}(y-1)^{m(j+1)-1}+\sum_{m\geq 2}(-1)^{m-1}(x-1)^{mi}(y-1)^{m(j+1)-1}} =$$

$$= x[x,y]^{(x-1)^i(y-1)^j-(x-1)^{i+k+1}(y-1)^j}[x,y]^{-(x-1)^i(y-1)^j} = x[x,y]^{-(x-1)^{i+k+1}(y-}$$

$$y \to y \to y[x,y]^{-(x-1)^{i+k}(y-1)^{j+1}}.$$

Thus, $\xi_{ij}^{-1}\chi_k^{-1}\xi_{ij}\chi_k = \left[x \to x[x,y]^{-(x-1)^{i+k+1}(y-1)^j},\ y \to y[x,y]^{-(x-1)^{i+k}(y-1)^{j+1}}\right]$.

We let $w = [x,y]^{(x-1)^{i+k}(y-1)^j}$ and $I_w = [x \to x^w, y \to y^w]$, then

$$\xi_{ij}^{-1}\chi_k^{-1}\xi_{ij}\chi_k = \left[x \to x[x,y]^{-(x-1)^{i+k+1}(y-1)^j},\ y \to y[x,y]^{-(x-1)^{i+k}(y-1)^{j+1}}\right] =$$

$$= \left[x \to x\left[[x,y]^{-(x-1)^{i+k}(y-1)^j}, x\right],\ y \to y\left[[x,y]^{-(x-1)^{i+k}(y-1)^j}, y\right]\right] =$$

$$= \left[x \to x[w^{-1},x],\ y \to y[w^{-1},y]\right] = I_w .$$

Therefore, $\xi_{ij}^{-1}\chi_k^{-1}\xi_{ij}\chi_k$ is an inner automorphism. This completes the proof of the theorem.

Remark. We observe that $\mathrm{Aut}(G)$, where $G = M_{2,c} = \langle x, y\rangle$, $c \geq 2$, is generated by τ, μ, ν, ξ_{ij}, χ_k, ψ_x, ψ_y, with certain defining relations τ^2, μ^2, $(\tau\mu)^4$, $[\nu,\mu,\nu]$, $(\tau\mu\tau\nu)^2$, $(\nu\tau\mu)^3$, $[\xi_{ij},\xi_{kl}]$, $[\chi_k,\chi_l]$, $[\xi_{ij},\chi_k][\psi_y,\psi_x]^{(\psi_x-1)^{i+k}(\psi_y-1)^j}$, $\forall i, j, k, l, \Psi'', \gamma_c(\Psi)$ (Ψ is the subgroup generated by ψ_x and ψ_y), $\xi_{0j}^{\tau}\chi_j\chi_{2j+1}^{-1}$, $\xi_{ij}^{\tau}\xi_{j+1,i-1}^{-1}[\psi_y,\psi_x]^{(\psi_x-1)^j(\psi_y-1)^{i-1}}$ for $i \neq 0$, $\chi_k^{\tau}\prod_{t\geq 1}\xi_{0,2^t k+2^t-1}$ and some other relations involving ξ_{ij}^{μ}, ξ_{ij}^{ν}'s and χ_k^{μ}, χ_k^{ν}'s. But, we leave this as an open problem.

Problem. Find a minimal set of generators and relators of $\mathrm{Aut}(G)$.

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