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ON SOME ASYMPTOTIC BEHAVIOUR FOR SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

ПРО АСИМПТОТИЧНУ ПОВЕДІНКУ РОЗВ'ЯЗКІВ ЛІНІЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ

We present some sufficient conditions for linear asymptotic equilibrium of linear differential equations in Hilbert and Banach spaces. The obtained results are applied to studying the asymptotic equivalence of two linear differential equations.

Наведено деякі достатні умови лінійної асимптотичної рівноваги лінійних диференціальних рівнянь у гільбертовому та банаховому просторах. Одержані результати застосовано для дослідження асимптотичної еквівалентності двох лінійних диференціальних рівнянь.

1. Introduction. Let us consider differential equation

$$\dot{x} = A(t)x \tag{1}$$

where $A(t)$ is a bounded linear operator on a Banach space B for each $t \geq t_0 \geq 0$ and is strongly continuous in t.

Definition 1. *The equation* (1) *is said to have a linear asymptotic equilibrium if its arbitrary solution* $x(t)$ *has a finite limits as* $t \to +\infty$ *and for each* $u_0 \in B$ *there exists an unique solution of* (1) *such that*

$$\lim_{t \to +\infty} x(t) = u_0.$$

Let us denote L_1 a set of functions that are absolutely integrable on R^n. In the case of finite dimension space R^n it is well known the following sufficient condition [1] for the linear asymptotic equilibrium of (1):

$$\| A(t) \| \in L_1. \tag{2}$$

A. Wintner [2] generalized this condition by

$$\| \tilde{A}(t) A(t) \| \in L_1 \quad \text{or} \quad \| A(t) \tilde{A}(t) \| \in L_1, \tag{3}$$

where

$$\tilde{A}(t) = \int_t^{+\infty} A(\tau)\, d\tau.$$

Authors of [3] and [4] presented another generalizations and extension of (2).

For the case of nonlinear equations the similar problems were investigated in [5 – 8].

In this paper we will give some generalization of the conditions (2) and (3) we applied the obtained results to problem of asymptotic equivalence.

2. The case of Hilbert spaces. Let us now consider the equation (1) in a Hilbert space H. We denote by $\langle \cdot, \cdot \rangle$ a scalar product in H and by $S(0, 1)$ the unit ball in it.

Definition 2. *We said that* $\| A(t)h \|$ *uniformly belongs to* L_1 *if for all* $h \in S(0, 1)$ *there exists a number* $T > 0$ *such that*

$$\int_T^{+\infty} \| A(t)h \| d\tau \leq q < 1.$$

Theorem 1. *Let the condition* $A(t) = A^*(t)$ *be satisfied and* $\|A(t)h\|$ *uniformly belong to* L_1. *Then the equation* (1) *in* H *has a linear asymptotic equilibrium.*

Proof. For arbitrary $u_0 \in H$ let us consider the functional

$$\eta_1(t, h) = \langle h_0, h\rangle - \int_t^{+\infty} \langle A(\tau)h_0, h\rangle d\tau, \quad t \geq T, \quad h \in H.$$

Since

$$\sup_{\|h\|\leq 1} |\eta_1(t, h)| \leq \|h_0\| + q\|h_0\|$$

$\eta_1(t, h)$ is a linear continuous functional in H. According to Riesz theorem, there exists in H elements $x_1(t)$ such that

$$\eta_1(t, h) = \langle x_1(t), h\rangle.$$

Clearly

$$\|x_1(t)\| \leq (1+q)\|h_0\|.$$

By setting $x_0(t) \equiv h_0$, we have

$$\left\|\frac{\Delta x_1(t)}{\Delta t} - A(t)x_0(t)\right\| = \sup_{\|h\|\leq 1}\left\|\left\langle\frac{\Delta x_1(t)}{\Delta t} - A(t)x_0(t), h\right\rangle\right\| \leq$$

$$\leq \left|\frac{1}{\Delta t}\int_t^{t+\Delta t} \|A(\tau)x_0(\tau) - A(t)x_0(t)\| d\tau\right| \to 0, \text{ as } \Delta t \to 0.$$

This shows that $x_1(t)$ is differentiable and

$$\dot{x}_1(t) = A(t)x_0(t).$$

Let us now consider the functional

$$\eta_2(t, h) = \langle h_0, h\rangle - \int_t^{+\infty} \langle A(\tau)x_1(\tau), h\rangle d\tau, \quad t \geq T, \quad h \in H.$$

It is easy to verify that $|\eta_1(t, h)| \leq (1 + q + q^2)\|h_0\|$.

