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MODEL BCS HAMILTONIAN AND APPROXIMATING HAMILTONIAN FOR AN INFINITE VOLUME. IV. TWO BRANCHES OF THEIR COMMON SPECTRA AND STATES*

МОДЕЛЬНИЙ ГАМІЛЬТОНІАН БКШ ТА АПРОКСИМУЮЧИЙ ГАМІЛЬТОНІАН ПРИ НЕСКІНЧЕННОМУ ОБ'ЄМІ. IV. ДВІ ГІЛКИ ЇХ СПІЛЬНИХ СПЕКТРІВ ТА СТАНІВ

We consider the model and approximating Hamiltonians directly in the case of an infinite volume. We show that each of these Hamiltonians has two branches of the spectrum and two systems of eigenvectors, which represent excitations of the ground states of the model and approximating Hamiltonians as well as the ground states themselves. On both systems of eigenvectors, the model and approximating Hamiltonians coincide with each other. In both branches of the spectrum, there is a gap between the eigenvalues of the ground and excited states.

Розглядаються модельний та апроксимуючий гамільтоніани безпосередньо при нескінченному об'ємі. Показано, що обидва гамільтоніани мають дві гілки спектра та дві системи власних векторів, які складаються з основних станів модельного та апроксимуючого гамільтоніанів та їх збуджень. На обох системах власних векторів модельний та апроксимуючий гамільтоніани збігаються. В обох гілках спектра існує щілина між власними значеннями основного та збуджених станів.

Introduction. The present work is a direct continuation of our previous works [1 – 3] devoted to the investigation of the spectrum and states of the model Hamiltonian of the BCS theory of superconductivity in a finite cube under periodic boundary conditions and the thermodynamic equivalence of this Hamiltonian and the Bogolyubov approximating Hamiltonian. In [1 – 3], we studied the spectrum of the ground and excited states of both model and approximating Hamiltonians asymptotically exactly as the volume of the cube tends to infinity and proved their thermodynamic equivalence in the following sense.

The averages (per unit volume) of the model and approximating Hamiltonians over the ground and excited states coincide with each other in the thermodynamic limit, i.e., in the case where the volume of the cube tends to infinity. This thermodynamic equivalence takes place both for the ground and excited states of the model Hamiltonian and for the ground and excited states of the approximating Hamiltonian. In this sense, the model and approximating Hamiltonians have two branches of the spectrum and two branches of eigenvectors.

In the present work, we consider the model and approximating Hamiltonians directly for an infinite volume in certain Hilbert spaces of translation-invariant functions. Earlier, we studied the model Hamiltonian directly for an infinite volume and showed that it differs from the free Hamiltonian only in the subspace of pairs [4 – 6]. We established that the model and approximating Hamiltonians coincide with each other on the ground state of the model Hamiltonian. In the present paper, we establish that they completely coincide in the following sense.

We consider the ground state of the model Hamiltonian and its excitations. We introduce the operators of creation and annihilation for which the ground state of the model

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Hamiltonian is a vacuum. The excited states are introduced by the action of the operators of creation on the ground state. We also introduce the excitations of pairs. These excitations form a basis in the Hilbert space of states of the model Hamiltonian, and, at the same time, they are its eigenvectors. On this Hilbert space, the model and approximating Hamiltonians coincide with each other, i.e., their actions on the elements of the Hilbert space coincide.

We also consider the ground and excited states of the approximating Hamiltonian. We introduce again operators of creation and annihilation for which the ground state of the approximating Hamiltonian is a vacuum. The excited states of the approximating Hamiltonian are introduced as a result of the action of the operators of creation on its ground state; they form a basis in the Hilbert space of states of the approximating Hamiltonian and are its eigenvectors. On this Hilbert space, the approximating and model Hamiltonians coincide with each other.

The operators of creation and annihilation that correspond to the ground states of the model and approximating Hamiltonians are not unitarily equivalent, and the corresponding Hilbert spaces generated by the action of the operators of creation on the ground states are different. Thus, *both Hamiltonians (model and approximating) have two branches of the spectrum and two systems of eigenvectors belonging to different Hilbert spaces.*

In both branches of the spectrum, there is a nonzero gap that separates the eigenvalues of the ground states from their excitations. The physical consequences of the presence of two systems of eigenvectors for the model and approximating Hamiltonians will be studied in a separate work.

We also want to note another, purely mathematical, aspect of this work. We express the ground states of the model and approximating Hamiltonians and their excitations by using the operators of creation, as is usually done in the Fock space. At the same time, both the ground states and their excitations do not belong to the Fock space, and if their norms are calculated as for elements of the Fock space, they diverge exponentially with volume. For this reason, in the present work we propose a completely new approach. It is based on the fact that the ground and excited states are completely determined by certain sequences of functions; we treat them as elements of a certain Hilbert space of pairs and excitations $\mathcal{H}^F \bigotimes \mathcal{H}^P$. The scalar product and norm in this space are finite and do not have volume divergences.

We define the action of the Hamiltonians on the ground and excited states by using the canonical anticommutation relations as in the case of the Fock space. However, the sequences of functions that characterize the result of the action of the Hamiltonians are again regarded as elements of the space $\mathcal{H}^F \bigotimes \mathcal{H}^P$. Thus, we again avoid volume divergences.

We use extensively that the operators in the model and approximating Hamiltonians that contains two operators of annihilations of electrons with opposite momenta and spin are proportional to the unit operator on the coherent states of pairs that represent the ground states of the model and approximating Hamiltonians. Actually we have given a new direct and complete proof of the same assertions used and proved by Bogolubov [7] and Haag [8].

Note that the elements of the space $\mathcal{H}^F \bigotimes \mathcal{H}^P$ and the action of the Hamiltonians on them are the thermodynamic limits of the corresponding elements in a *finite volume* as this volume tends to infinity. We use the same notation as in [1–3] and impose the same restrictions on the potential. We continue the enumeration of sections, theorems, and formulas.

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15. Hilbert space. *1. Hilbert space of pairs.* Consider the operators of creation $a^+(k,\sigma)$ and annihilation $a(k,\sigma)$ of electrons with momentum k and spin σ. Momenta $k \in R^3$ and $\sigma = \pm 1$. We use the following denotation

$$\overline{k} = (k, \sigma),$$

and

$$\int d\overline{k} \ldots = \sum_{\sigma=\pm 1} \int dk \ldots .$$

The operators $a^+(k,\sigma)$ and $a(k,\sigma)$ satisfy the following canonical anticommutation relations

$$\{a^+(k_1,\sigma_1), a(k_2,\sigma_2)\} = \delta(k_1 - k_2)\delta_{\sigma_1,\sigma_2} \tag{15.1}$$

where $\delta(k_1 - k_2)$ is δ-function and $\delta_{\sigma_1,\sigma_2}$ is Kronecker symbol. The rest of anticommutators is equal to zero. We will also use the following denotation

$$a^+(k,1) = a^+(k), \qquad a(k,1) = a(k),$$
$$a^+(-k,-1) = a^+(-k), \qquad a(-k,-1) = a(-k).$$

Introduce the following state of pairs

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \int f_n(k_1,\ldots,k_n) a^+(k_1) a^+(-k_1) \ldots a^+(k_n) a^+(-k_n) |0\rangle dk_1 \ldots dk_n. \tag{15.2}$$

The state f is defined by the sequence

$$f = (f_0, f_1(k_1), \ldots, f_n(k_1,\ldots,k_n), \ldots),$$

of symmetric functions $f_n(k_1,\ldots,k_n) = f_n(k_{i_1},\ldots,k_{i_n})$, where $(i_1,\ldots,i_n)$ is some permutation of numbers $(1,\ldots,n)$, and $f_n(k_1,\ldots,-k_i,\ldots,k_n) = f_n(k_1,\ldots,k_i,\ldots \ldots, k_n)$, i. e. f_n is also even function.

We introduce the following scalar product of two states f and g

$$g = \sum_{n=0}^{\infty} \frac{1}{n!} \int g_n(k_1,\ldots,k_n) a^+(k_1) a^+(-k_1) \ldots a^+(k_n) a^+(-k_n) |0\rangle dk_1 \ldots dk_n$$

(where $g_n(k_1,\ldots,k_n)$ are again symmetric even functions)

$$(f,g) = \sum_{n=0}^{\infty} \frac{1}{n!} \int \overline{f_n(k_1,\ldots,k_n)} g(k_1,\ldots,k_n) dk_1 \ldots dk_n \tag{15.3}$$

and norm $\|f\| = (f,f)^{1/2}$.

Denote by $\mathcal{H}^P$ the Hilbert space with elements f (15.2), scalar product (15.3), and with norm $\|f\| < \infty$. We say that $\mathcal{H}^P$ is the Hilbert space of pairs.

The Hilbert space $\mathcal{H}^P$ can be obtained from states of the Hilbert space $\mathcal{H}_V^P$ in which element f and g are defined as follows $[1-3]$

$$f=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\dots,k_n} f_n(k_1,\dots,k_n)a_{k_1}^+a_{-k_1}^+\dots a_{k_n}^+a_{-k_n}^+|0\rangle,$$
$$g=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k_1,\dots,k_n} g_n(k_1,\dots,k_n)a_{k_1}^+a_{-k_1}^+\dots a_{k_n}^+a_{-k_n}^+|0\rangle. \tag{15.4}$$

Summation in (15.4) is carried out over the momenta $k=\dfrac{2\pi}{L}n$, $n=(n_1,n_2,n_3)$, n_i are natural numbers, $i=1,2,3$, $V=L^3$. The scalar product of f and g is defined as follows

$$(f,g)_V'=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{V^n}\sum_{k_1\neq\dots\neq k_n}\overline{f_n(k_1,\dots,k_n)}g_n(k_1,\dots,k_n)=$$
$$=\sum_{n=0}^{\infty}\frac{1}{V^n}\sum\nolimits'_{k_1\neq\dots\neq k_n}\overline{f_n(k_1,\dots,k_n)}g_n(k_1,\dots,k_n) \tag{15.5}$$

(see details in $[1-3]$).

One can consider $(f,g)_V'$ as corresponding to (15.3) integral sums with the elementar infinitesimal volume $1/V$ and functions $f_n(k_1,\dots,k_n)$, $g_n(k_1,\dots,k_n)$ equal to zero on all hyperplanes $k_i=k_j$, $(i,j)\subset(1,\dots,n)$. It is obvious that

$$\lim_{V\to\infty}(f,g)_V'=(f,g). \tag{15.6}$$

2. *Coherent states of pairs.* Consider the following special state of pairs

$$\Phi=\sum_{n=0}^{\infty}\frac{1}{n!}\int f(k_1)\dots f(k_n)a^+(k_1)a^+(-k_1)\dots a^+(k_n)a^+(-k_n)|0\rangle dk_1\dots dk_n=$$
$$=e^{\int f(k)a^+(k)a^+(-k)dk}|0\rangle \tag{15.7}$$

with $\int|f(k)|^2dk<\infty$. We say that Φ is coherent state of pairs with wave function $f(k)$, the same for all pairs.

