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H.-P. Blatt (Kathol. Univ. Eichstätt-Ingolstadt, Germany)

THE INFLUENCE OF POLES ON EQUIOSCILLATION IN RATIONAL APPROXIMATION

ВПЛИВ ПОЛЮСІВ НА ЕКВІОСЦИЛЯЦІЇ У РАЦІОНАЛЬНОМУ НАБЛИЖЕННІ

The error curve for rational best approximation of $f \in C[-1,1]$ is characterized by the well-known equioscillation property. Contrary to the polynomial case, the distribution of these alternations is not governed by the equilibrium distribution. It is known that these points need not to be dense in $[-1,1]$. The reason is the influence of the distribution of the poles of the rational approximants. In this paper, we generalize the results known so far to situations where the requirements for the degrees of numerators and denominators are less restrictive.

Крива похибок для раціонального найкращого наближення $f \in C[-1,1]$ характеризується відомою властивістю еквіосциляцій. На відміну від поліноміального випадку розподіл цих змін знаку не визначається рівноважним розподілом. Відомо, що ці точки не обов'язково мають бути щільними в $[-1,1]$, що зумовлено впливом розподілу полюсів раціональних наближень. У даній роботі узагальнено відомі результати на випадки, де на степені чисельників та знаменників накладаються менш жорсткі умови.

1. Introduction. Let $f \in C[-1,1]$ be a real-valued function and let $\mathcal{R}_{n,m}$ denote the family of real rational functions with numerator in $\mathcal{P}_n$ and denominator in $\mathcal{P}_m$, where $\mathcal{P}_k$ is the set of algebraic polynomials of degree at most $k, k \in \mathbb{N}_0$. For each pair of nonnegative integers (n,m), there exists a unique function $r^*_{n,m} \in \mathcal{R}_{n,m}$ that is the best uniform approximation to f on $I=[-1,1]$ in the sense that

$$\|f - r^*_{n,m}\| < \|f - r\| \quad \text{for all} \quad r \in \mathcal{R}_{n,m}, \quad r \neq r^*_{n,m},$$

where $\|\cdot\|$ denotes the $\sup\mathrm{norm}$ on I. Writing $r = p_n/q_m$, where $p_n \in \mathcal{P}_n$ and $q_m \in \mathcal{P}_m$ have no common factor and q_m is monic, the defect of r is defined by

$$d_{n,m}(r) := \min(n - \deg p_n, m - \deg q_m). \tag{1}$$

Let us define

$$l(r) = n + m + 1 - d_{n,m}(r); \tag{2}$$

then $l(r)$ is the dimension of the tangential space with respect to the coefficients of the numerator and denominator as parameter space. We write $r^*_{n,m} := p^*_n/q^*_m$ with no common factors and define for abbreviation

$$l_{n,m} := l(r^*_{n,m}).$$

Then it is well known that the best approximation of f is characterized by the following equioscillation property:

There exist $l_{n,m}+1$ points $x_k^{(n,m)}$,

$$-1 \le x_0^{(n,m)} < \ldots < x_{l_{n,m}}^{(n,m)} \le 1,$$

such that

$$\lambda_{n,m}(-1)^k(f - r^*_{n,m})(x_k^{(n,m)}) = \|f - r^*_{n,m}\|,\ 0 \le k \le l_{n,m}, \tag{3}$$

where $\lambda_{n,m} = +1$ or $\lambda_{n,m} = -1$ is fixed. Such a point set $\{x_k^{(n,m)}\}$ is called *alternation set.* In general, it is not unique. Therefore, in the following, we denote by

$$A_{n,m} = A_{n,m}(f) = \{x_k^{(n,m)}\}_{k=0}^{l_{n,m}}$$

an arbitrary but fixed alternation set for the best approximation $r^*_{n,m}$ of f out of $\mathcal{R}_{n,m}$.