Therefore $\eta_2(t, h)$ is a linear continuous functional in H. Hence there exists $x_2(t) \in H$ such that

$$\eta_2(t, h) = \langle x_2(t), h\rangle$$

and

$$\|x_2(t)\| \leq (1 + q + q^2)\|h_0\|.$$

Moreover we have

$$\left\|\frac{\Delta x_2(t)}{\Delta t} - A(t)x_1(t)\right\| = \sup_{\|h\|\leq 1}\left|\frac{1}{\Delta t}\int_t^{+\Delta\infty} \langle A(\tau)x_1(\tau) - A(t)x_1(t), h\rangle d\tau\right| \leq$$

$$\leq \left|\frac{1}{\Delta t}\int_t^{t+\Delta t} \|A(\tau)x_1(\tau) - A(t)x_1(t)\| d\tau\right| \to 0 \tag{4}$$

when $\Delta t \to 0$.

This means that $x_2(t)$ is differentiable and

$$\dot{x}_2(t) = A(t)x_1(t).$$

Continuing this process we construct a linear continuous functional

$$\eta_n(t) = \langle h_0, h\rangle - \int_t^{+\infty} \langle A(\tau)x_{n-1}(\tau), h\rangle d\tau$$

which has the following properties:

$$\eta_n(t, h) = \langle x_n(t), h\rangle,$$

$$\| x_n(t)\| \leq (1 + q + \ldots + q^n)\| h_0 \| \leq \frac{1}{1-q}\| h_0 \|, \tag{5}$$

$$\dot{x}_n(t) = A(t)x_{n-1}(t). \tag{6}$$

Thus, we obtain a sequence of differentiable function $\{x_n(t)\}$. This sequence uniformly converges on $[T, +\infty]$. To prove this statement it suffice to show that

$$\| x_n(t) - x_{n-1}(t)\| \leq \| h_0 \| q^n. \tag{7}$$

In fact, for $n = 1$ we have

$$\| x_1(t) - x_0(t)\| = \sup_{\| h\| \leq 1} |\langle x_1(t) - x_0(t), h\rangle| \leq$$

$$\leq \sup_{\| h\| \leq 1} \int_T^{+\infty} \| A(\tau)h\| \| x_0(\tau)\| d\tau \leq \| h_0 \| q$$

i. e. the formula (7) is valid for $n = 1$. Let us now assume that (7) is valid for $n \geq 1$. Then

$$\| x_{n+1}(t) - x_n(t)\| = \sup_{\| h\| \leq 1} |\langle x_{n+1}(t) - x_n(t), h\rangle| =$$

$$= \sup_{\| h\| \leq 1} \left| \int_T^{+\infty} \langle x_n(\tau) - x_{n-1}(\tau), A(\tau)h\rangle d\tau \right| \leq$$

$$\leq \sup_{\| h\| \leq 1} \int_T^{+\infty} \| x_n(\tau) - x_{n-1}(\tau)\| \| A(\tau)h\| \leq \| h_0 \| q^{n+1}$$

i. e. the formula (7) is valid for $n + 1$. Let us set now

$$x(t) = \lim_{n\to+\infty} x_n(t)$$

and show that $x(t)$ is differentiable on $[T, +\infty)$. In fact

$$\| \dot{x}_n(t) - \dot{x}_{n-1}(t)\| = \sup_{\| h\| \leq 1} |\langle A(t)x_{n-1}(t) - A(t)x_{n-2}(t), h\rangle| \leq$$

$$\leq \sup_{\| h\| \leq 1} \| x_{n-1}(t) - x_{n-2}(t)\| \| A(t)h\| =$$

$$= \| x_{n-1}(t) - x_{n-2}(t)\| \sup_{\| h\| \leq 1} \| A(t)h\|. \tag{8}$$

Let $t \in [T, T_1]$. Since $A(t)$ is strongly continuous on $[T, T_1]$, according to uniformly bounded principle there exists $c > 0$ such that

$$\| A(t)h\| \leq c\| h\|$$

for all $t \in [T, T_1]$, $h \in H$.

