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FIRST-PASSAGE PROBABILITIES FOR RANDOMLY EXCITED MECHANICAL SYSTEMS BY A SELECTIVE MONTE CARLO SIMULATION METHOD

РОЗРАХУНОК ЙМОВІРНОСТЕЙ ПЕРШОГО ДОСЯГНЕННЯ ДЛЯ ВИПАДКОВО ЗБУРЕНИХ МЕХАНІЧНИХ СИСТЕМ ЗА ДОПОМОГОЮ СЕЛЕКТИВНОГО МЕТОДУ МОДЕЛЮВАННЯ МОНТЕ-КАРЛО

In this paper, Monte Carlo methods applied for the reliability assessment of structures under stochastic excitations are further advanced, i.e., by leading the generated samples towards the low probability range which is practically not accessible by direct Monte Carlo methods. Based on criteria denoting those realizations which lead most likely to failure, a simulation technique called the "Russian Roulette and Splitting" (RR&S) is presented and discussed briefly. In a numerical example, the RR&S procedure is compared with direct Monte Carlo simulation method (MCS) demonstrating comparative accuracy.

Продовжено модифікацію методів Монте-Карло для оцінки надійності структур при стохастичних збуреннях, а саме, згенеровані виборки зводяться до множини значень невеликої ймовірності, що практично неможливо при застосуванні прямих методів Монте-Карло. На основі критеріїв визначення тих реалізацій, що скоріш за все закінчуються невдачею, описано і стисло проаналізовано метод моделювання „Російська рулетка та розщеплення". Наведено приклад, в якому метод „Російська рулетка та розщеплення" порівнюється з прямим методом моделювання Монте-Карло і визначається їх порівняльна точність.

Introduction. In the design of structural systems, safety is an important issue to be considered. Reliability, which is defined as the probability that the system meets some specified demands for a specified time period under specified environmental conditions, is used as a probabilistic measure to evaluate the reliability of structural systems. The performance function of a structure system must be determined to describe the system's behavior and to identify the relationship between the basic parameters in the system. Simulation has been used to assess the reliability of structural systems.

When making considerations about structural safety, it is essential to appreciate that measure of safety based on a general probabilistic model in general does not express a physical property of the structure in its environments of actions. Rather the safety measure is a decision variable that embraces the applied knowledge about the strength properties of the structure in relation to the actions on the structure. The value of the safety measure therefore changes with the amount and quality information on the basis of which it is calculated. With this philosophy in mind, the structural reliability theory becomes a design decision tool based on scientific methods rather than being a scientific theory itself aiming at a description "truth of nature" [1].

The evaluation of the reliability of large technological systems such as nuclear power plants, chemical plants, offshore platforms is of paramount importance. These systems generally consist of various subsystems, such as electronic, mechanical and structural systems. They are affected by external hazards, such as earthquakes, as well as internal loading conditions. While the reliability of electronic systems, i.e., their components, can be determined by experimental procedures, this is not feasible for most of mechanical components and structures. For both mechanical components and structures, analytical and numerical procedures to predict the failure rates have to be utilized. For a view of the present state of development, i.e., the procedures used for determining the reliability of mechanical components in various technological areas, the reader is referred, e.g., to [2]. In developing procedures, it is important to keep in mind that, for both mechanical as well as probabilistic modeling, the state of the art procedures should be utilized .

The concept of limit state. If a structural failure can be attributed to a single uniquely defined mechanical cause, the corresponding mathematical model is said to define a failure mode in the space of relevant physical variables. This failure mode may be an idealization of the only possible way of failure of a single structural element, e.g. as in the case of a tension bar. It may also be an idealization of just one of several ways of failure of a structural element or system, e.g. as in the case of a reinforced concrete beam, that essentially may fail in bending, in shear, or by sliding of the reinforcement at a support (debonding). Usually it is possible to define the physical formulation space such that the failure mode defines a division of it in two parts separated by a smooth surface. The two parts are the safe set and the failure set relative to the failure mode. The surface may usually be defined uniquely by a differentiable function g of the physical variables as being the set of points for which g is zero. Note that while the surface is uniquely defined by the failure mode, the function g is not. In spite of this ambiguity, the function g is often called the failure function. The unique surface of its zero points is called the failure surface or the limit state. Conventionally, g is defined to be positive in the safe set.