Let $\nu_{n,m}$ denote the normalized counting measure of $A_{n,m}$, i.e.,

$$\nu_{n,m}([\alpha,\beta]) := \frac{\#\{x_k^{(n,m)} : \alpha \le x_k^{(n,m)} \le \beta\}}{l_{n,m}+1}. \tag{4}$$

Kadec [1] has shown that there exists a subsequence Λ of $\mathbb{N}$ such that

$$\nu_{n,0} \xrightarrow{*} \mu \quad \text{as} \quad n \in \Lambda, \quad n \longrightarrow \infty, \tag{5}$$

where μ is the equilibrium measure of $[-1,1]$, i.e., the density of μ on I is

$$d\mu(x) = \frac{dx}{\sqrt{1-x^2}}.$$

For rational approximation, Borwein et al. [2] have proved that denseness on $[-1,1]$ holds for a subsequence of alternation sets $A_{n,m}$ whenever $m = m(n)$ and $\dfrac{n}{m(n)} \longrightarrow \kappa > 1$ as $n \longrightarrow \infty$. Moreover, they have shown in the case $\lim\limits_{n\to\infty} \dfrac{m(n)}{n} = 0$ that there exists $\Lambda \subset \mathbb{N}$ such that

$$\nu_{n,m(n)} \xrightarrow{*} \quad \text{as} \quad n \in \Lambda, \quad n \longrightarrow \infty.$$

More quantitative results were obtained by Kroó and Peherstorfer in [3]. Namely, let us denote by $N_{n,m}(\alpha,\beta)$ the number of points of $A_{n,m}$ in $[\alpha,\beta]$. Then the main result can be stated as follows: let $m(n) < n$, then

$$\frac{N_{n,m(n)}(\alpha,\beta)}{n-m(n)} \ge \mu([\alpha,\beta]) - c\sqrt{\frac{\log n}{n-m(n)}}, \tag{6}$$

where c is an absolute constant independent of f and n.

Braess et al. [4] considered the case $m(n) = n + \kappa$, $\kappa \in \mathbb{Z}$, fixed. Their results were based on the number $\gamma_n(\varepsilon)$ of poles of best approximants lying outside an ε-neighbourhood of $[-1,1]$. Roughly speaking, if $\gamma_n(\varepsilon)$ is sufficiently big, then there is a connection of the distribution of $A_{n,m}$ with the equilibrium distribution μ.

The intimate relation between $A_{n,m}(f)$ and the poles of $r^*_{n,m}$ was investigated in [5]. To be precise, let f be not a rational function and let n and $m(n)$ satisfy

$$m(n) \le n; \qquad m(n) \le m(n+1) \le m(n)+1. \tag{7}$$

Moreover, let

$$Q_n(x) = q^*_{m(n)}(x)q^*_{m(n+1)}(x) = \prod_{i=1}^{\kappa_n}(x-y_i) \tag{8}$$

be the product of the denominators of $r^*_{n,m(n)}$ and $r^*_{n+1,m(n+1)}$, then

$$\tau_n(\Delta) := \frac{\#\{y_i : y_i \in A\}}{\kappa_n} \quad (A \subset \mathbb{C})$$

denotes the normalized counting measure of all finite poles of $r^*_{n,m(n)}$ and $r^*_{n+1,m(n+1)}$ counted with their multiplicities. Then it was proved in [5] that there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_{n,m(n)} - \alpha_n \widehat{\tau}_n - (1-\alpha_n)\mu \xrightarrow{\ *\ } 0 \quad \text{as} \quad n \to \infty, \quad n \in \Lambda, \tag{9}$$

in the weak*-topology, where

$$\alpha_n = \frac{\kappa_n}{l_{n,m(n)} + 1}$$

and $\widehat{\tau}_n$ denotes the balayage measure of τ_n onto $[-1,1]$. The purpose of the present paper is to obtain a convergence result of type (9), where the restriction $m(n) \le n$ is weakened to $m(n) \le n+1$. We point out that this weaker condition implies that the original proof in [5] has to be substantially modified. Moreover, the weaker condition $m(n) \le n+1$ allows to apply and to understand examples of [2].