From (7) and (8) we deduce

$$\| \dot{x}_n(t) - \dot{x}_{n-1}(t) \| \le c \| h_0 \| q^{n-1}.$$

This prove that the sequence $\{x_n(t)\}$ uniformly converges on $[T, T_1]$ and hence $x(t)$ is differentiable. Since

$$\dot{x}_n(t) = A(t) x_{n-1}(t),$$

making $n \to +\infty$ we obtain

$$\dot{x}(t) = A(t) x(t)$$

i. e. $x(t)$ is a solution of (1). We prove now $x(t) \to h_0$ when $t \to +\infty$. In fact

$$\| x_n(t) - h_0 \| = \sup_{\|h\| \le 1} |\langle x_n(t) - h_0, h \rangle| \le$$

$$\le \sup_{\|h\| \le 1} \int_T^{+\infty} \| x_{n-1}(\tau) \| \| A(\tau) h \| d\tau \le \frac{\| h_0 \|}{1-q} q$$

where $q \to 0$ when $t \to +\infty$. That means $x_n(t) \to h_0$ as $t \to +\infty$. Since $x_n(t)$ uniformly converges to $x(t)$, $x(t)$ tends also to h_0 as $t \to +\infty$.

We show now that there exists an unique solution of (1) which tends to h_0 as $t \to$ $\to +\infty$. Suppose contrarily that there exists other solution $y(t)$ of (1) which has this property. Then $x(t) - y(t) \to 0$ as $t \to +\infty$. Clearly

$$\langle x(t) - y(t), h \rangle = \langle x_0 - y_0, h \rangle + \int_{t_0}^{t} \langle A(\tau) x(\tau) - A(\tau) y(\tau), h \rangle d\tau =$$

$$= \langle x_0 - y_0, h \rangle + \int_{t_0}^{t} \langle x(\tau) - y(\tau), A(\tau h) \rangle d\tau$$

and $x_0 - y_0 \ne 0$.

Therefore

$$|\langle x(t) - y(t), h \rangle| \ge |\langle x_0 - y_0, h \rangle| - \int_{t_0}^{t} \| x(\tau) - y(\tau) \| \| A(\tau) h \| d\tau. \quad (9)$$

Since $x(t) - y(t) \to 0$ as $t \to +\infty$ we can suppose

$$\| x(\tau) - y(\tau) \| \le \| x(t_0) - y(t_0) \| = \| x_0 - y_0 \|$$

for all $\tau \ge t_0$.

By virtue (9) we have then

$$|\langle x(t) - y(t), h \rangle| \ge |\langle x_0 - y_0, h \rangle| - \| x_0 - y_0 \| \int_{t_0}^{t} \| A(\tau) h \| d\tau.$$

Making $t \to +\infty$ we obtain

$$\| x_0 - y_0 \| q \ge |\langle x_0 - y_0, h \rangle|.$$

Hence $\| x_0 - y_0 \| q \ge \| x_0 - y_0 \|$.

This is a contraduction.

Let $x(t)$ be an any solution of (1). Then

$$\| x(t) \| \leq \| x_0 \| + \sup_{\|h\| \leq 1} \int_{t_0}^{t} \| x(\tau) \| \| A(\tau) h \| d\tau .$$

Let we denote

$$\gamma = \sup_{\|h\| \leq 1} \int_{t_0}^{t} \| x(\tau) \| \| A(\tau) h \| d\tau .$$

There exists a sequence $\{ h_n \} \subset S(0, 1)$ such that

$$\gamma - \varepsilon_n \leq \int_{t_0}^{t} \| x(\tau) \| \| A(\tau) h_n \| d\tau \leq \gamma,$$

where $\varepsilon_n \to 0$ as $n \to +\infty$. Hence

$$\| x(t) \| \leq \| x_0 \| + \varepsilon_n + \int_{t_0}^{t} \| x(\tau) \| \| A(\tau) h_n \| d\tau .$$

By virtue of Gronwall – Bellman lemma we have then

$$\| x(t) \| \leq (\varepsilon_n + \| x_0 \|) \exp \int_{t_0}^{t} \| A(\tau) h_n \| d\tau .$$