We have

$$(\Phi,\Phi)=\sum_{n=0}^{\infty}\frac{1}{n!}\int|f(k_1)|^2\dots|f(k_n)|^2dk_1\dots dk_n=e^{\int|f(k)|^2dk}. \tag{15.8}$$

Denote by Φ_0 the following coherent state

$$\Phi_0=e^{\int f_0(k)a^+(k)a^+(-k)dk}|0\rangle \tag{15.9}$$

where $f_0(k)$ is the eigenfunction of the operator H_2 with the lowest eigenvalue E_0 [1],

$$f_0(k)=cv(k)\Big/\left(E_0-\frac{2k^2}{2m}+2\mu\right),\quad c=\left(\int\left[v^2(k)\Big/\left(E_0-\frac{2k^2}{2m}+2\mu\right)^2\right]dk\right)^{-\frac12}.$$

By Φ_0^a denote the following coherent state

$$\Phi_0^a=e^{\int f_0^a(k)a^+(k)a^+(-k)dk}|0\rangle \tag{15.10}$$

where

$$f_0^a(k) = -\left(\left(\varepsilon^2(k)+c^2v^2(k)\right)^{1/2} - \varepsilon(k)\right)^{1/2}\left(\left(\varepsilon^2(k)+c^2v^2(k)\right)^{1/2} + \varepsilon(k)\right)^{-1/2} \quad (15.10')$$

and $\varepsilon(k) = \dfrac{k^2}{2m} - \mu$. The constant c shall be determined later (see Section 16).

Note that the states Φ (15.7) as well as f (15.2) do not belong to the Fock space. It follows directly from the following, equivalent to (15.2) and (15.7), representation of f and Φ

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \int f_n(k_1, \dots, k_n)\delta(k_1 + k_1') \dots \delta(k_n + k_n') \times$$
$$\times a^+(k_1)a^+(k_1') \dots a^+(k_n)a^+(k_n')|0\rangle dk_1 dk_1' \dots dk_n dk_n', \quad (15.2')$$
$$\Phi = \sum_{n=0}^{\infty} \frac{1}{n!} \int f(k_1)\delta(k_1 + k_1') \dots f(k_n)\delta(k_n + k_n') \times$$
$$\times a^+(k_1)a^+(k_1') \dots a^+(k_n)a^+(k_n')|0\rangle dk_1 dk_1' \dots dk_n dk_n'. \quad (15.7')$$

Obviously functions $f_n(k_1, \dots, k_n)\delta(k_1 + k_1') \dots \delta(k_n + k_n')$, $f(k_1)\delta(k_1 + k_1') \dots$ $\dots f(k_n)\delta(k_n + k_n')$ are not square integrable with respect to $(k_1, k_1', \dots, k_n, k_n')$. Norms of f and Φ calculated as elements of usual Fock space are equal to

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} V^n \int |f_n(k_1, \dots, k_n)|^2 dk_1 \dots dk_n,$$
$$\|\Phi\|^2 = \sum_{n=0}^{\infty} \frac{1}{n!} V^n \int |f(k_1)|^2 dk = e^{V\int |f(k)|^2 dk} \quad (15.11)$$

where $V = V(R^3)$ is the volume of threedimensional space R^3.

Consider the following state

$$\Phi_m = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{m!} \int \chi_m(q_1, \dots, q_m) f(k_1) \dots f(k_n) a^+(q_1)a^+(-q_1) \dots$$
$$\dots a^+(q_m)a^+(-q_m)a^+(k_1)a^+(-k_1) \dots a^+(k_n)a^+(-k_n)|0\rangle dq_1 \dots dq_m dk_1 \dots dk_n =$$
$$= \sum_{n=m}^{\infty} \frac{1}{n!} \int f_n(k_1, \dots, k_n) a^+(k_1)a^+(-k_1) \dots a^+(k_n)a^+(-k_n)|0\rangle dk_1 \dots dk_n, \quad (15.12)$$

where

$$f_n(k_1, \dots, k_n) = \mathrm{sym}[\chi_m(k_m) f(k_{m+1}) \dots f(k_n)],$$

and $\chi_m(q_1, \dots, q_m)$ is symmetric square integrable function. The state Φ_m is a particular case of state f (15.2).

*3. **States of pairs with excitations.*** Consider the following state

$$f_l = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{l!} \int \Psi_l(p_1, \dots, p_l) f_n(k_1, \dots, k_n) a^+(p_1) \dots a^+(p_l) \times$$
$$\times a^+(k_1)a^+(-k_1) \dots a^+(k_n)a^+(-k_n)|0\rangle dp_1 \dots dp_l dk_1 \dots dk_n =$$
$$= \frac{1}{l!} \int \Psi_l(p_1, \dots, p_l) a^+(p_1) \dots a^+(p_l) dp_1 \dots dp_l f \quad (15.13)$$

where function $\Psi_l(p_1, \dots, p_l)$ is antysymmetric and square integrable. We suppose, for the sake of simplicity, that the operators $a^+(p_1), \dots, a^+(p_l)$ correspond to electrons with spin $+1$. Generalisation to case with some numbers of electorns with spin -1 is obvious.

The norm of state f_l is defined as follows

$$\|f_l\|^2 = (f_l, f_l) =$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{l!} \int |\Psi_l(p_1, \dots, p_l)|^2 |f_n(k_1, \dots, k_n)|^2 dp_1 \dots dp_l dk_1 \dots dk_n =$$

$$= \frac{1}{l!} \int |\Psi_l(p_1, \dots, p_l)|^2 dp_1 \dots dp_l \|f\|. \tag{15.14}$$

The scalar product of two different states (15.3) is obvious. For example, if

$$g_l = \frac{1}{l!} \int h_l(p_1, \dots, p_l) a^+(p_1) \dots a^+(p_l) dp_1 \dots dp_l g$$

then

$$(f_l, g_l) = \frac{1}{l!} \int \overline{\Psi_l(p_1, \dots, p_l)} h_l(p_1, \dots, p_l) dp_1 \dots dp_l (f, g). \tag{15.15}$$

The states f_l with $\|f_l\| < \infty$ belong to the Hilbert space $\mathcal{H}_l^F \bigotimes \mathcal{H}^P$, where $\mathcal{H}_l^F$ is the l-particle Fock space of fermions.

Note that states with pairs with excitations f_l (15.13) and $\mathcal{H}_l^F \bigotimes \mathcal{H}^P$ can be obtained from corresponding states of $\mathcal{H}_{l,V}^F \bigotimes \mathcal{H}_V^P$ in which states f_l and g_l are defined as follows

$$f_l = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{l!} \sum_{p_1 \neq \dots \neq p_l \neq k_1 \neq \dots \neq k_n} \Psi_l(p_1, \dots, p_l) f_n(k_1, \dots, k_n) a_{p_1}^+ \dots a_{p_l}^+ \times$$
$$\times a_{k_1}^+ a_{-k_1}^+ \dots a_{k_n}^+ a_{-k_n}^+ |0\rangle,$$

$$g_l = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{l!} \sum_{p_1 \neq \dots \neq p_l \neq k_1 \neq \dots \neq k_n} h_l(p_1, \dots, p_l) g_n(k_1, \dots, k_n) a_{p_1}^+ \dots a_{p_l}^+ \times$$
$$\times a_{k_1}^+ a_{-k_1}^+ \dots a_{k_n}^+ a_{-k_n}^+ |0\rangle \tag{15.16}$$

and their scalar product as follows

$$(f_l, g_l)'_V = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{1}{l!} \frac{1}{V^{l+n}} \sum_{p_1 \neq \dots \neq p_l \neq k_1 \neq \dots \neq k_n} \overline{\Psi_l(p_1, \dots, p_l)} h_l(p_1, \dots, p_l) \times$$
$$\times \overline{f_n(k_1, \dots, k_n)} g_n(k_1, \dots, k_n) \tag{15.17}$$

(see details in [1, 2]).

One can consider $(f_l, g_l)'_V$ as corresponding to (15.15) integral sums with the elementar infinitesimal volume $\frac{1}{V}$ and functions $\Psi_l(p_1, \dots, p_l) f_n(k_1, \dots, k_n)$, $h_l(p_1, \dots, p_l) g_n(k_1, \dots, k_n)$ equal to zero on all hyperplanes where some pairs of momenta coinside. It is obvious that

$$\lim_{V \to \infty} (f_l, g_l)'_V = (f_l, g_l), \qquad \lim_{V \to \infty} (\|f_l\|'_V)^2 = \|f_l\|^2. \tag{15.18}$$

In what follows we will consider scalar products and norms for infinite $\Lambda = R^3$ as the thermodynamic limit (15.18) of corresponding scalar products and norms defined for finite Λ as above by (15.17).

16. Two methods of determination of the spectra and eigenvectors of the approximating Hamiltonian in infinite volume. *1. The first method.* Consider the approximating Hamiltonian in infinite volume [7]

$$H_a = \int \left(\frac{p^2}{2m} - \mu\right) a^+(\bar{p})a(\bar{p})d\bar{p} + c\int v(p)a^+(p)a^+(-p)dp +$$
$$+c\int v(p)a(-p)a(p)dp - c^2g^{-1}VI, \quad V = V(R^3). \tag{16.1}$$

It can be formally obtained from $H_{a,\Lambda}$ in finite cube Λ

$$H_{a,\Lambda} = \sum_{\bar{p}} \left(\frac{p^2}{2m} - \mu\right) a_{\bar{p}}^+ a_{\bar{p}} + c(V)\sum_p v_p a_p^+ a_{-p}^+ +$$
$$+c(V)\sum_p v_p a_{-p}a_p - g^{-1}c^2(V)VI, \quad V = V(\Lambda), \tag{16.2}$$

by the following replacements:

$$\lim_{V\to\infty} = \frac{V^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} a_k^+ = a^+(k), \qquad \lim_{V\to\infty} = \frac{V^{\frac{1}{2}}}{(2\pi)^{\frac{3}{2}}} a_k = a(k),$$
$$\lim_{V\to\infty} = \frac{V}{(2\pi)^3}\delta_{k+k'} = \delta(k+k'),$$
$$\lim_{V\to\infty} \frac{(2\pi)^3}{V}\sum_k \ldots = \int dk \ldots, \qquad \lim_{V\to\infty} c(V) = c.$$

The constant c and $c(V)$ should be obtained from condition of minimum of energy of ground states of H_a and $H_{a,\Lambda}$ respectively.