Uncertainty modelling. The uncertainty sources that are relevant for the reliability evaluation may be classified according to their nature into physical uncertainty, statistical uncertainty, knowledge uncertainty, and model uncertainty. Physical uncertainty may be subdivided into the inherent uncertainly of the physical properties of the object itself (e.g., the natural fluctuations of the strength parameters through a specimen of material or from specimen to specimen, fluctuations of the wind pressure at a point of a building facade) and the inherent uncertainty of the measuring device. Both types of uncertainty admit probabilistic modelling in the relative frequency sense and theoretical distributions may be fitted to observed empirical distributions.

These various uncertainty sources, all being relevant for practical structural reliability evaluations, show that it is philosophically most satisfying to consider all input probabilities to a reliability model as being credibility (subjective probabilities) assessed by the user of the model or by a code committee.

Another conclusion is that several of the distribution types of a reliability model may, in principle, be unverifiable with respect to their detailed shape. They are simply formal elements that serve as vehicles of the information fed into the model by the user. This is a substantial argument for the requirement of having code specifications of distribution types to be used in the reliability evaluation. Furthermore, the shape of the asymptotic distribution depends on the very extreme tail of the generating distribution. The physical mechanisms that govern the shapes of the extreme tails are usually different from those governing the central part of the generating distribution. Most often these extreme tail mechanisms are not know due to the rareness by which they become active. Thus, the extreme value argument cannot be decisive for the code standardization.

General remarks. When applying direct MCS, the cumulative distribution function (CDF) of the response should be estimated from the statistics of all realizations $X_n(t)$ as given in the following relation:

$$\mathrm{CDF}(\mathbf{x};t) \cong F_S(x;t) = \sum_{n=1}^{N} I[\mathbf{X}_n(t),\mathbf{x}]\cdot w_n(t), \tag{1}$$

where $\mathbf{x}$ denotes a state vector, t the time, $\mathbf{X}_n$ the state vector of the n-th realization, $w_n(t)$ the weight and discrete probability of the n-th realization at time t, and N the sample size. The indicator function $I[\mathbf{X}_n, \mathbf{x}]$ assumes the value one in the case where all components of the n-th state vector $\mathbf{X}_n$ are smaller than the components of the vector $\mathbf{x}$, i.e., $\mathbf{X}_{n,k} < \mathbf{x}_k$, $k = 1, \ldots, n$. Otherwise, the indicator function assumes the value zero. In the case where direct MCS is applied, all weights are constant and assume the value $w_n(t) = 1/N$. Hence, in order to obtain estimates also in the very low probability range, the procedure must be able to modify the weight to values much smaller than $1/N$.

Methods of computing element failure probabilities. Monte Carlo simulation is occasionally the only practicable way of obtaining estimates of probability of failure. The principle of Monte Carlo simulation is simply to interpret the integral of the probability density over the failure set as the expectation of a random variable Z that takes the value 1 for outcomes in the failure set and the value 0 otherwise. By repeated random experiments performed by the use of a computer, a large number N of outcomes of the vector of basic variables X are generated according to the density function $f(x)$ of X. The average of the corresponding sample of size N of values of Z is an unbiased estimate of the failure probability P. Furthermore, the empirical standard deviation of the sample divided by $(N)^{1/2}$ is an estimate of the standard deviation of this estimate. However, the coefficient of variation of the estimator is $[(1-P)/(PN)]^{1/2}$ showing that if P is very small, as is typically the case in structural reliability applications, this simulation procedure requires the generation of a very large sample in order to give a reasonably accurate estimate of P. The RR&S simulation technique allows to overcome this disadvantage and to receive estimations of small probabilities of the first passage using considerably smaller sample size.

When solving problems of transport theory, the phase space can be divided into "interesting" and "less interesting" areas with the help of the so-called borders of splitting. It is desirable to restrict the number of realizations of random walks falling into "less interesting" areas, as the probability of contribution of such areas in the estimation is so small, or because of the nonrelevant realizations which may significantly increase the variance of the sampling distribution.