Borwein et al. [2] have proved in the case $m(n) = n+1$ that there exists a function with no alternation points in a certain interval.

It is a challenge to generalize results of type (8) to $m(n) > n+1$.

2. Main results. We assume that $m(n)$ depends on the parameter $n \in \mathbb{N}$. Let

$$E_{n,m(n)} := \inf_{r \in \mathcal{R}_{n,m(n)}} \|f - r\| = \|f - r^*_{n,m(n)}\|$$

and define for abbreviation

$$\begin{gathered} r^*_n := r^*_{n,m(n)}, \quad p^*_n = p^*_{n,m(n)}, \quad q^*_n = q^*_{n,m(n)}, \\ E_n = E_{n,m(n)}, \quad l_n = l_{n,m(n)}, \quad d_n = d_{n,m(n)}, \\ x_k^{(n)} := x_k^{(n,m(n))}, \quad k = 0,1,\ldots,l_n. \end{gathered}$$

Again, we use the normalized counting measure ν_n of the alternation set $\{x_k^{(n)}\}_{k=0}^{l_n}$ and the normalized counting measure τ_n of the union of the (finite) poles of r^*_n and r^*_{n+1}. All poles are counted with their multiplicities. For any finite Borel measure ν, the logarithmic potential of ν is defined by

$$U^\nu(z) := \int \log\frac{1}{|z-t|}\, d\nu(t).$$

A crucial role is played by the balayage measure $\widehat{\tau}_n$ of τ_n onto $[-1,1]$. $\widehat{\tau}_n$ is the unique measure supported on $[-1,1]$, for which $\|\widehat{\tau}_n\| = \|\tau_n\|$ and

$$U^{\widehat{\tau}_n}(z) = U^{\tau_n}(z) + c, \quad z \in [-1,1],$$

where

$$c = \int G(t,\infty)\, d\tau_n(t)$$

and $G(z,a)$ denotes Green's function of $\Omega = \overline{\mathbb{C}} \setminus [-1,1]$ with pole at $a \in \Omega$ (cf. [6]). Furthermore, $\widehat{\tau}_n$ has the following properties:

a) $U^{\tau_n}(z) \le U^{\tau_n}(z) + c$, $z \in \mathbb{C}$;

b) if h is continuous on $\overline{\mathbb{C}}$ and harmonic in Ω, then $\displaystyle\int h d\tau_n = \int h d\widehat{\tau}_n$.

Our main result can be formulated in the following theorem.

Theorem. *Let f be not a rational function and let the parameters $m(n)$, $n \in \mathbb{N}$, satisfy*

$$m(n) \le n+1, \quad m(n) \le m(n+1) \le m(n)+1. \tag{10}$$

Then there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\nu_n - \alpha_n \widehat{\tau}_n - (1-\alpha_n)\mu \xrightarrow{\ *\ } 0 \quad \textit{as} \quad n \to \infty, \quad n \in \Lambda,$$

where

$$\alpha_n = \frac{\deg q_n^* + \deg q_{n+1}^*}{l_n + 1}.$$

We note that condition (10) is less restrictive than (7).

It is possible to formulate the above result in a more concise manner such that only the alternation counting measure ν_n and pole counting measures of r_n^* and r_{n+1} are involved. Let

$$R_n = r_{n+1}^* - r_n^* = \frac{p}{q},$$

where p and q have no common divisor. Then the degree of $\dfrac{p}{q}$ is defined by

$$\deg \frac{p}{q} := \max(\deg p, \deg q).$$

Then the number of zeros, resp. poles, of R_n in the closed complex plane $\overline{\mathbb{C}}$ is $\deg R_n$, where all zeros and poles are counted with their multiplicity.