The Hamiltonian H_a can be diagonalized

$$H_a = \int E(p)\alpha^+(\bar{p})\alpha(\bar{p})d\bar{p} + C(c)VI \tag{16.1'}$$

where the new operators of creation $\alpha^+(\bar{k})$ and annihilation $\alpha(\bar{k})$ satisfy the same canonical anticommutation relations (15.1) as the operators $a^+(\bar{k}), a(\bar{k})$ and are expressed through the operators $a^+(\bar{k})$, $a(\bar{k})$ by the following formulae

$$\alpha^+(k) = u(k)a^+(\bar{k}) + w(k)a(-k), \qquad \alpha^+(-k) = u(k)a^+(-k) - w(k)a(k),$$
$$\alpha(k) = u(k)a(k) + w(k)a^+(-k), \qquad \alpha(-k) = u(k)a(-k) - w(k)a^+(k), \tag{16.3}$$
$$u(k) = \frac{1}{\sqrt{2}}\left(1 + \varepsilon(k)\left(\varepsilon^2(k) + c^2v^2(k)\right)^{-\frac{1}{2}}\right)^{\frac{1}{2}},$$
$$w(k) = \frac{1}{\sqrt{2}}\left(1 - \varepsilon(k)\left(\varepsilon^2(k) + c^2v^2(k)\right)^{-\frac{1}{2}}\right)^{\frac{1}{2}},$$
$$C(c) = \int \left[\varepsilon(k) - \left(\varepsilon^2(k) + c^2v^2(k)\right)^{-\frac{1}{2}}\right] dk - g^{-1}c^2,$$
$$E(k) = \left(\varepsilon^2(k) + c^2v^2(k)\right)^{\frac{1}{2}}, \qquad \varepsilon(k) = \frac{k^2}{2m} - \mu.$$

Note that canonical transformations (16.3) are not unitary equivalent because the operator of multiplication by functions $u(k)$ and $w(k)$ are not the Hilbert – Schmidt class and the necessary conditions of unitary equivalence are not fulfilled. For system in finite Λ transformations (16.5) are unitary equivalent because the domain D (support of the potential $v(k)$) contains only finite number of quasimomenta and, due to the Fermi-statistic, system under consideration has finite degrees of freedom.

The Hamiltonian $H_{a,\Lambda}$ can also be diagonalized

$$H_{a,\Lambda} = \sum_{\bar{p}} E(p)\alpha^+_{\bar{p}}\alpha_{\bar{p}} + C(c(V))VI, \quad V = V(\Lambda),$$

$$C(c(V)) = \frac{(2\pi)^3}{V}\sum_k [\varepsilon(k) - (\varepsilon^2(k) + c^2(V)v_k^2)^{\frac{1}{2}}] - g^{-1}c^2(V). \tag{16.2'}$$

It is easy to check that state Φ_0^a is the vacuum state for the operators $\alpha^+(\bar{k}), \alpha(k)$

$$\alpha(k)\Phi_0^a = (u(k)f_0^a(k)a^+(-k) + w(k)a^+(-k))\Phi_0^a(k) = 0,$$
$$\alpha(-k)\Phi_0^a = -(u(k)f_0^a(k)a^+(k) + w(k)a^+(k))\Phi_0^a(k) = 0.$$

Therefore

$$H_a\Phi_0^a = C(c)V\Phi_0^a,$$

i.e. Φ_0^a is eigenvector of H_a with (infinite) eigenvalue $C(c)V$. If one introduces the renormalized Hamiltonian

$$H_{a,r} = H_a - C(c)VI = \int E(k)\alpha^+(\bar{k})\alpha(\bar{k})d\bar{k} \tag{16.4}$$

then Φ_0^a becomes its eigenvector with eigenvalue zero

$$H_{a,r}\Phi_0^a = 0.$$

Define the following excited eigenvectors (eigenstates)

$$\varphi^a(\bar{p_1},\dots,\bar{p_l}) = \alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\Phi_0^a, \quad l \geq 1,$$

$$\varphi^a(\bar{p_1},\dots,\bar{p_l},q_1,-q_1,\dots,q_m,-q_m) = \alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\times$$
$$\times\alpha^+(q_1)\alpha^+(-q_1)\dots\alpha^+(q_m)\alpha^+(-q_m)\Phi_0^a, \quad l+m \geq 1. \tag{16.5}$$

They are eigenvector of H_a and $H_{a,r}$

$$H_a\varphi^a(\bar{p_1},\dots,\bar{p_l},q_1,-q_1,\dots,q_m,-q_m) =$$
$$= \left(\sum_{i=1}^{l} E(p_i) + 2\sum_{i=1}^{m} E(q_i) + C(c)V\right)\varphi^a(\bar{p_1},\dots,\bar{p_l},q_1,-q_1,\dots,q_m,-q_m). \tag{16.6}$$

We consider eigenstates $\varphi^a(\bar{p_1},\dots,\bar{p_l},\ q_1,-q_1,\dots,q_m,-q_m)$ with different $(\bar{p_1},\dots,\bar{p_l},\ q_1,\dots,q_m)$ or different l, m as orthogonal ones, because they are orthogonal for finite cube Λ. They are normalized on δ-functions for $\Lambda = R^3$.

Consider normalized excited states

$$\varphi_{lm}^a = \frac{1}{l!\,m!}\int \Psi_l(p_1,\dots,p_l)\chi_m(q_1,\dots,q_m)\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\times$$
$$\times\alpha^+(q_1)\alpha^+(-q_1)\dots\alpha^+(q_m)\alpha^+(-q_m)d\bar{p_1}\dots d\bar{p_l}dq_1\dots dq_m\Phi_0^a \tag{16.7}$$

where functions $\Psi_l(p_1,\dots,p_l)$ and $\chi_m(q_1,\dots,q_m)$ are antisymmetric and symmetric respectively and have supports in D (support of $v(k)$). The function $\chi_m(q_1,\dots,q_m)$ is orthogonal to the function $f_0^a(k)$ with respect to all variables $q_1,\dots,q_m$, i.e.

$$\int \bar{f}_0^a(q)\chi_m(q, q_1, \dots, q_m)dq = 0 \tag{16.8}$$

due to symmetricity of $\chi_m(q_1, \dots, q_m)$. Note that condition of orthogonality (16.8) is not necessary but it will be important in the next section. The reason why the function χ_m is orthogonal to f_0^a is that we do not want to have excited pairs in the same state as in ground state.

The norm of φ_{lm}^a is defined as follows

$$\|\varphi_{lm}^a\|^2 = \frac{1}{l!}\frac{1}{m!}\int |\Psi_l(p_1, \dots, p_l)|^2 dp_1 \dots dp_l \times$$
$$\times \int |\chi_m(q_1, \dots, q_m)|^2 dq_1 \dots dq_m \|\Phi_0^a\|^2 \tag{16.9}$$

and corresponding scalar product. It means that $\varphi_{lm} \subset \mathcal{H}_l^F \bigotimes \mathcal{H}_m^P \bigotimes \Phi_0^a$. (For motivation of formula (16.9) see (16.9′) and (16.10).)

Now we show how one can construct the function $\chi_m(q_1, \dots, q_m)$ orthogonal to $f_0^a(k)$. The function $f_0^a(k)$ depends only on $|k|$. Then one can construct desired function $\chi_m(q_1, \dots, q_m)$ as symmetric product of m functions $\chi_1(q_1) \dots \chi_m(q_m)$ and functions $\chi_i(q)$ are product of two functions, one of them depends on $|k|$ and the second depends on variables $0 \le \theta \le \pi, 0 \le \varphi \le 2\pi$ and is orthogonal to unit on unite sphere. For example $\chi_i(q) = \chi_i(|q|)Y_{ml}(\theta, \varphi)$ where $Y_{ml}(\theta, \varphi)$ is spherical function, $|m| + l \ge 1$.

Obviously that

$$\int f_0^a(k)\chi_i(k)dk = \int f_0^a(|k|)\chi(|k|)k^2 dk \int_0^{\pi}\int_0^{2\pi} Y_{ml}(\theta, \varphi)\sin\theta d\theta d\varphi = 0,$$

$$\int v(k)\chi_i(k)dk = 0, \quad i = 1, \dots, m.$$

Note that $\chi_i(k)$ is also orthogonal to $v(k) = v(|k|)$ and to any functions that depend only on $|k|$.

We summarize the obtained above results in the following theorem.

Theorem 15. *The approximating Hamiltonian H_a has eigenvectors $\varphi^a(\bar{p}_1, \dots \dots, \bar{p}_l, q_1, -q_1, \dots, q_m, -q_m)$ (16.5) with eigenvalues $\sum_{i=1}^l E(p_i) + 2\sum_{i=1}^m E(q_i) + C(c)V$. The eigenvectors $\varphi^a(\bar{p}_1, \dots, \bar{p}_l, q_1, -q_1, \dots, q_m, -q_m)$ are orthogonal basis in the Hilbert space $\mathcal{H}_l^F \bigotimes \mathcal{H}_m^P \bigotimes \Phi_0^a$.*

Note that there is the gap $E(p)$ in the spectrum of H_a (the difference between the eigenvalues of the exitation $\alpha^+(p)\Phi_0^a$ and the ground state Φ_0^a).

We introduce the state φ_{lm}^a in finite cube Λ

$$\varphi_{lm,\Lambda}^a = \frac{1}{l!\,m!}\sum_{(p)_l \neq (q)_m} \Psi_l(p_1, \dots, p_l)\chi_m(q_1, \dots, q_m)\alpha_{\bar{p}_1}^+ \dots \alpha_{\bar{p}_l}^+ \times$$
$$\times \alpha_{q_1}^+ \alpha_{-q_1}^+ \dots \alpha_{q_m}^+ \alpha_{-q_m}^+ \Phi_0^a \tag{16.7′}$$

with norm

$$(\|\varphi_{lm,\Lambda}^a\|_V')^2 = \frac{1}{l!}\frac{1}{m!}\frac{1}{V^{l+m}}\sum_{(p)_l \neq (q)_m} |\Psi_l(p_1, \dots, p_l)|^2 |\chi_m(q_1, \dots, q_m)|^2 \times$$
$$\times \sum_{n=0}^{\infty}\frac{1}{n!}\frac{1}{V^n}\sum_{(k)_n \neq (\bar{p})_l \neq (q)_m} |f_0^a(k_1)|^2 \dots |f_0^a(k_n)|^2. \tag{16.9′}$$

Note that in (16.9′) we have $p_1 \neq \ldots \neq p_l \neq q_1 \neq \ldots \neq q_m \neq k_1 \neq \ldots \ldots \neq k_n$, because states with different $(\bar{p})_l$ or $(q)_m$ are orthogonal [2]. We have the same denotations for the ground states Φ_0, Φ_0^a for finite and infinite cube.