The process of down-crossing from the more interesting area into the less interesting is called the "Russian Roulette" technique. The later must not affect seriously the statistics of the sample. Playing "Russian Roulette" means that each realization survives with the probability $P_n(t) = 1/m_i$ (where m_i is an integer) and ceases to exist with probability $1 - P_n(t)$. The survival probabilities might be either constant for all realizations or variable. When comparing with equation (1), this leads to a modified sampling distribution $F_{\bar{S}}(x;t) \neq F_S(x;t)$, i.e.,

$$F_{\bar{S}}(x;t) = \sum_{n=1}^{N} I[X_n(t), x]\, J_n(t)\, \overline{w}_n(t), \qquad (2)$$

where $J_n(t)$ are independent random variables expressing "survival" or "death" of the realization $X_n(t)$,

$$J_n(t) = \begin{cases} 1 & \text{with} \quad P = P_n(t) \ \text{(survival)}; \\ 0 & \text{with} \quad P = 1 - P_n(t) \ \text{(death)}, \end{cases} \qquad (3)$$

and $\overline{w}_n(t)$ denotes the weight of the samples after playing „Russian Roulette".

Then, it will be shown that the expectation of the sampling distribution after applying „Russian Roulette", $F_{\bar{S}}(x;t)$, is equal to the sampling distribution of the nonreduced sample, i.e.,

$$\mathrm{E}\left\{F_{\bar{S}}(x;t)\right\} = F_S(x;t), \qquad (4)$$

provided that the weights are modified to

$$\overline{w}_n(t) = \frac{w_n(t)}{P_n(t)}. \qquad (5)$$

Hence, taking into account the relation $\mathrm{E}\{J_n(t)\} = P_n(t)$, the validity of equation (4) is easily verified:

$$\mathrm{E}\left\{F_{\bar{S}}(x;t)\right\} = \mathrm{E}\left\{\sum_{n=1}^{N} I[X_n(t), x] J_n(t)\,\overline{w}_n(t)\right\} =$$

$$= \sum_{n=1}^{N} I[X_n(t), x]\,\mathrm{E}\{J_n(t)\}\,\frac{w_n(t)}{P_n(t)} = \sum_{n=1}^{N} w_n(t)\, I[X_n(t), x] = F_S(x;t). \tag{6}$$

Thus, the property of equation (4) shows that the sampling distribution $F_{\bar{S}}(x;t)$ similarly to the unreduced sampling distribution $\{F_S(x;t)\}$ remains an unbiased estimate for the true distribution function of the stochastic response $\mathrm{CDF}(x;t)$. However, the variance of the sampling distribution $\mathrm{Var}\left\{F_{\bar{S}}(x;t)\right\}$ of the reduced sample will be in general larger than the variance $\mathrm{Var}\{F_S(x;t)\}$ of the unreduced sampling distribution. Hence, reducing the sample size by „Russian Roulette" increases the variance of the sampling distribution. This is the price one hase to pay for having the possibility to direct by splitting the realization into regions of interest [3].

Since the researcher is interested in increasing the sample functions in "interesting" areas, the splitting procedure will be used as follows: when, during random walks in phase space, the realization crosses the i-th border of splitting into the more interesting area, the realization is splitted on m_i identical sample functions, where the weight of each new generated realization is on m_i times less than the weight of the initial one. At further simulation, m_i new independent sample functions will be obtained due to the differences in random loading. Thus, the chances of crossing the following border of splitting and falling into even more interesting area are increased on the whole, the technique of splitting is able to increase considerably the number of sample functions which are capable to cross into interesting areas [4].

The aforementioned RR&S simulation technique was proposed by von Neumann and Ulam in 1945 [5, 6]. The procedure was intensively used in nuclear physics for the numerical solution of the neutron transport problem and reactor shielding [7, 8].