We define the normalized pole counting measure $\sigma_{\mathrm{pole},n}$ of R_n in $\overline{\mathbb{C}}$ by

$$\sigma_{\mathrm{pole},n}(A) = \frac{\#\{\text{poles of } R_n \text{ in } A\}}{\deg R_n} \quad (A \subset \overline{\mathbb{C}})$$

and the normalized zero counting measure $\sigma_{\mathrm{zero},n}$ of R_n in $\mathbb{C}$ by

$$\sigma_{\mathrm{zero},n}(A) = \frac{\#\{\text{zeros of } R_n \text{ in } A\}}{\deg R_n} \quad (A \subset \overline{\mathbb{C}}).$$

Corollary. *Under the conditions of Theorem 1, there exists $\Lambda \subset \mathbb{N}$ such that*

$$\widehat{\sigma}_{\mathrm{zero},n} - \widehat{\sigma}_{\mathrm{pole},n} \xrightarrow{\ *\ } 0 \quad \textit{as} \quad n \to \infty, \quad n \in \Lambda.$$

Especially,

$$\nu_n - \widehat{\sigma}_{\mathrm{pole},n} \xrightarrow{\ *\ } 0 \quad \text{as} \quad n \to \infty, \quad n \in \Lambda,$$

if $\displaystyle\lim_{n\to\infty} \frac{m(n)}{n} \le 1$.

Let us discuss the second part of the corollary in Kadec's case, i.e., $(n, m(n)) = (n, 0)$. Then $R_n = p_{n+1}^* - p_n^*$ and p_n, p_{n+1}^* are the best approximating polynomials to f with respect to $\mathcal{P}_n$, resp. $\mathcal{P}_{n+1}$ and R_n has a pole of multiplicity $n+1$ at ∞ if $p_n^* \ne p_{n+1}^*$.

Now, for the Dirac measure δ_∞ at the point at ∞ we know that the balayage mesure $\widehat{\delta}_\infty$ is just the equilibrium measure μ (cf. [6]). Moreover, all zeros of $p^*_{n+1} - p^*_n$ are separating the alternation points. Hence, $\widehat{\sigma}_{\mathrm{zero},n} = \sigma_{\mathrm{zero},n}$ and

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} \sigma_{\mathrm{zero},n} = \lim_{\substack{n\to\infty\\ n\in\Lambda}} \nu_{n,0} = \mu$$

from the corollary. That is Kadec's result (5).

3. Proofs. Since $\lim\limits_{n\to\infty} E_n = 0$, by a well-known argument, there exists a subsequence $\Lambda \subset \mathbb{N}$ such that

$$\frac{E_n + E_{n+1}}{E_n - E_{n+1}} \le n^2 \quad \text{for} \quad n \in \Lambda \tag{11}$$

(cf. [7, p. 243], Lemma 7.3.3). In particular, for $n \in \Lambda$ we have $r^*_n \ne r^*_{n+1}$ and, by (3),

$$(-1)^k (r^*_{n+1} - r^*_n)(x^{(n)}_k) \ge E_n - E_{n+1} \tag{12}$$

for $0 \le k \le l_n$, where we have assumed without loss of generality that the number $\lambda_{n,m(n)} = 1$ in (3). Writing

$$R_n = r^*_{n+1} - r^*_n = \frac{p^*_{n+1}q^*_n - p^*_n q^*_{n+1}}{q^*_n q^*_{n+1}} = \frac{P_n}{q^*_n q^*_{n+1}} = \frac{P_n}{Q_n},$$

we obtain

$$(-1)^k R_n(x^{(n)}_k) \ge E_n - E_{n+1}, \quad 0 \le k \le l_n. \tag{13}$$

In the following, we assume that a_n is the highest coefficient of $P_n(x)$, i.e.,

$$P_n(x) = a_n x^{l_n} + \dots\ .$$

By (13), P_n or $R_n = \dfrac{P_n}{Q_n}$ has at least l_n zeros in $(-1,1)$. Since $r^*_n \ne r^*_{n+1}$, condition (10) implies that all zeros of P_n are in $(-1,1)$. As in [5], our next intention is to reconstruct the polynomial Q_n by interpolation at the points $x^{(n)}_k$, $0 \le k \le l_n$. Since

$$\kappa_n = \deg Q_n = \deg q^*_n + \deg q^*_{n+1} \le m(n) - d_n + n + 2 = l_n + 1,$$

the degree of Q_n is, in general, too big to be reconstructed by interpolation at $x^{(n)}_k$, $0 \le$ $\le k \le l_n$.