It is obvious that

$$\lim_{V\to\infty}(\|\varphi^a_{lm,\Lambda}\|'_V)^2 = \|\varphi^a_{lm}\|^2,$$
$$\lim_{V\to\infty}\|H_{a,r,\Lambda}\varphi^a_{lm,\Lambda}\|'_V = \|H_{a,r}\varphi^a_{lm}\|^2 \tag{16.10}$$

because for finite cube Λ

$$H_{a,r,\Lambda}\varphi^a_{lm,\Lambda} = \frac{1}{l!\,m!}\sum_{(p)_l\neq(q)_m}\left[\sum_{i=1}^{l}E(p_i)+2\sum_{i=1}^{m}E(q_i)\right]\Psi_l(\bar{p_1},\ldots,\bar{p_l})\times$$
$$\times\chi_m(q_1,\ldots,q_m)\alpha^+_{\bar{p_1}}\ldots\alpha^+_{\bar{p_l}}\alpha^+_{q_1}\alpha^+_{-q_1}\ldots\alpha^+_{q_m}\alpha^+_{-q_m}\Phi^a_0 \tag{16.11}$$

and we have the same expression $H_{a,r}\varphi^a_{lm}$ for $\Lambda = R^3$ but the sums $\sum_{(}\bar{p})_l \neq (q)_m \ldots$ are replaced by the corresponding integrals $\int d(p)_l d(q)_m \ldots$. Recall that $E(p), E(q)$ are bounded in D. We have proved the following theorem.

Theorem 16. *The Hamiltonian* $H_{a,r,\Lambda}$ *in finite cube* Λ (16.2′), (16.11) *converges to the Hamiltonian* $H_{a,r}$ *with* $\Lambda = R^3$ *on the states* $\varphi^a_{lm,\Lambda}$ *in sence of* (16.10).

2. The second method. It consists in the following. We define the second ground state Φ_0 as an eigenvector of two operators. The first operator is

$$I_1 = c\int v(p)a(-p)a(p)dp - c^2g^{-1}VI, \tag{16.12}$$

the second operator is

$$I_2 = \int\left(\frac{p^2}{2m}-\mu\right)a^+(\bar{p})a(\bar{p})d\bar{p} + c\int v(p)a^+(p)a(-p)dp, \tag{16.13}$$

c is a constant to be defined later.

Note that the sum of two operators (16.12), (16.13) is equal to H_a.

The eigenvalue problem for the first operator and the ground state Φ_0 is formulated as follows

$$I_1\Phi_0 = \left(c\int v(k)a(-k)a(k)dk - c^2g^{-1}VI\right)\Phi_0 = 0. \tag{16.14}$$

It follows from (16.14) that Φ_0 is the following coherent state

$$\Phi_0 = e^{\int f(k)a^+(k)a^+(-k)dk}|0\rangle$$

with an arbitrary function $f(k)$ and constant $c = g\int v(k)f(k)dk$. One puts $\delta(o) = V$, as its commonly accepted. (We use that in Φ_0 the pairs with the same momenta are absent.) The term $-c^2g^{-1}VI$ compensates divergent as the volume V result of action of the operator $c\int v(k)a(-k)a(k)dk$ on Φ_0. It resembles counterterms in quantum field theory.

To determine the function $f(k)$ we postulate that Φ_0 (or its components with n-pairs) is an eigenvector of the second operator (16.13)

$$I_1\Phi_0 = \left[\int\left(\frac{k^2}{2m}-\mu\right)a^+(\bar{k})a(\bar{k})d\bar{k} + c\int v(k)a^+(k)a^+(-k)dk\right]\Phi_0 = E\Phi_0. \tag{16.15}$$

Eigenvalue problem (16.12) is reduced to the following set of equations

$$\sum_{i=1}^{N}\left[\left(\frac{2k_i^2}{2m}-2\mu\right)f(k_1)\dots f(k_n)+f(k_1)\dots cv(k_i)\dots f(k_n)\right]= \\ =Ef(k_1)\dots f(k_n),\quad n\geq 1. \tag{16.16}$$

It follows from (16.16) that eigenvalue problem (16.15) is degenerated and for given n has eigenvalue $E=E_n=nE_0$ where $f(k)$ is solution of equations

$$\left(\frac{2k^2}{2m}-2\mu\right)f(k)+cv(k)=E_0f(k) \tag{16.17}$$

or

$$f(k)=\frac{cv(k)}{E_0-\dfrac{2k^2}{2m}+2\mu}.$$

Substituting $f(k)$ in expression for the constant c one obtains equation for E_0

$$1=g\int\frac{v^2(k)dk}{E_0-\dfrac{2k^2}{2m}+2\mu}$$

that has the unique negative solution E_0 such that $\min\limits_{k\in D}|E_0-\frac{2k^2}{2m}+2\mu|>\Delta>0$ i.e. E_0 is the lowest eigenvalue of eigenvalue problem (16.17) and Δ is the gap in the spectrum (see detail in [1], Sect. 6). We will use the function $f(k)$ (16.17) normalized to unity and denote it by

$$f_0(k)=\frac{v(k)}{E_0-\dfrac{2k^2}{2m}+2\mu}\left(\int\frac{v^2(k)dk}{\left(E_0-\dfrac{2k^2}{2m}+2\mu\right)^2}\right)^{-\frac{1}{2}}.$$

Thus from equation (16.15) one obtains Φ_0 and solution of the set of equations (16.16) with function $f(k)=f_0(k)$ (15.9).

The coherent state Φ_0 is eigenvector of the operator

$$\int\left(\frac{p^2}{2m}-\mu\right)a^+(\bar p)a(\bar p)d\bar p+c\int v(p)a^+(p)a^+(-p)dp-\frac{E_0}{2}N,$$
$$N=\int a^+(\bar p)a(\bar p)d\bar p$$

with eigenvalue zero.

We obtained the second ground state Φ_0 of the approximating Hamiltonian H_a.

Now introduce new operators of creation and annihilation for which Φ_0 is the vacuum. It is easy to check that the following operators

$$\alpha^+(k)=u(k)a^+(k)+w(k)a(-k),\qquad \alpha(k)=u(k)a(k)+w(k)a^+(-k),$$
$$\alpha^+(-k)=u(k)a^+(-k)-w(k)a(k),\qquad \alpha(-k)=u(k)a(-k)-w(k)a^+(k),$$
$$u(k)=(1+f_0^2(k))^{-\frac{1}{2}},\qquad w(k)=-f_0(k)(1+f_0^2(k))^{-\frac{1}{2}} \tag{16.18}$$

with $f_0(k)$ defined according (16.17), have the property

$$\alpha(\pm k)\Phi_0=0,$$

i.e. Φ_0 is the vacuum for operators (16.18).

Define the excited states of the vacuum Φ_0

$$\varphi(\bar{p_1},\dots,\bar{p_l})=\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\Phi_0 \tag{16.19}$$

and suppose that $\bar{p_i}\neq-\bar{p_j}$ i.e., that there are not operators with opposite momenta and spin.

Repeating the calculations analogical to Φ_0 one can prove that

$$\begin{gathered}\left(H_a-\frac{E_0}{2}N\right)\Phi_0=0,\\ \left(H_a-\frac{E_0}{2}N\right)\varphi(\bar{p_1},\dots,\bar{p_l})=\left(\sum_{i=1}^{l}\varepsilon(p_i)-\frac{E_0}{2}l\right)\varphi(\bar{p_1},\dots,\bar{p_l})\end{gathered} \tag{16.20}$$

i.e., Φ_0 and its excited states $\varphi(\bar{p_1},\dots,\bar{p_l})$ are eigenvectors of $H_a-\dfrac{E_0}{2}N$.

It is obvious that excited states with n pairs in Φ_0

$$\varphi(\bar{p_1},\dots,\bar{p_l})^n=\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\frac{1}{n!}\int f_0(k_1)\dots f_0(k_n)a^+(k_1)a^+(-k_1)\dots$$
$$\dots a^+(k_n)a^+(-k_n)|0\rangle dk_1\dots dk_n=\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\Phi_0^n$$

are also eigenvectors of I_2 with eigenvalues $nE_0+\sum_{i=1}^{l}\varepsilon(p_i)$.

In proving (16.20) it was used the fact that the operator $c\int v(k)a(-k)a(k)dk$ acts only on Φ_0 due to absence in $\varphi(\bar{p_1},\dots,\bar{p_l})$ pairs of operators $\alpha^+(p_i)\alpha^+(-p_i)$ and the following formula

$$\varphi(\bar{p_1},\dots,\bar{p_l})=\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})\Phi_0=$$
$$=(1+f_0^2(p_i))^{\frac{1}{2}}\dots(1+f_0^2(p_l))^{\frac{1}{2}}a^+(\bar{p_1})\dots a^+(\bar{p_l})\Phi_0$$

(see [2], Sect. 10). The system of excited states $\varphi(\bar{p_1},\dots,\bar{p_l})$ is orthogonal.

Note that

$$\alpha^+(p_1)\alpha^+(-p_1)\Phi_0=(-f_0(p_1)+a^+(p_1)a^+(-p_1))\Phi_0 \tag{16.21}$$

therefore

$$\left[c\int v(k)a(-k)a(k)dk-c^2g^{-1}VI\right](\alpha^+(p_1)\alpha^+(-p_1)\Phi_0)=$$
$$=\left[-f_0(p_1)c^2g^{-1}V\Phi_0+cv(p_1)V\Phi_0\right]\neq0. \tag{16.21′}$$

This means that state (16.21) with one (or more) excited pairs can not be eigenvector of H_a in the framework of the second method. Later in the end of the next section we will construct a proper excitations of pairs.

The obtained above results we summarize in the following theorem.

Theorem 17. *The approximating Hamiltonian* $H_a-\dfrac{E_0}{2}N$ *has the orthogonal system of eigenvectors* $\varphi(\bar{p_1},\dots,\bar{p_l})$, $l\geq0$, *with eigenvalues* $\sum_{i=1}^{l}\varepsilon(p_i)-\dfrac{E_0}{2}l$. *The Hamiltonian* I_2 *has the orthogonal system of eigenvectors* $\varphi(\bar{p_1},\dots,\bar{p_l})^n$ *with eigenvalues* $nE_0+\sum_{i=1}^{l}\varepsilon(p_i)$, $\bar{p_l}\neq-\bar{p_j}$.