Application: Flexural-torsional of a rectangular beam. As an application, the first-passage probabilities of an elastic beam were determined during time $[0, t]$ for double-sided symmetrical barriers located at levels $\pm 3 \cdot 10^{-4}$, $\pm 4 \cdot 10^{-4}$ and $\pm 5 \cdot 10^{-4}$ m. The elastic beam is simply supported, uniform, narrow, rectangular of length L subjected to a stochastically varying concentrated load $P(t)$ acting at the center of the beam cross-section as shown in Fig. 1. For nonfollower force, the lateral deflection $u(t)$ and the angle of twist $\psi(t)$ of a transverse cross-section z = constant are governed by the equations of motion [9]:

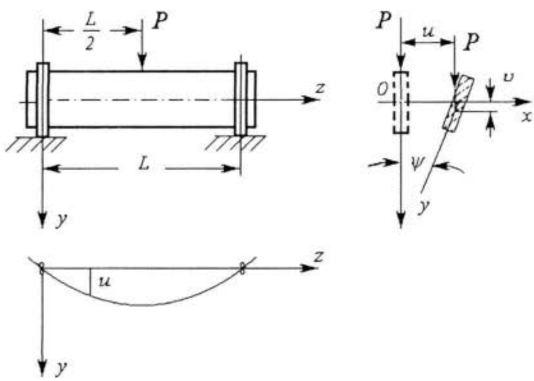

Fig. 1. Loaded rectangular beam in flexural-torsional deformation.

$$\mathrm{E}I_y \frac{\partial^4 u}{\partial z^4} + \frac{\partial^2 (M_x \psi)}{\partial z^2} + m \frac{\partial^2 u}{\partial t^2} + D_u \frac{\partial u}{\partial t} = 0,$$
$$M_x \frac{\partial^2 u}{\partial z^2} - GJ \frac{\partial^2 \psi}{\partial z^2} + mr^2 \frac{\partial^2 \psi}{\partial t^2} + D_\psi \frac{\partial \psi}{\partial t} = 0 \qquad (7)$$

where

$$M_x = \begin{cases} \frac{1}{2} Pz, & 0 \le z \le \frac{1}{2} L; \\ \frac{1}{2} P(L - z), & \frac{1}{2} L \le z \le L, \end{cases}$$

and $\mathrm{E}I_y$, GJ denote the relevant flexural and torsional rigidities of the cross-section, r the polar radius of gyration of the cross-section, D_u, D_ψ the viscous damping coefficients, and m the mass per unit length.

The conditions of simple support at the ends imply the boundary conditions:

$$u(0, t) = u(L, t) = u''(0, t) = u''(L, t) = 0,$$
$$\psi(0, t) = \psi(L, t) = 0. \qquad (8)$$

Considering the fundamental mode, the above boundary conditions are satisfied by taking

$$u(z, t) = Krq_1(t) \sin \frac{\pi z}{L}, \qquad \psi(z, t) = q_2(t) \sin \frac{\pi z}{L}. \qquad (9)$$

Substituting (9) in the equations of motion (7) and using the Galerkin method, we get

$$\ddot{q}_1 + 2\beta_1 \dot{q}_1 + \omega_1^2 q_1 - \frac{1}{K} \omega_1 \omega_2 \xi(t) q_2 = 0,$$
$$\ddot{q}_2 + 2\beta_2 \dot{q}_2 + \omega_2^2 q_2 - K \omega_1 \omega_2 \xi(t) q_1 = 0, \qquad (10)$$

where

$$\omega_1^2 = \frac{\mathrm{E}I_y}{m} \left(\frac{\pi}{L}\right)^4, \qquad \omega_2^2 = \frac{GJ}{mr^2} \left(\frac{\pi}{L}\right)^2, \qquad 2\beta_1 = \frac{D_u}{m},$$
$$2\beta_2 = \frac{D_\psi}{mr^2}, \qquad \xi(t) = \frac{P(t)}{P_{cr}}, \qquad P_{cr} = \frac{8mr L \omega_1 \omega_2}{4 + \pi^2}. \qquad (11)$$

Here, P_{cr} is the value of the critical nonfollower at which static buckling will occur. By choosing the constant $K = -(\omega_2/\omega_1)^{1/2}$, one obtains

$$\ddot{q}_1 + 2\beta_1 \dot{q}_1 + \omega_1^2 q_1 + \omega_1 k_{12} \xi(t) q_2 = 0,$$
$$\ddot{q}_2 + 2\beta_2 \dot{q}_2 + \omega_2^2 q_2 + \omega_2 k_{21} \xi(t) q_1 = 0, \qquad (12)$$

where $k_{12} = k_{21} = (\omega_1 \omega_2)^{1/2} = k_N$, $\xi(t)$ is the Gaussian white noise with intensity $I = 2\pi S_0$ and zero mean, ω_1, ω_2 are natural frequencies.