In the case $\kappa_n \le l_n$, we can use the method of proof in [5]. Therefore, we can restrict ourselves in the following to the case $\kappa_n = l_n + 1$.

First, we have to modify the polynomial $P_n(x)$: Let ξ_n be such that

$$\xi_n \ge n \max(1, |y_1|, |y_2|, \dots, |y_{\kappa_n}|), \tag{14}$$

where $y_1, \dots, y_{\kappa_n}$ are all zeros of $Q_n(x)$ in $\mathbb{C}$. Then we define

$$\widetilde{P}_n(x) := (x - \xi_n)P_n(x) \quad \text{and} \quad \widetilde{R}_n := \frac{\widetilde{P}_n}{Q_n}. \tag{15}$$

Then

$$\deg \widetilde{P}_n = \deg Q_n = l_n + 1$$

and we can reconstruct Q_n by interpolation at the points $x_k^{(n)}$, $0 \le k \le l_n$, and at the point ξ_n. We obtain

$$Q_n(z) = \sum_{k=0}^{l_n} \frac{Q_n(x_k^{(n)})w(z)}{(z - x_k^{(n)})w'(x_k^{(n)})} + \frac{Q_n(\xi_n)w(z)}{(z-\xi_n)w'(\xi_n)}, \tag{16}$$

where

$$w(z) = (z-\xi_n)\prod_{k=0}^{l_n}(z - x_k^{(n)}). \tag{17}$$

For $z \neq \xi_n$, $x_k^{(n)}$, $0 \le k \le l_n$, relation (16) can be written as

$$\frac{Q_n(z)}{w(z)} = \sum_{k=0}^{l_n} \frac{Q_n(x_k^{(n)})}{(z - x_k^{(n)})w'(x_k^{(n)})} + \frac{Q_n(\xi_n)}{(z-\xi_n)w'(\xi_n)}. \tag{18}$$

By definition, we have

$$\|R_n\| \le \|f - r_n^*\| + \|f - r_{n+1}^*\| \le E_n + E_{n+1}$$

and, therefore,

$$\left\|\widetilde{R}_n\right\| \le (\xi_n + 1)(E_n + E_{n+1}). \tag{19}$$

Moreover, by (13) we get

$$(-1)^{k+1}\widetilde{R}_n(x_k^{(n)}) \ge (\xi_n - 1)(E_n - E_{n+1}). \tag{20}$$

Next, we consider the function

$$h(z) := \log\left|\widetilde{R}_n(z)\right| - \sum_{i=1}^{\kappa_n} G(z, y_i) + G(z, \xi_n).$$

The function $h(z)$ is subharmonic in $\overline{\mathbb{C}}$; hence, the maximum principle applies and

$$h(\infty) \le \max_{z\in I} h(z) = \max_{z\in I}\log\left|\widetilde{R}_n(z)\right| = \log\|\widetilde{R}_n\|,$$

and we obtain

$$h(\infty) = \log|a_n| - \sum_{i=1}^{\kappa_n} G(\infty, y_i) + G(\infty, \xi_n).$$

Therefore, with (19)

$$\log|a_n| \le \log\Big((\xi_n+1)(E_n+E_{n+1})\Big) + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n). \tag{21}$$