In the next section it will be shown that $\varphi(\bar{p_1},\dots,\bar{p_l})$ are eigenvectors of the model Hamiltonian with the same eigenvalues. The system $\varphi(\bar{p_1},\dots,\bar{p_l})$ can be used as a basis of the Hilbert space $\mathcal{H}^F\bigotimes\Phi_0$ with the following normalized elements

$$\varphi_l=\frac{1}{l!}\int\psi_l(\bar{p_1},\dots,\bar{p_l})\varphi(\bar{p_1},\dots,\bar{p_l})d\bar{p_1}\dots d\bar{p_l}=$$

$$= \frac{1}{l!} \int \psi_l(\bar{p_1}, \dots, \bar{p_l}) \alpha^+(\bar{p_1}) \dots \alpha^+(\bar{p_l}) dp_1 \dots dp_l \Phi_0 \tag{16.22}$$

where $\psi_l(\bar{p_1}, \dots, \bar{p_l})$ is antisymmetric and

$$\frac{1}{l!} \int |\psi_l(\bar{p_1}, \dots, \bar{p_l})|^2 d\bar{p_1} \dots d\bar{p_l} < \infty.$$

The scalar product of two element φ_l and g_l, where

$$g_l = \frac{1}{l!} \int h_l(\bar{p_1}, \dots, \bar{p_l}) \alpha^+(\bar{p_1}) \dots \alpha^+(\bar{p_l}) d\bar{p_1} \dots d\bar{p_l} \Phi_0,$$

is defined as follows

$$\begin{gathered} (\varphi_l, g_l) = \frac{1}{l!} \int \overline{\psi_l(\bar{p_1}, \dots, \bar{p_l})} h_l(\bar{p_1}, \dots, \bar{p_l}) d\bar{p_1} \dots d\bar{p_l} (\Phi_0, \Phi_0), \\ \|\varphi_l\|^2 = \frac{1}{l!} \int |\psi_l(\bar{p_1}, \dots, \bar{p_l})|^2 d\bar{p_1} \dots d\bar{p_l} (\Phi_0, \Phi_0). \end{gathered} \tag{16.23}$$

We have

$$\left(H_a - \frac{E_0}{2} N \right) \varphi_l =$$

$$= \frac{1}{l!} \int \left(\sum_{i=1}^{l} \varepsilon(p_i) - \frac{E_0}{2} l \right) \psi_l(\bar{p_1}, \dots, \bar{p_l}) \varphi(\bar{p_1}, \dots, \bar{p_l}) d\bar{p_1} \dots d\bar{p_l} d\bar{p_1} \dots d\bar{p_l} \Phi_0. \tag{16.24}$$

In proving (16.24) we suppose that $\psi_l(\bar{p_1}, \dots, \bar{p_l})$ is equal to zero if some $\bar{p_i} = -\bar{p_j}$.

Remark. In what follows we will use the vacuum states $\Phi_{0,\Lambda}$ and $\Phi^a_{0,\Lambda}$ in the cube Λ with norm equal to unity $\dfrac{\Phi_0}{\|\Phi_0\|'_V}$, $\dfrac{\Phi^a_0}{\|\Phi^a_0\|'_V}$ and with the same denotation Φ_0 and Φ^a_0.

Consider again excited state $\varphi(p_1, \dots, p_l)$ in finite cube Λ

$$\varphi_{\bar{p_1}, \dots, \bar{p_l}} = \alpha^+_{\bar{p_1}} \dots \alpha^+_{p_l} \Phi_0 = \prod_{i=1}^{l} (1 + f_0^2(p_i))^{\frac{1}{2}} a^+_{\bar{p_1}} \dots a^+_{\bar{p_l}} \times$$

$$\times \prod_{k \neq (p)_l} (1 + f_0(k) a^+_k a^+_{-k}) |0\rangle \prod_k \left(1 + \frac{1}{V} f_0^2(k) \right)^{-\frac{1}{2}}.$$

Recall that in order to calculate $(\|\varphi_{\bar{p_1}, \dots, \bar{p_l}}\|'_V)^2$ we use canonical commutation relations and in obtained expression multiply all $f_0^2(k)$ by $\dfrac{1}{V}$ (see for detail [2], Sect. 10). Then one obtains

$$(\|\varphi_{\bar{p_1}, \dots, \bar{p_l}}\|'_V)^2 = \prod_{i=1}^{l} \left(1 + \frac{1}{V} f_0^2(p_i) \right) \prod_{i=1}^{l} \left(1 + \frac{1}{V} f_0^2(p_i) \right)^{-1} = 1.$$

One can also calculate norm of $\varphi_{\bar{p_1}, \dots, \bar{p_l}}$ using canonical commutation relations for $\alpha(\bar{p}), \alpha^+(\bar{p})$ and the fact that Φ_0 is the vacuum, i.e., $\alpha(\bar{k})\Phi_0 = 0$. Then one obtains again

$$(\|\varphi_{\bar{p_1}, \dots, \bar{p_l}}\|'_V)^2 = (\Phi_0, \Phi_0)'_V = 1.$$

Performed above calculations show that one obtains the same results by using the operators $\alpha^+(\bar{k})$, $\alpha(\bar{k})$ or the operators $a^+(\bar{k})$, $a(\bar{k})$.

Note that the norm of the state

$$a^+_{\bar{p}_1}\dots a^+_{\bar{p}_l}\Phi_0 = a^+_{\bar{p}_1}\dots a^+_{\bar{p}_l}\prod_{k\neq(p)_l}(1+f_0(k)a^+_k a^+_{-k})|0\rangle\prod_k\left(1+\frac{1}{V}f_0^2(k)\right)^{-\frac{1}{2}}$$

is not equal to unity for finite Λ, but becomes equal to unity in the limit $V\to\infty$. It follows from formula

$$(\|a^+_{\bar{p}_1}\dots a^+_{\bar{p}_l}\Phi_0\|'_V)^2 = \prod_{k\neq(p)_l}\left(1+\frac{1}{V}f_0^2(k)\right)\prod_k\left(1+\frac{1}{V}f_0^2(k)\right)^{-1} = \\ = \prod_{i=1}^{l}\left(1+\frac{1}{V}f_0^2(p_i)\right)^{-1}.$$

It is obvious that $\lim_{V\to\infty}\|a^+_{\bar{p}_1}\dots a^+_{\bar{p}_l}\Phi_0\|'_V = 1$.

17. Two methods of determination of the spectra and eigenvectors of the model Hamiltonian BCS. *1. The first method.* Consider the model BCS Hamiltonian [9] for infinite cube $\Lambda = R^3$

$$H = \int\left(\frac{p^2}{2m}-\mu\right)a^+(\bar{p})a(\bar{p})d\bar{p} + \\ +\frac{g}{V}\int v(p)v(p')a^+(p)a^+(-p)a(-p')a(p')dpdp' = H_0 + H_I \tag{17.1}$$

where $V = V(R^3)$ is the volume of the three-dimensional space R^3.

The model Hamiltonian (17.1) has a rigorous meaning in the Hilbert space of translation-invariant functions and its spectra has been investigated in detail [4 – 6]. We present a short review of these results.

Let us consider the following coherent state

$$\Phi_0 = e^{\int f_0(k)a^+(k)a^+(-k)dk}|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\int f_0(k_1)\dots f_0(k_n)\times \\ \times a^+(k_1)a^+(-k_1)\dots a^+(k_n)a^+(-k_n)dk_1\dots dk_n|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\Phi_0^n \tag{17.2}$$

and determine the normalized to unity function $f_0(k)$ from conditions that each Φ_0^n is an eigenvector of H with the lowest eigenvalue. From these conditions we obtain

$$\sum_{i=1}^{n}\left(\frac{2k_i^2}{2m}-2\mu\right)f_0(k_1)\dots f_0(k_n) + \sum_{i=1}^{n}\int v(k)f_0(k)dk f_0(k_1)\dots \overset{i}{v}(k_i)\dots f_0(k_n) = \\ = E_n f_0(k_1)\dots f_0(k_n), \tag{17.3}$$

i.e. $H\Phi_0^n = E_n\Phi_0^n$.

In obtaining (17.3) we again used the identity $\frac{1}{V}\delta(0) = 1$, and the fact that, according to the Fermi statistics, in Φ_0 pairs with the same momenta are absent.

By using the method of separation of variables one concludes that $f_0(k)$ is the solution of the equations

$$\left(\frac{2k^2}{2m}-2\mu\right)f_0(k) + cv(k) = E_0 f_0(k), \qquad c = \int v(k)f_0(k)dk, \\ 1 = g\int\frac{v^2(k)}{-\frac{2k^2}{2m}+2\mu+E_0}dk. \tag{17.4}$$

It was shown in the previous section that for considered potential there exists unique solution of the last equation (17.4) $E_0 < 0$ that is divided from the rest of spectra by the gap $|\Delta| > 0$ (see [1], Sect. 6). The normalized to unity function $f_0(k)$ is the following

$$f_0(k) = \frac{v(k)}{E_0 - \frac{2k^2}{2m} + 2\mu} \left(\int \frac{v^2(k)dk}{\left(E_0 - \frac{2k^2}{2m} + 2\mu\right)^2} \right)^{-\frac{1}{2}} \tag{17.5}$$

and $|E_0 - \frac{2k^2}{2m} + 2\mu| \geq \Delta > 0$. The eigenvalue $E_n = nE_0$. The coherent state Φ_0 is completely determined.

If one consider the renormalized Hamiltonian

$$H_r = H - \frac{E_0}{2}N, \qquad N = \int a^+(\bar{p})a(\bar{p})d\bar{p}$$

then the coherent state Φ_0 is its eigenvector with eigenvalue zero

$$H_r\Phi_0 = 0. \tag{17.6}$$

We can repeat the calculation from the previous section, introduce the operators of creation and annihilation of quasiparticles (16.18) for which Φ_0 is the vacuum. The excited states

$$\varphi(\bar{p_1}, \ldots, \bar{p_l}) = \alpha^+(\bar{p_1}) \ldots \alpha^+(\bar{p_l})\Phi_0$$

are the eigenvectors of H_r with eigenvalues $\sum_{i=1}^{l} \varepsilon(p_i) - \frac{E_0}{2}l$. The states $\alpha^+(\bar{p_1}) \ldots$ $\ldots \alpha^+(\bar{p_l})\Phi_0^n$ are the eigenvectors of H with eigenvalues $nE_0 + \sum_{i=1}^{l} \varepsilon(p_i)$. (We suppose, as in previous section, that $p_i \neq -p_j$ for all $1 \leq i \leq l$.)