Consequently

$$\| x(t) \| \leq \| x_0 \| e^{\gamma} \leq \| x_0 \| e^{q} < +\infty .$$

Let $t_1, t_2 \geq N$ where N is a sufficiently large. Then

$$\left\| \int_{t_1}^{t_2} A(\tau) x(\tau)\, d\tau \right\| = \sup_{\|h\| \leq 1} \left| \left\langle \int_{t_1}^{t_2} A(\tau) x(\tau)\, d\tau, h \right\rangle \right| \leq$$

$$\leq \sup_{\|h\| \leq 1} \int_{t_1}^{t_2} \| x(\tau) \| \| A(\tau) h \| d\tau \leq \| x_0 \| e^{q} \varepsilon .$$

This means that

$$\int_{t_0}^{+\infty} A(\tau) x(\tau)\, d\tau$$

converges and consequently $x(t)$ has a finite limit at infinity. This completes the proof of theorem.

***Remark* 1.** Theorem 1 is also true in the case of nonself-adjoint operator $A(t)$ under the condition that $\| A(t) h \|$ and $\| A^{*}(t) h \|$ belong to L_1 uniformly for all $h \in S(0, 1)$.

3. The case of Banach spaces. First let us prove the following theorem.

Theorem 2. *The equation* (1) *has a linear asymptotic equilibrium if and only if the equation*

$$\frac{dU}{dt} = A(t) U \tag{10}$$

considered in $[B]$ *has a solution* $V(t)$ *which tends to* I *as* $t \to +\infty$ *and which has a* $V^{-1}(t) \in [B]$ *for* $t \geq t_0 \geq 0$.

Proof. Let $U(t)$ be a Cauchy operator of (1) $(U(t_0)=I)$. As well known, the solution $x(t)$ of (1) has a form

$$x(t) = U(t)x_0, \qquad x_0 \in B.$$

According to supposition, the operator

$$U(+\infty)x_0 = x(+\infty)$$

is define on B. Moreover $U(+\infty)$ is a bijection on B. By Banach theorem there exists a $U_{+\infty}^{-1} \in [B]$.

We set now $V(t) = U(t)U_{+\infty}^{-1}$.

It is easy to verify that $V(t)$ satisfies all required properties.

Inversely let $V(t)$ be a solution of (10) with required in theorem properties. For each $u_0 \in B$, $V(t)u_0$ is a solution of (1) tended to u_0 as $t \to +\infty$. Let $x(t) = U(t)x_0$ be an arbitrary solution of (1) and $T \geq t_0$. Then

$$y(t) = V(t)V^{-1}(T)U(T)x_0$$

is a solution of (1) which tends to $V^{-1}(T)U(T)x_0$ as $t \to +\infty$. Since

$$x(T) = U(T)x_0 = y(T),$$

we have then $x(t) \equiv y(t)$ i. e. any solution of (1) has a finite limit at infinity. For each $u_0 \in B$ there exists only one solution of (1) tended to u_0 as $t \to +\infty$. In fact, if there exists other solution $x(t) = U(t)x_0$ which tends to u_0 as $t \to +\infty$ then

$$U(t)x_0 - V(t)u_0 \to 0 \quad \text{as} \quad t \to +\infty$$

or

$$V(t)V^{-1}(T)U(T)x_0 - V(t)u_0 \to 0 \quad \text{as} \quad t \to +\infty.$$

Hence

$$V^{-1}(T)U(T)x_0 = u_0.$$

Consequently

$$U(T)x_0 = V(T)u_0,$$

i. e.

$$x(t) \equiv V(t)u_0.$$

We note that the equation (1) has an linear asymptotic equilibrium in R^+ if only if it has this property on $[t_0, +\infty)$ where t_0 may be large enough. Let us introduce now the operators

$$P_n(t) = \sum_{k=1}^{n} \tilde{A}^k(t)A(t) + \sum_{k=2}^{n} \frac{d}{dt}\left[\tilde{A}^k(t)\right],$$

$$Q_n(t) = \sum_{k=1}^{n} A(t)\tilde{A}^k(t) - \sum_{k=2}^{n} \frac{d}{dt}\left[\tilde{A}^k(t)\right].$$

Theorem 3. *Let the conditions*

$$\| P_n(t) \| \in L_1 \quad or \quad \| Q_n(t) \| \in L_1$$

be satisfied for some nonnegative integer n. Then the equation (1) *has a linear asymptotic equilibrium.*