Next, let us consider the approximation of the function $\widetilde{P}_n(x)$ at the points

$$x_k^{(n)}, \quad 0 \le k \le l_n,$$

with interpolation at the zero ξ_n with respect to $\mathcal{P}_{l_n}$ and the weight function $\dfrac{1}{Q_n(x)}$. It turns out that de la Vallée Poussin's theorem implies together with (20) that the minimal error ρ satisfies

$$\rho \ge (\xi_n - 1)(E_n - E_{n+1}). \tag{22}$$

On the other hand, for any $P \in \mathcal{P}_{l_n}$ with $P(\xi_n) = 0$, we have

$$\rho = \frac{\left|\sum_{k=0}^{l_n} \beta_k \big(\widetilde{P}_n - P\big)(x_k^{(n)})\right|}{\sum_{k=0}^{l_n} \left|\beta_k Q_n(x_k^{(n)})\right|}, \tag{23}$$

where

$$\frac{1}{\beta_k} = w'\big(x_k^{(n)}\big) = \Big(x_k^{(n)} - \xi_n\Big) \prod_{i \neq k} \Big(x_k^{(n)} - x_i^{(n)}\Big). \tag{24}$$

Now, fix the polynomial $P \in \mathcal{P}_{l_n}$ by $P(\xi_n) = 0$ and $P\big(x_k^{(n)}\big) = \widetilde{P}_n\big(x_k^{(n)}\big)$, $1 \le k \le l_n$.
Hence,

$$(\widetilde{P}_n - P)(x) = a_n (x - \xi_n) \prod_{k=1}^{l_n} (x - x_k^{(n)})$$

and, therefore,

$$(\widetilde{P}_n - P)(x_0^{(n)}) = a_n (x_0^{(n)} - \xi_n) \prod_{k=1}^{l_n} (x_0^{(n)} - x_k^{(n)}).$$

By (22) – (24) we obtain

$$\rho = \frac{|a_n|}{\sum_{k=0}^{l_n} \left|\beta_k Q_n(x_k^{(n)})\right|} \ge (\xi_n - 1)(E_n - E_{n+1}).$$

Using representation (18), for $z \notin I$ we get

$$\left|\frac{Q_n(z)}{w(z)}\right| \le D(z) \sum_{k=0}^{l_n} |\beta_k Q_n(x_k^{(n)})| + \frac{1}{|z - \xi_n|} \left|\frac{Q_n(\xi_n)}{w'(\xi_n)}\right| \le$$

$$\le D(z) \frac{|a_n|}{(\xi_n - 1)(E_n - E_{n+1})} + \frac{1}{|z - \xi_n|} \left|\frac{Q_n(\xi_n)}{w'(\xi_n)}\right|, \tag{25}$$

where

$$D(z) = \max_{0 \le k \le l_n} \left|z - x_k^{(n)}\right|^{-1}.$$

Since $\kappa_n = l_n + 1 \le 2n + 3$, for $n \ge 2$ we obtain

$$\left|\frac{Q_n(\xi_n)}{w'(\xi_n)}\right| = \left|\frac{\prod_{i=1}^{\kappa_n} (\xi_n - y_i)}{\prod_{k=0}^{l_n} (\xi_n - x_k^{(n)})}\right| \le \left[\frac{\xi_n (1 + 1/n)}{\xi_n - 1}\right]^{\kappa_n} \le$$

$$\le \left(\frac{1 + 1/n}{1 - 1/\xi_n}\right)^{\kappa_n} \le \left(\frac{1 + 1/n}{1 - 1/n}\right)^{\kappa_n} \le c_1, \tag{26}$$

where c_1 is independent of n.