The proof of these statements follows directly from representation

$$\varphi(\bar{p_1}, \ldots, \bar{p_l}) = (1 + f_0^2(p_1))^{\frac{1}{2}} \ldots (1 + f_0^2(p_l))^{\frac{1}{2}} a^+(\bar{p_1}) \ldots a^+(\bar{p_l})\Phi_0$$

and from an observation that

$$H_I\varphi(\bar{p_1}, \ldots, \bar{p_l}) = (1 + f_0^2(p_1))^{\frac{1}{2}} \ldots (1 + f_0^2(p_l))^{\frac{1}{2}} a^+(\bar{p_1}) \ldots a^+(\bar{p_l})H_I\Phi_0$$

in virtue of $\bar{p_i} \neq -\bar{p_j}$ (see also [2], Sect. 8). We summarize the above obtained results in the following theorem.

Theorem 18. *The renormalized model Hamiltonian BCS* $H_r = H - \frac{E_0}{2}N$ *has eigenvectors* Φ_0, $\varphi(\bar{p_1}, \ldots, \bar{p_l}), \bar{p_l} \neq -\bar{p_j}, l \geq 1$, *with eigenvalues* 0, $\sum_{i=1}^{l} \varepsilon(p_i) - \frac{E_0}{2}l$ *respectively. The Hamiltonian* H *has eigenvectors* $\Phi_0^n, \varphi^n(\bar{p_1}, \ldots, \bar{p_l})$, $n \geq 0, l > 1$, *with eigenvalues* $nE_0, nE_0 + \sum_{i=1}^{l} \varepsilon(p_i)$ *respectively. Note that* $\left|\sum_{i=1}^{l} \varepsilon(p_i) - \frac{E_0}{2}l\right| > \frac{\Delta}{2}l$, *i.e., there is the gap in the spectra of the operator* H_r.

Consider the renormalized excited state

$$\Phi_l = \frac{1}{l!} \int \Psi_l(\bar{p_1}, \ldots, \bar{p_l})a^+(\bar{p_1}) \ldots a^+(\bar{p_l})d\bar{p_1} \ldots d\bar{p_l}\Phi_0 =$$

$$= \frac{1}{l!} \int \Psi_l(p_1, \ldots, p_l)d\bar{p_1} \ldots d\bar{p_l}\Phi_{(\bar{p})_l}, \qquad (\bar{p})_l = (\bar{p_1}, \ldots, \bar{p_l})$$

and corresponding state in a finite cube Λ

$$\Phi_{l,\Lambda} = \frac{1}{l!} \sum_{\bar{p_1},\dots,\bar{p_l}} \Psi_l(\bar{p_1},\dots,\bar{p_l}) a^+_{\bar{p_1}} \dots a^+_{\bar{p_l}} \Phi_{0,\Lambda} = \frac{1}{l!} \sum_{\bar{p_1},\dots,\bar{p_l}} \Psi_l(\bar{p_1},\dots,\bar{p_l}) \Phi_{(\bar{p})_l,\Lambda}.$$

(Note that in the previous papers $[1-5]$ we denote $\Phi_{0,\Lambda}$, $\Phi_{(\bar{p})_l}, \Lambda$ by Φ_0, $\Phi_{(p)_l}$ because we considered systems only in a finite cube Λ.)

We have

$$\|\Phi_l\|^2 = \frac{1}{l!} \int |\Psi_l(\bar{p_1},\dots,\bar{p_l})|^2 d\bar{p_1} \dots d\bar{p_l} \|\Phi_0\|^2,$$

$$(\|\Phi_{l,\Lambda}\|'_V)^2 = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{p_1 \neq \dots \neq p_l \neq (k_n)} \frac{1}{V^l} |\Psi_l(p_1,\dots,p_l)|^2 \times$$
$$\times \frac{1}{n!} \sum_{k_1 \neq \dots \neq k_n} \frac{1}{V^n} |f_0(k_1)|^2 \dots |f_0(k_n)|^2.$$

Obviously

$$\lim_{V\to\infty} (\|\Phi_{l,\Lambda}\|'_V)^2 = \|\Phi_l\|^2. \tag{17.7}$$

Further we have

$$H_r \Phi_l = \frac{1}{l!} \int \left(\sum_{i=1}^{l} \varepsilon(p_i) - \frac{E_0}{2} l \right) \Psi_l(\bar{p_1},\dots,\bar{p_l}) a^+(\bar{p_1}) \dots a^+(\bar{p_l}) d\bar{p_1} \dots d\bar{p_l} \Phi_0,$$

$$H_{r,\Lambda} \Phi_{l,\Lambda} = \sum_{n=0}^{\infty} \frac{1}{n!} \Bigg\{ \sum_{\substack{\bar{p_1},\dots,\bar{p_l} \\ k_1,\dots,k_n}} \Psi_l(\bar{p_1},\dots,\bar{p_l}) \Bigg[\Bigg(\sum_{i=1}^{l} \varepsilon(p_i) + \sum_{i=1}^{n} 2\varepsilon(k_i) -$$
$$- \frac{E_0(L)}{2}(l+2n) \Bigg) \Psi_l(\bar{p_1},\dots,\bar{p_l}) f_0(k_1) \dots f_0(k_n) +$$
$$+ \frac{g}{V} \sum_{i=1}^{n} \sum_{p} v(k_i) v(p) f_0(k_1) \dots \overset{i}{\overline{f_0}}(p) \dots f_0(k_n) \Bigg] \Bigg\} a^+_{\bar{p_1}} \dots a^+_{\bar{p_l}} \times$$
$$\times a^+_{k_1} a^+_{-k_1} \dots a^+_{k_n} a^+_{-k_n} |0\rangle + \frac{1}{l!} \sum_{(p_1,\dots,p_l)} \Psi(p_1,\dots,p_l) B_{(p)_l} \Phi_{(p)_l,\Lambda} \tag{17.8}$$

(see [2], Sect. 8, formulae (8.3), (8.13)).

We have from (17.7), (17.8)

$$\lim_{V\to\infty} (\|H_{r,\Lambda} \Phi_{l,\Lambda}\|'_V)^2 = \|H_r \Phi_l\|^2 \tag{17.9}$$

because

$$\lim_{V\to\infty} \left\| \frac{1}{l!} \sum_{(p_1,\dots,p_l)} \psi(p_1,\dots,p_l) B_{(p)_l} \Phi_{(p)_l,\Lambda} \right\|'_V = 0$$

(see [2], Sect. 8, formulae (8.14)).

We also take into account that

$$\left(\frac{2k^2}{2m} - 2\mu - E_0 \right) f_0(k) + c v(k) = 0, \quad c = \int v(k) f_0(k) dk, \qquad \lim_{l\to\infty} E_0(L) = E_0.$$

We have proved the following theorem.

Theorem 19. *The Hamiltonian* $H_{r,\Lambda}$ *in finite cube* Λ *converges to the Hamiltonian* H_r *in the whole space* $\Lambda = R^3$ *on excited states in sence* (17.9).

One can consider the excited states φ_l and $\varphi_{l,\Lambda}$ with the operators $\alpha^+(\bar{p_1}),\ldots$ $\ldots\alpha^+(\bar{p_l})$ or $\alpha^+_{\bar{p_1}},\ldots,\alpha^+_{\bar{p_l}}$ instead of the operators $a^+(\bar{p_1}),\ldots,a^+(p_l)$ or $a^+_{\bar{p_1}},\ldots$ $\ldots,a^+_{\bar{p_l}}$. The Theorem 19 is true also in this case. It is sufficient to replace $\Psi_l(\bar{p_1},\ldots,\bar{p_l})$ by $\Psi_l(\bar{p_1},\ldots,\bar{p_l})\prod_{i=1}^{l}(1+f_0(p_i)^2)^{\frac{1}{2}}$.

*2. **Excited states of pair.*** As was shown in the previous subsection equation for eigenvalue of one pair has a unique solution E_0

$$1=g\int\frac{v^2(k)dk}{-\dfrac{2k^2}{2m}+2\mu+E_0}$$

and corresponding eigenvector

$$f_0(k)=\frac{v(k)}{E_0-\dfrac{2k^2}{2m}+2\mu}\left(\int\frac{v^2(k)dk}{\left(E_0-\dfrac{2k^2}{2m}+2\mu\right)^2}\right)^{-\frac{1}{2}}$$

such that $\int dk|f^0(k)|^2<\infty$.

The rest of eigenvectors belong to continuous spectra and we determine them using equation for eigenvectors and from condition of orthogonality to $f_0(k)$ and to $v(k)$. Namely, we represent eigenvectors that correspond to the continuous spectra $-\omega\le E\le$ $\le\omega$ as follows

$$f_E(k)=f_E(|k|)Y_{ml}(\theta,\varphi),\quad |m|+l\ge 1.$$

Then equation for eigenvector $f_E(k)$

$$\left(\frac{2k^2}{2m}-2\mu\right)f_E(k)+v(k)\int v(k)f_E(k)dk=Ef_E(k)$$

is reduced to the following equation

$$\left(\frac{2k^2}{2m}-2\mu\right)f_E(k)=Ef_E(k) \tag{17.10}$$

due to the condition of orthogonality $\int f_E(k)v(k)dk=0$ (see Section 16, formula (16.8)). From (17.10) it follows that

$$\left(\frac{2k^2}{2m}-2\mu\right)f_E(|k|)=Ef_E(|k|)$$

and

$$f_E(|k|)=\delta\left(\frac{2k^2}{2m}-2\mu-E\right).$$

Thus, solution of equation (17.10) is

$$f_E(k)=\delta\left(\frac{2k^2}{2m}-2\mu-E\right)Y_{ml}(\theta,\varphi),\quad |m|+l\ge 1. \tag{17.11}$$

The general excited state of pair is superposition of functions (17.11)

$$f(k)=\int\limits_{-\omega}^{\omega}dE\delta\left(\frac{2k^2}{2m}-2\mu-E\right)\sum_{l,m}c_{l,m}Y_{ml}(\theta,\varphi) \tag{17.12}$$

with

$$\sum_{l,m,\, l+|m|\geq 1} |c_{l,m}|^2 < \infty.$$

Note that $f_E(k)$ are orthogonal to $f_0(k)$.

Now construct the general excited state of the ground state Φ_0 with l electrons (or quasiparticles) and m excited pairs

$$\varphi_{l,m} = \frac{1}{l!}\frac{1}{m!}\int \Psi_l(\bar{p_1},\dots,\bar{p_l})\,\mathrm{sym}(f_1(q_1)\dots f_m(k_m))a^+(\bar{p_1})\dots a^+(\bar{p_l})\times$$
$$\times a^+(q_1)a^+(-q_1)\dots a^+(q_m)a^+(-q_m)d\bar{p_1}\dots d\bar{p_l}dq_1\dots dq_m\Phi_0, \quad l+m\geq 1, \tag{17.13}$$

where

$$f_i(q) = \delta\left(\frac{2k^2}{2m} - 2\mu - E_i\right)\sum_{l,m} c^i_{l,m}Y_{ml}(\theta,\varphi), \quad -\omega\leq E_i\leq\omega,$$
$$E_i\neq E_j, \quad (i,j)\subset(1,\dots,m).$$

Note that the functions $f_1(q_1),\dots,f_m(q_m)$ are generalized ones.