Proof. Let us denote by Ω a set of continuous on $[t_0, +\infty)$ operator functions $X(t)$ satisfied the inequality

$$\|X(t)\| \leq R \quad \text{for} \quad t \geq t_0.$$

It is easy to verify that Ω is a close set. Let us consider now the operator

$$(SX)(t) = I - \sum_{k=1}^{n} \tilde{A}^k(t)X(t) + \int_t^{+\infty} P_n(\tau)X(\tau)\,d\tau, \qquad t \geq t_0, \qquad x(t) \in \Omega.$$

We prove that for t_0 large enough, S is a retractile operator on Ω. In fact, since

$$\tilde{A}(t) \to 0 \quad \text{as} \quad t \to +\infty$$

for t_0 large enough we have

$$\left\| \sum_{k=1}^{n} \tilde{A}^k(t)X(t) \right\| \leq \varepsilon_1 \|X(t)\| \leq \varepsilon_1 R,$$

$$\left\| \int_t^{+\infty} P_n(\tau)X(\tau)\,d\tau \right\| \leq R \int_t^{+\infty} \|P_n(\tau)\|\,d\tau \leq \varepsilon_2 R.$$

Hence $\|(SX)(t)\| \leq 1 + \varepsilon_1 R + \varepsilon_2 R \leq R$, i. e. $S: \Omega \to \Omega$. For any $X(t), Y(t) \in \Omega$ we have

$$\|SX - SY\| \leq \sup_{t \geq t_0} \left\| \sum_{k=1}^{n} \tilde{A}^k(t)[X(t) - Y(t)] \right\| +$$

$$+ \sup_{t \geq t_0} \left\| \int_t^{+\infty} P_n(\tau)[X(\tau) - Y(\tau)]\,d\tau \right\| \leq \varepsilon_1 \|X - Y\| + \varepsilon_2 \|X - Y\| = \alpha \|X - Y\|$$

where $\alpha = \varepsilon_1 + \varepsilon_2 < 1$ for t_0 large enough. Thus S is a retractile operator on Ω.

Consequently there exists a $X(t) \in \Omega$ such that

$$X(t) \equiv I - \sum_{k=1}^{n} \tilde{A}^k(t)X(t) - \int_t^{+\infty} P_n(\tau)X(\tau)\,d\tau.$$

It is easy to verify that

$$X(t) \to I \quad \text{as} \quad t \to +\infty.$$

We prove now that $X(t)$ is a solution of equation (10). In fact

$$\dot{X}(t) = -\sum_{k=1}^{n} \tilde{A}^k(t)\dot{X}(t) - \sum_{k=1}^{n} \frac{d}{dt}\left[\tilde{A}^k(t)\right]X(t) + P_n(t)X(t) =$$

$$= -\sum_{k=1}^{n} \tilde{A}^k(t)\dot{X}(t) + A(t)X(t) + \sum_{k=1}^{n} \tilde{A}^k(t)X(t)$$

hence

$$\left[I + \sum_{k=1}^{n} \tilde{A}^k(t)\right]\dot{X}(t) = \left[I + \sum_{k=1}^{n} \tilde{A}^k(t)\right]A(t)X(t).$$

For $t \geq t_0$ and t_0 is large enough operator

$$I - \sum_{k=1}^{n} \tilde{A}^k(t)$$

is inversive.

Therefore $\dot{X}(t) = A(t)X(t)$. By virtue of Theorem 2, we obtain the required statement. Let now $\|Q_n(t)\| \in L_1$. By analogous proof we can show that the operator

$$(SX)(t) = I + \sum_{k=1}^{n} X(t)\tilde{A}^k(t) - \int_t^{+\infty} X(\tau)\tilde{Q}_n(\tau)d\tau, \quad t \geq t_0,$$

is retractile on Ω for t_0 large enough. Hence there exists $X(t) \in \Omega$ such that

$$X(t) \equiv I + \sum_{k=1}^{n} X(t)\tilde{A}^k(t) - \int_t^{+\infty} X(\tau)Q_n(\tau)d\tau.$$

It is easy to verify that $X(t) \to I$ as $t \to +\infty$ and $X(t)$ satisfies the equation

$$\dot{X} = -XA(t).$$

Since $X(t)$ converges to I as $t \to +\infty$, $X^{-1}(t)$ converges also to I as $t \to +\infty$. The proof of theorem now is followed from Theorem 2 and from that $X^{-1}(t)$ is a solution of (10).