The following numerical data were utilized: $\omega_1 = 1{,}5$ rad/sec, $\omega_1 = 2$ rad/sec, $I = 1\,\mathrm{m}^2/\mathrm{sec}^3$.

The cumulative distribution functions of the first passage probabilities of the system response $q_1(t)$ for barriers located at levels $\pm 3 \cdot 10^{-4}$, $\pm 4 \cdot 10^{-4}$ and $\pm 5 \cdot 10^{-4}$ m are presented in Fig. 2 in logarithmic scale.

In Fig. 2, the results obtained with the help of the RR&S technique are shown by solid lines and the results obtained by the straightforward MCS are shown by dotted lines.

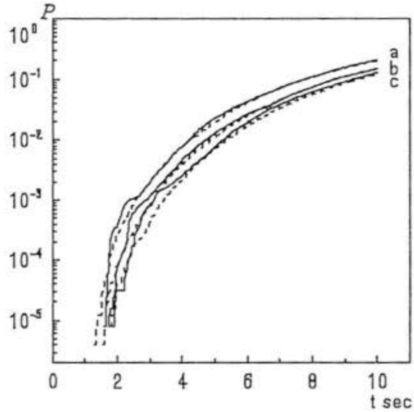

Fig. 2. Estimates of first-passage probabilities versus time for the double-sided barriers located at levels $\pm 3 \cdot 10^{-4}$ m (a), $\pm 4 \cdot 10^{-4}$ m (b), and $\pm 5 \cdot 10^{-4}$ m (c).

In the first case (barriers at level $\pm 3 \cdot 10^{-4}$ m), 6 borders of splitting were located at $\pm 0{,}75 \cdot 10^{-4}$, $\pm 1{,}5 \cdot 10^{-4}$, $\pm 2{,}25 \cdot 10^{-4}$ m, the number of splitting on each of the borders $m_1 = m_2 = m_3 = 5$; the cumulative distribution function for this problem is shown in Fig. 2 (curve a) by solid lines.

In the second case (barriers at level $\pm 4 \cdot 10^{-4}$ m), 6 borders of splitting were located at $\pm 1 \cdot 10^{-4}$, $\pm 2 \cdot 10^{-4}$, $\pm 3 \cdot 10^{-4}$ m, the number of splitting on each of the borders $m_1 = m_2 = m_3 = 5$; the cumulative distribution function for this case is shown in Fig. 2 (curve b) by solid lines.

In the third case (barriers at level $\pm 5 \cdot 10^{-4}$ m), 6 borders of splitting were located at $\pm 1{,}25 \cdot 10^{-4}$, $\pm 2{,}5 \cdot 10^{-4}$, $\pm 3{,}75 \cdot 10^{-4}$, the number of splitting on each of the borders $m_1 = m_2 = m_3 = 5$; the cumulative distribution function for this case is shown in Fig. 2 (curve c) by solid lines.

In all three cases, the initial sample size for the RR&S procedure $n = 2\,000$ and, for the straightforward MCS, $N = 250\,000$. The analysis of the Fig. 2 shows good coincidence of the results of both procedures for all barriers. The following table contains the time ratio $T_{\mathrm{MCS}}/T_{\mathrm{RR\&S}}$, where T is the computational time necessary to obtain the estimates P.

Time Ratio-MCS to RR&S

Barrier level	$\pm 3 \cdot 10^{-4}$ m	$\pm 4 \cdot 10^{-4}$ m	$\pm 5 \cdot 10^{-4}$ m
$\dfrac{T_{\mathrm{MCS}}}{T_{\mathrm{RR\&S}}}$	11,80	37,29	18,18

Commentary. From the results mentioned in Table, we note that the RR&S simulation technique offers a considerable gain in computational time to obtain the estimates P, comparatively to direct MCS technique.

Conclusion. The carried out research shows that the proposed "Russian Roulette and Splitting" simulation technique allows effectively to access the failure probabilities of construction in low probability range. It has been applied successfully to a structural element with two-degrees of freedom subjected to stochastic excitation, showing the feasibility of this approach to realistic engineering structures.

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