We have

$$H_r\varphi_{l,m} = \frac{1}{l!m!}\int \Psi_l(\bar{p_1},\dots,\bar{p_l})\,\mathrm{sym}(f_1(q_1)\dots f_m(q_m))\times$$
$$\times\left[\sum_{i=1}^{l}\left(\varepsilon(p_i) - \frac{E_0}{2}\right) + \sum_{i=1}^{m}(2\varepsilon(q_i) - E_0)\right]a^+(\bar{p_1})\dots a^+(p_l)\times$$
$$\times a^+(q_1)a^+(-q_1)\dots a^+(q_m)a^+(-q_m)d\bar{p_1}\dots d\bar{p_l}dq_1\dots dq_m\Phi_0, \tag{17.14}$$
$$2\varepsilon(q_i) = E_i, \quad H_r\Phi_0 = 0.$$

If one replaces the operators of creation of electrons $a^+(\bar{p_1})\dots a^+(\bar{p_l})$ by the operators of creations of quasiparticles $\alpha^+(\bar{p_1})\dots\alpha^+(\bar{p_l})$ then formula (17.13) will be true, it is sufficient to put under integral sign the factor $\prod_{i=1}^{l}(1+f_0^2(p_i))^{\frac{1}{2}}$.

Remark. We have already constructed excited states of pair and general excited state of the ground state Φ_0 namely $\varphi_{l,m}$ (17.13). Now we are able to investigate the operator $H_a - \dfrac{E_0}{2}N$ on $\varphi_{l,m}$ by the second method (see Section 16). It follows from the orthogonality $v(k)$ and $f_i(k)$ that $\int v(q)f_i(q)dq = 0$ and we put $\lim_{V\to\infty} V\int v(q)f_i(q)dq = 0$. It is again $\bar{p_i}\neq\bar{p_j}$ for all pair $(i,j)\subset(1,\dots,l)$.

Then the operator $c\int v(p)a(-p)a(p)dp$ acts only on Φ_0 in $\varphi_{l,m}$ and its action cancels with the operator $c^2g^{-1}VI$, i.e. $I_1\varphi_{lm} = 0$.

Obviously that

$$\left(H_a - \frac{E_0}{2}N\right)\varphi_{l,m} =$$
$$= \left[\int\left(\frac{p^2}{2m} - \mu - \frac{E_0}{2}\right)a^+(\bar{p})a(\bar{p})d\bar{p} + c\int v(p)a^+(p)a^+(-p)dp\right]\varphi_{l,m} =$$
$$= \frac{1}{l!m!}\int \psi_l(\bar{p_1},\dots,\bar{p_l})\mathrm{sym}(f_1(q_1)\dots f_m(q_m))\times$$
$$\times\left[\sum_{i=1}^{l}\left(\varepsilon(p_i) - \frac{E_0}{2}\right) + \sum_{i=1}^{m}(2\varepsilon(q_i) - E_0)\right]a^+(\bar{p_1})\dots a^+(\bar{p_l})\times$$
$$\times a^+(q_1)a^+(-q_1)\dots a^+(q_m)a^+(-q_m)d\bar{p_1}\dots d\bar{p_l}dq_1\dots dq_m\Phi_0 = H_r\varphi_{l,m}. \tag{17.15}$$

It follows from (17.14), (17.15) that the excited state

$$\varphi(\bar{p_1},\dots,\bar{p_l})_m = a^+(\bar{p_1})\dots a^+(\bar{p_l})\frac{1}{m!}\int \mathrm{sym}(f_1(q_1)\dots f_m(q_m))\times$$
$$\times a^+(q_1)a^+(-q_1)\dots a^+(q_m)a^+(-q_m)dq_1\dots dq_m\Phi_0$$

is eigenvector of the Hamiltonians $H_a - \frac{E_0}{2}N$, H_r with eigenvalues $\left[\sum_{i=1}^{l}\left(\varepsilon(p_i) - \frac{E_0}{2}\right) + \sum_{i=1}^{m}(E_i - E_0)\right]$, i.e.

$$\left(H_a - \frac{E_0}{2}N\right)\varphi(\bar{p_1},\dots,\bar{p_l})_m =$$
$$= \left[\sum_{i=1}^{l}\left(\varepsilon(p_i) - \frac{E_0}{2}\right) + \sum_{i=1}^{m}(E_i - E_0)\right]\varphi(\bar{p_1},\dots,\bar{p_l})_m,$$
$$H_r\varphi(\bar{p_1},\dots,\bar{p_l})_m = \left[\sum_{i=1}^{l}\left(\varepsilon(p_i) - \frac{E_0}{2}\right) + \sum_{i=1}^{m}(E_i - E_0)\right]\varphi(\bar{p_1},\dots,\bar{p_l})_m. \qquad (17.16)$$

3. The second method. Consider the ground state Φ_0^a (15.10) of the approximating Hamiltonian H_a and the action of the model Hamiltonian H on Φ_0^a. We obtain by analogy with (17.3)

$$H\Phi_0^a = \sum_{n=1}^{\infty}\int\left\{\sum_{i=1}^{n}\left[\left(\frac{2k_i^2}{2m} - 2\mu\right)f_0^a(k_1)\dots f_0^a(k_i)\dots f_0^a(k_n) + \right.\right.$$
$$\left.\left. + f_0^a(k_1)\dots c_1\,\overset{i}{\overline{v(k_i)}}\,f_0^a(k_1)\dots f_0^a(k_n)\right]\right\}a^+(k_1)a^+(-k_1)\dots a^+(k_n)a^+(-k_n)|0\rangle \qquad (17.17)$$

where

$$c_1 = g\int v(p)f^a(p)dp.$$

Note that, according to (15.10′) the constant c_1 is function depending on c, i.e. $c_1 = c_1(c)$.

It is easy to show by direct calculation that

$$H\Phi_0^a = \left[\int\left(\frac{p^2}{2m} - \mu\right)a^+(\overline{p})a(\overline{p})d\overline{p} + c_1\int v(p)a^+(p)a^+(-p)dp\right]\Phi_0^a \qquad (17.18)$$

(see, for example, (17.3) with $f_0^a(k)$ instead of $f_0(k)$). The state Φ_0^a is coherent one and, as in Section 16, formulae (16.14), one has

$$\left[c_1\int v(p)a(-p)a(p)dp - g^{-1}c_1^2VI\right]\Phi_0^a = 0. \qquad (17.19)$$

Taking into account equalities (17.18), (17.19) one obtains

$$H\Phi_0^a = \left[\int\left(\frac{p^2}{2m} - \mu\right)a^+(\overline{p})a(-\overline{p})d\overline{p} + c_1\int v(p)a^+(p)a^+(-p)dp + \right.$$
$$\left. + c_1\int v(p)a(-p)a(p)dp - g^{-1}c_1^2VI\right]\Phi_0^a = H_a\Phi_0^a. \qquad (17.20)$$

If one uses the operators of creation and annihilation of quasiparticles $\alpha^+(\overline{k})$, $\alpha(\overline{k})$ defined according (16.3) then one obtains

$$H\Phi_0^a = H_a\Phi_0^a = \int \left[E(p)\alpha^+(\overline{p})\alpha(\overline{p})d\overline{p} + C(c_1)VI\right]\Phi_0^a = C(c_1)V\Phi_0^a,$$
$$\alpha(\overline{p})\Phi_0^a = 0, \qquad C(c_1) = \int \left[\varepsilon(k) - (\varepsilon(k) + c_1^2 v^2(k))^{1/2}\right]dk - g^{-1}c_1^2. \tag{17.21}$$

Note that $C(c_1)$ depends on c_1 the same way as $C(c)$ depends on c.

Earlier in Section 16 the constant c was defined from the condition of minimum of $C(c)$ with respect to c. Now we define the constant c_1 from the condition of selfconsistene $c_1 = g\int v(p)f^a(p)dp$ where in $f^a(p)$ we put the same constant c_1. It follows that the constant $c_1 > 0$ should satisfy the following equation of selfconsistence for $v(p) = v > 0$

$$c_1 = -gv\int\sqrt{\frac{c_1^2}{(\sqrt{\varepsilon^2(k)+c_1^2}+\varepsilon(k))^2}}dk = -gvc_1\int\frac{dk}{\sqrt{\varepsilon^2(k)+c_1^2}+\varepsilon(p)}$$

or

$$1 = -gv\int\frac{dk}{\sqrt{\varepsilon^2(k)+c_1^2}+\varepsilon(k)}.$$

Deriving this equation we put $\sqrt{c_1^2} = c_1$, $c_1 > 0$ because integrand $vf^a(k) = -v\dfrac{w(k}{u(k)}$ should be negative.

Note that in this case the constant c_1 defined from equation of selfconsistence but not from the condition of minimum of energy of the ground state.

It follows from (17.21) that Φ_0^a is also eigenvector of H with the eigenvalue $C(c_1)V$.

Now consider the action of H on excited states

$$\varphi^a(\overline{p}_1,\ldots,\overline{p}_l) = \alpha^+(\overline{p}_1)\ldots\alpha^+(\overline{p}_l)\Phi_0^a, \tag{17.22}$$

with $\bar{p}_i \neq -\bar{p}_j$, $(i,j)\subset(1,\ldots,l)$.