***Remark* 2.** The results (2) and (3) can be obtained from this theorem for $n = 0$ and $n = 1$ respectively.

4. Asymptotic equivalence.

Definition 3. *The equations*

$$\dot{x} = A(t)x, \tag{11}$$

$$\dot{y} = B(t)x \tag{12}$$

are said to be asymptotically equivalent if to each solution $x(t)$ of (11) there exists a solution $y(t)$ of (12) such that

$$\| x(t) - y(t) \| = o(1) \quad as \quad t \to +\infty \tag{13}$$

and conversely, to each solution $y(t)$ of (12) there exists a solution $x(t)$ of (11) satisfying the relation (13).

The asymptotic equivalence of equations are considered in [7 – 9]. We now apply the results of Sections 3, 4 to obtain some sufficient conditions for asymptotic equivalence of (11) and (12). Let us denote the Cauchy operators to the equations (11), (12) with initial conditions

$$X(0) = Y(0) = I$$

by $X(t)$, $Y(t)$ respectively. We set $C(t) = Y^{-1}(t)[A(t) - B(t)]Y(t)$.

Theorem 4. *Let the condition*

$$\| Y(t) \| \leq M, \quad t \geq 0,$$

be holded and let the operator $C(t)$ satisfies the conditions of Theorem 1. *Then the equation* (11), (12) *considered in Hilbert space H are asymptotically equivalent.*

Proof. Let $x(t)$ be arbitrary solution of (11). It is easy to verify that $z(t) = Y^{-1}(t)x(t)$ is a solution of the equation

$$\dot{z} = C(t)z. \tag{14}$$

According to Theorem 1, the equation (13) has a linear asymptotic equilibrium. Hence there exists

$$z_\infty = \lim_{t \to +\infty} z(t).$$

The function $y(t) = Y(t)z_\infty$ is a solution of (12). Moreover

$$\| x(t) - y(t) \| = \| Y(t)z(t) - Y(t)z_\infty \| \leq M \| z(t) - z_\infty \| \to 0 \quad \text{as} \quad t \to +\infty.$$

Inversely, let $y(t)$ be an any solution of (12). Then $y(t) = Y(t)y_0$. By definition of linear asymptotic equilibrium there exists a solution $z(t)$ of (14) such that

$$y_0 = \lim_{t \to +\infty} z(t).$$

Then the function $x(t) = Y(t)z(t)$ is a solution of (11). Moreover

$$\| x(t) - y(t) \| = \| Y(t)z(t) - Y(t)z_0 \| \leq M \| z(t) - z_0 \| \to 0$$

as $t \to +\infty$. The theorem is completely proved.

Let us set now

$$\tilde{C}(t) = \int_t^{+\infty} C(\tau)\, d\tau$$

if the integral is converged,

$$G_n(t) = \sum_{k=1}^{n} \tilde{C}^k(t) C(t) + \sum_{k=2}^{n} \frac{d}{dt}\left[\tilde{C}^k(t) \right],$$

$$H_n(t) = \sum_{k=1}^{n} C(t) \tilde{C}^k(t) - \sum_{k=2}^{n} \frac{d}{dt}\left[\tilde{C}^k(t) \right].$$

Theorem 5. *Let the condition* $\| Y(t) \| \leq M$ *be satisfied and let*

$$\| G_n(t) \| \in L_1 \quad \text{or} \quad \| H_n(t) \| \in L_1$$

for some nonnegative integer n.

Then the equations (11), (12) *are asymptotically equivalent.*

The proof is analogous to that of Theorem 4.

Remark **3.** From the process of proof we see that the condition of boundedness of $Y(t)$ can be replaced by one of $X(t)$. In particular, if one of the equations (11) and (12) is stable on right-hand and on left-hand (see [4, p. 167]) and $\| C(t) \| \in L_1$ then they are asymptotically equivalent. In fact, the conditions of Theorem 5 hold in this case for $n = 0$ (we see then $G_0(t) = H_0(t) = C(t)$).

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