In the following, we consider the level line

$$\Gamma_{1/n} := \left\{ z \in \mathbb{C} : G(z, \infty) = \log\left(1 + \frac{1}{n}\right) \right\}$$

of Green's function $G(z,\infty)$. Then for $z \in \Gamma_{1/n}$, $n \geq 2$, we have

$$\log \frac{1}{|z-\xi_n|} \leq \log \frac{1}{|\xi_n|} + c_2,$$

where c_2 is independent of n. Since

$$\lim_{z\to\infty} \left(G(z,\infty) + \log \frac{1}{2} - \log|z| \right) = 0$$

and $\lim\limits_{n\to\infty} \xi_n = \infty$, there exists $c_3 > 0$ such that

$$\log \frac{1}{|z-\xi_n|} \leq -G(\xi_n,\infty) + c_3$$

for all $n \geq 2$. Then from (26) we obtain for $z \in \Gamma_{1/n}$ and $n \geq 2$ that

$$\log \left(\frac{1}{|z-\xi_n|} \left| \frac{Q_n(\xi_n)}{w'(\xi_n)} \right| \right) \leq c_4 - G(\xi_n,\infty), \tag{27}$$

where $c_4 > 0$ is independent of n.

Define for abbreviation

$$A_n := \frac{|a_n|}{(\xi_n - 1)(E_n - E_{n+1})}. \tag{28}$$

Then inequality (21) together with (11) implies

$$\log A_n \leq \log \frac{\xi_n + 1}{\xi_n - 1} + \log \frac{E_n + E_{n+1}}{E_n - E_{n+1}} + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n) \leq$$

$$\leq \log \frac{n+1}{n-1} + \log(n^2) + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n)$$

and for $z \in \Gamma_{1/n}$

$$\log(D(z)A_n) \leq c_5 \log n + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n) \tag{29}$$

for $n \in \Lambda$, $n \geq 2$, with some constant c_5 independent of n.

Comparing the right-hand sides of (27) and (29), we conclude from (25) that, for $n \in \Lambda$, $n \geq 2$, and $z \in \Gamma_{1/n}$,

$$\log \left| \frac{Q_n(z)}{w(z)} \right| \leq c_6 \log n + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n) \tag{30}$$

with an absolute constant c_6 independent of n.

The last inequality can be written with the logarithmic potentials $U^{\nu_n}(z)$, $U^{\tau_n}(z)$ and the Dirac measure δ_{ξ_n} at the point ξ_n as

$$U^{\nu_n}(z) - \alpha_n U^{\tau_n}(z) + \frac{1}{l_n+1} U^{\delta_{\xi_n}}(z) \leq$$

$$\leq \frac{1}{l_n+1} \left(c_6 \log n + \sum_{i=1}^{\kappa_n} G(\infty, y_i) - G(\infty, \xi_n) \right).$$

Next, we use the balayage measure $\widehat{\delta}_{\xi_n}$ of δ_{ξ_n} onto the interval $[-1,1]$. Since

$$U^{\widehat{\delta}_{\xi_n}}(z) \le U^{\delta_{\xi_n}}(z) + G(\infty,\xi_n), \quad z \in \mathbb{C},$$

we obtain for $z \ne \xi_n$

$$U^{\nu_n}(z) - U^{\tau_n}(z) + \frac{1}{l_n+1} U^{\widehat{\delta}_{\xi_n}}(z) \le \frac{1}{l_n+1}\left(c_6 \log n + \sum_{i=1}^{\kappa_n} G(\infty,y_i)\right). \quad (31)$$

Taking into account that we can choose the point ξ_n arbitrarily large on the positive real axis and

$$\lim_{\xi_n\to\infty} \widehat{\delta}_{\xi_n} = \mu$$

in the weak*-sense (cf. [6], Chapter II, formula 4.46), we can choose ξ_n such that

$$|U^{\widehat{\delta}_{\xi_n}} - U^{\mu}(z)| < \frac{1}{n}, \quad z \in \Gamma_{1/n}.$$

Then we obtain for $z \in \Gamma_{1/n}$ that

$$U^{\nu_n}(z) - U^{\tau_n}(z) \le c\,\frac{\log n}{n} + \frac{1}{\kappa_n}\sum_{i=1}^{\kappa_n} G(\infty,y_i).$$

The last inequality is of the same structure as inequality (30) in [5]. Hence, the remaining proof follows the same lines as in this paper and is therefore omitted.

The proof of the corollary is left to the reader.

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