Taking into account that H_I acts only on Φ_0^a due to the condition $\bar{p}_i \neq -\bar{p}_j$ one obtains an analog of (17.16)

$$H\varphi^a(p_1,\ldots,p_l) = \sum_{n=0}^{\infty}\frac{1}{n!}\int\Bigg\{\sum_{i=1}^{l}\varepsilon(p_i)f_0^a(k_1)\ldots f_0^a(k_n) +$$
$$+\sum_{i=1}^{n}\left[2\varepsilon(k_i)f_0^a(k_1)\ldots f_0^a(k_i)\ldots f_0^a(k_n) + f_1^a(k_1)\ldots c_1 v(k_i)\ldots f_1^a(k_n)\right]\Bigg\}\times$$
$$\times\prod_{i=1}^{l}\left(1+(f_0^a(p_i))^2\right)^{1/2}a^+(\overline{p}_1)\ldots a^+(\overline{p}_l)a^+(k_1)a^+(-k_1)\ldots$$
$$\ldots a^+(k_n)a^+(-k_n)|0\rangle dk_1\ldots dk_n =$$
$$= \left[\int\left(\frac{p^2}{2m}-\mu\right)a^+(\overline{p})a(\overline{p})d\overline{p} + c_1\int v(k)a^+(k)a^+(-k)dk\right]\varphi^a(\overline{p}_1,\ldots,\overline{p}_l). \tag{17.18$'$}$$

Taking into account that the operator $c_1\int v(k)a(-k)a(k)dk$ acts only on Φ_0^a in $\varphi^a(\overline{p}_1,\ldots,\overline{p}_l)$ due to the condition $\overline{p}_i \neq -\overline{p}_j$ one obtains an analog of (17.19), namely

$$\left[c_1\int v(k)a(-k)a(k)dk - g^{-1}c_1^2VI\right]\varphi^a(\overline{p}_1,\ldots,\overline{p}_l) = 0. \qquad (17.19')$$

From equalities (17.18′), (17.19′) one conclude that

$$\varphi^a(\overline{p}_1,\ldots,\overline{p}_l) = H_a\varphi^a(\overline{p}_1,\ldots,\overline{p}_l) =$$
$$= \int\left[E(p)\alpha^+(\overline{p})\alpha(\overline{p})d\overline{p} + C(c_1)VI\right]\varphi^a(\overline{p}_1,\ldots,\overline{p}_l) =$$
$$= \left(\sum_{i=1}^{l}E(p_i) + C(c_1)V\right)\varphi^a(\overline{p}_1,\ldots,\overline{p}_l). \qquad (17.23)$$

The above obtained results can be summarized in the following theorem.

Theorem 20. *The model Hamiltonian H coincides with the approximating Hamiltonian H_a on the ground and excited states Φ_0^a, $\varphi^a(\overline{p}_1,\ldots,-\overline{p}_l)$, $e\geq 1$, $\overline{p}_i \neq \overline{p}_j$, of the Hamiltonian H_a and formulae* (17.21), (17.23) *hold.*

Note that in our previous papers [1 – 3] it has been proved that

$$\lim_{V\to\infty}\frac{1}{V}\big(\Phi_0,(H_\Lambda - H_{a,\Lambda})\Phi_0\big) = 0, \qquad \lim_{V\to\infty}\frac{1}{V}(\Phi_0^a,(H_\Lambda - H_{a,\Lambda})\Phi_0^a) = 0$$

and analogous equality for excited states of Φ_0 and Φ_0^a.

In present paper we established directly for $\Lambda = R^3$ that

$$H\Phi_0 = H_a\Phi_0, \qquad H\Phi_0^a = H_a\Phi_0^a$$

and analogous equalities for excited states for Φ_0 and Φ_0^a.

These differences are connected with the following circumstances. For finite volume

$$\left(\sum_p v_pa_{-p}a_p - g^{-1}c^2VI\right)\Phi_0 =$$
$$= c\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{k_1,\ldots,k_n}\frac{1}{n!}f_0(k_1)\ldots f_0(k_n)a^+_{k_1}a^+_{-k_1}\ldots a^+_{k_n}a^+_{-k_n}|0\rangle\times$$
$$\times\sum_{k=k_1,\ldots,k=k_n}v_kf_0(k) = B_1\Phi_0, \qquad c = \frac{1}{V}\sum_k v_kf_0(k). \qquad (17.24)$$

We have estimate $\|B_1\Phi_0\|^2 \leq v^2(\alpha f^2 + 2\alpha^2f^6e^\alpha f^2)$, $f = \sup_k|f_0(k)|$ and therefore

$$\lim_{V\to\infty}\frac{1}{V}\|B_1\Phi_0\| = 0$$

and

$$\lim_{V\to\infty}\frac{1}{V^\delta}\|B_1\Phi_0\| = 0$$

even for arbitrary small $\delta > 0$ (see [1], formulae (7.7), (7.8)).

The operator B_1 is connected with Fermi statistic, according to which in Φ_0 pairs with the same momenta are absent and it compensate these absent momenta in $c = \frac{1}{V}\sum_k v_kf_0(k)$.

For infinite volume

$$c\int v(p)a(-p)a(p)dp\Phi_0 = g^{-1}c^2V\Phi_0$$

because $\int v(k)f_0(k)dk = c$ even if the function $f_0(k)$ is equal to zero on hyperplanes $k = k_1, \dots, k = k_n$ (according to Fermi statistic).

4. Ground state with excited pairs. Consider the following state with ground state Φ_0^a, m excited pairs, and l quasiparticles

$$\varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m = \frac{1}{m!} \int \operatorname{sym}\big(f_1(q_1) \dots f_m(q_m)\big) a^+(q_1)a^+(-q_1) \dots$$
$$\dots a^+(q_m)a^+(-q_m)dq_1 \dots dq_m \varphi^a(\overline{p}_1, \dots, \overline{p}_l), \qquad p_i \neq p_j, \tag{17.25}$$

where $f_1(q_1), \dots, f_m(q_m)$ are the excited states of pairs defined by formulae (17.11) with $E_1, \dots, E_m$.

Consider the model Hamiltonian H on $\varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m$. We have

$$H\varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m = \frac{1}{m!} \int \left(\sum_{i=0}^{m} E_i\right) \operatorname{sym}\big(f_1(q_1) \dots f_m(q_m)\big) a^+(q_1)a^+(-q_1) \dots$$
$$\dots a^+(q_m)a^+(-q_m)dq_1 \dots dq_m \varphi^a(\overline{p}_1, \dots, (\overline{p}_l) +$$
$$+\frac{1}{m!} \int \operatorname{sym}\big(f_1(q_1), \dots, f_m(q_m)\big) a^+(q_1)a^+(-q_1) \dots$$
$$\dots a^+(q_m)a^+(-q_m)dq_1 \dots dq_m H\varphi^a(\overline{p}_1, \dots, \overline{p}_l) =$$
$$= \left(\sum_{i=0}^{m} E_i\right) \varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m + \frac{1}{m!} \int \operatorname{sym}\big(f_1(q_1), \dots, f_m(q_m)\big) \times$$
$$\times a^+(q_1)a^+(-q_1) \dots a^+(q_m)a^+(-q_m)dq_1 \dots dq_m H_a \varphi^a(\overline{p}_1, \dots, \overline{p}_l) =$$
$$= \left(\sum_{i=0}^{m} E_i\right) \varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m + \left(\sum_{i=1}^{m} E(p_i) + C(c_1)V\right) \varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m. \tag{17.26}$$

Recall that we used in (17.26) the formulae (17.23).

We summarize the above obtained results in the following theorem.

Theorem 21. *The excited state* $\varphi^a(\overline{p}_1, \dots, \overline{p}_l)_m$ (17.25) *of the ground state* Φ_0^a *with* m *excited pairs with wave functions* $f_1(q_1), \dots, f_m(q_m)$ (17.11), (17.12) *and* l *quasiparticles with momenta* $\bar{p_1}, \dots, \bar{p_l}, \overline{p}_1 \neq -\overline{p}_l$, *is the eigenvector of the model Hamiltonian* H *with eigenvalue*

$$\sum_{i=0}^{m} E_i + \sum_{i=0}^{m} E(p_i) + C(c_a)V$$

and formulae (17.26) *holds.*

Using the same calculation as in Section 16 (see formulae (16.21), (16.21 ′)) one can show that the excitations

$$\alpha^+(q_1)\alpha^+(-q_1) \dots \alpha^+(q_m)\alpha^+(-q_m)\varphi^a(p_1, \dots, p_l), \quad l \geq 0,$$

are not the eigenvectors of the model Hamiltonian H, but they are eigenvectors of the approximating Hamiltonian H_a.

Earlier we showed that the state (16.21) $\alpha^+(p_1)\alpha^+(-p_1)\Phi_0$ can not be an eigenvector of H_a in the framework of the second method of H_a, i.e. it can not be an eigenvector of H. Note that $\varphi(\overline{p}_1, \dots, \overline{p}_l)^n$ is eigenvector of H but not of H_a.

Thus, there are some eigenvectors of H that are not eigenvectors of H_a and vice versa.

5. Concluding remarks. In the given paper we used the following approach to investigation of the model and approximating Hamiltonians directly for infinite volume. The ground states Φ_0, Φ_0^a and their excitations are represented by the operators of creations $a^+(\overline{k})$ or $\alpha^+(\overline{k})$ as usual elements of the Fock spaces. But we consider the sequences of functions that define Φ_0, Φ_0^a and their excitations as elements of the Hilbert space $\mathcal{H}^F \otimes \mathcal{H}^P$ and calculate scalar products and norms of these sequences in $\mathcal{H}^F \otimes \mathcal{H}^P$.

The ground states Φ_0, Φ_0^a and their excitations do not belong to the usual Fock space.

We define the action of the model and approximating Hamiltonians as usual, using canonical anticommutation relations as in the case of the Fock space. But results of action are again regarded as elements of the space $\mathcal{H}^F \otimes \mathcal{H}^P$.

Using above described approach we avoid divergences connected with infinite volume (see Section 15, formulae (15.11)).

For example, the average energy of the model Hamiltonian H over the ground state Φ_0 calculated in $\mathcal{H}^F \otimes \mathcal{H}^P$ is equal to

$$\frac{(\Phi_0, H\Phi_0)}{(\Phi_0, \Phi_0)} = \frac{1}{(\Phi_0, \Phi_0)} \sum_{n=1}^{\infty} \frac{1}{n!} \int f_0(k_1)\dots f_0(k_n) \sum_{i=1}^{n} \left[\left(\frac{2k_i^2}{2m} - \mu \right) f_0(k_1)\dots f_0(k_n) + \right.$$
$$\left. + \int f_0(k)v(k)dk f_0(k_1)\dots v(k_i)\dots f_0(k_n) \right] dk_1\dots dk_n =$$
$$= \frac{1}{(\Phi_0, \Phi_0)} \sum_{n=1}^{\infty} \frac{1}{n!} nE_0 \int f_0^2(k_1)\dots f_0^2(k_n) dk_1\dots dk_n = E_0 \int f_0^2(k)dk = E_0. \tag{17.27}$$

Thus, the average energy of H over the ground state Φ_0 is finite for infinite volume.

If one repeats the same calculation for the average energy of H over the ground state Φ_0 as in the usual Fock space one obtains $VE_0 \int f^2(k)dk = VE_0$.

This means that the average energy of H over Φ_0 per volume calculated in the usual Fock space is equal to the same average calculated in the space $\mathcal{H}^F \otimes \mathcal{H}^P$ but not per volume.

In the next paper we will investigate the model Hamiltonian proposed by Thirring and Ilieva [10, 11] directly for infinite $\Lambda = R^3$.

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