UDC 519.21

H. Airault (Univ. de Picardie), **P. Malliavin** (Univ. Paris VI, France)

REGULARIZED BROWNIAN MOTION ON THE SIEGEL DISK OF INFINITE DIMENSION

РЕГУЛЯРИЗОВАНИЙ БРОУНІВСЬКИЙ РУХ НА НЕСКІНЧЕННОВИМІРНОМУ ДИСКУ СІГЕЛА

We construct a process of Brownian motion on the Siegel disk of infinite dimension.

Побудовано процес броунівського руху на нескінченновимірному диску Сігела.

The theory of groups of infinite dimension and their homogeneous spaces needs integration theory fitting to the underlying algebraic structure. In fact sometimes it exists on the Lie algebra a canonic Hilbertian structure induced by a canonic cocycle. This Hilbertian structure defines a formal canonic Laplacian. Then a natural question is to provide an effective construction of the corresponding heat process. This have been done in [1] for the diffeomorphism group of the circle. We shall make below a preliminary study of the case of Siegel disk of infinite dimension.

1. Siegel disk in infinite dimension and its Kählerian metric. We consider the space V of real valued C^1-functions defined on the circle with mean value equal to 0. On V we define a bilinear alternate form

$$\omega(u, v) = \frac{1}{\pi}\int_0^{2\pi} uv'\,d\theta, \tag{$1_{\rm i}$}$$

which is canonic in the sense that it is the unique alternate bilinear form invariant under the action of orientation preserving diffeomorphism of the circle. As consequence it is possible to obtain a representation of the diffeomorphism group into the symplectic group consisting of automorphism of V which preserves the symplectic form ω. We introduce on V a complex structure defined by the Hilbert transform

$$\mathcal{J}\colon\ \sin(k\theta) \mapsto \cos(k\theta), \qquad \cos(k\theta) \mapsto -\sin(k\theta).$$

We define on V an Hilbertian metric

$$\| u\|^2 = -\omega(u, \mathcal{J}u);$$

then

$$\left|\sum_{k>0} a_k\cos(k\theta) + b_k\sin(k\theta)\right|^2 = \sum_k k(a_k^2 + b_k^2). \tag{$1_{\rm ii}$}$$

Then $\mathcal{J}$ is an orthogonal transformation of V.

We denote $H = V\otimes G$; then H can be identified with complex valued function defined on the circle having mean value 0; on H the operation of conjugation $f\mapsto \bar f$ is well defined. The orthogonal transformation $\mathcal{J}$ can be diagonalized in H; as $\mathcal{J}^2 = -1$ only appears the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$. We denote H^+ the eigenspace associated to the eigenvalue $\sqrt{-1}$; then we can identify H^+ to the vectors of type $(1, 0)$ that is the vectors of the form $v - \sqrt{-1}\,\mathcal{J}(v)$, $v\in V$. We can also identify H^+ with the functions having an holomorphic extension inside the unit disk. Then define $H^- = \overline{H}^+$; then H^- can be identified with the functions on the circle which possess an holomorphic extension outside the unit disk which is regular at the point at ∞ of the complex plane. The bilinear form ω extends to a bilinear form $\tilde{\omega}$ defined on H and we have

$$\tilde{\omega}(w, w') = 0 \quad \text{if} \quad w, w' \in H^+ \quad \text{or} \quad w,\ w' \in H^-.$$

We define a symmetric C-bilinear form on $H \times H$ by

$$\langle h_1, h_2 \rangle = (h_1 | \overline{h}_2), \quad \text{then} \quad (h_1, h_2) = \langle h_1, \overline{h}_2 \rangle. \tag{1_{iii}}$$

Then

$$\tilde{\omega}(h_1, h_2) = \langle h_1^+, h_2^- \rangle - \langle h_1^-, h_2^+ \rangle. \tag{1_{iv}}$$

Given $A \in \mathrm{End}(H)$ we denote A^T the transposed defined by

$$\langle A h_1, h_2 \rangle = \langle h_1, A^T h_2 \rangle.$$

Given $a \in \mathrm{End}(H^+)$, then the matrix $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ makes possible to identify $\mathrm{End}(H^+) \subset \mathrm{End}(H)$; then $a^T \in \mathrm{End}(H)$ is well defined; furthermore we have through the duality coupling

$$\langle a h^+, h^- \rangle = \langle h^+, a^T h^- \rangle;$$

which means that $a^T \in \mathrm{End}(H^-)$. The adjoint $a^\dagger \in \mathrm{End}(H^+)$ is defined by

$$(a w_1 | w_2) = (w_1 | a^\dagger w_2) \qquad \forall\, w_1, w_2 \in H^+.$$

The conjugation operator sends $H^+ \mapsto H^-$ therefore $\overline{a} \in \mathrm{End}(H^-)$ and we have the fact that the adjoint is obtained by conjugation followed by transposition

$$a^\dagger = (\overline{a})^T = \overline{a^T}.$$

A linear endomorphism $\tilde{U}$ of V extends to an endomorphism U of H. Denoting π^+, π^- the projection of H on H^+, H^- we introduce

$$a := \pi^+ \tilde{U} \pi^+; \qquad b := \pi^+ \tilde{U} \pi^-.$$

The fact that the endomorphism $\tilde{U}$ commutes with the conjugation it is equivalent to the fact that the second line of its associated matrix is the conjugate of the first line that is:

$$U = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} := U_{a,b}. \tag{2_0}$$

The conservation of the symplectic form (1_i) by $U_{a,b}$ is equivalent to

$$(\overline{a})^T (a + b) - b^T (\overline{a} + \overline{b}) = \pi^+, \qquad (\overline{b})^T (a + b) - a^T (\overline{a} + \overline{b}) = -\pi^-,$$

we remark that the first relation is the conjugate of the second. Therefore we have only to take care of the second relation which by splitting on the components H^+, H^- gives

$$a^T \overline{a} - b^\dagger b = \pi^-, \quad \text{and its conjugate} \quad a^\dagger a - b^T \overline{b} = \pi^+, \tag{2_i}$$

$$a^T \overline{b} - b^\dagger a = 0, \quad \text{and its conjugate} \quad a^\dagger b - b^T \overline{a} = 0. \tag{2_{ii}}$$

We define *Symplectic Group of infinite order,* let $\mathrm{Sp}(\infty)$ as the group of matrices $U_{a,b}$ invertible and preserving the symplectic form (1_i) and such that

$$\mathrm{trace}(b^\dagger b) := \|b\|_2^2 < \infty. \tag{2_{iii}}$$

The preservation of the symplectic form is equivalent to the relations (2_{i}), (2_{ii}).

In finite dimension the invertibility of $U_{a,b}$ is implied by the relations (2_{i}), (2_{ii}). In infinite dimension this is no more the case and this invertibility is equivalent to the following relations

$$\overline{a}\, a^T - \overline{b}\, b^T = \pi^-, \qquad \overline{b}\, a^\dagger - \overline{a}\, b^\dagger = 0. \tag{2_{iv}}$$

Then $\mathrm{Sp}(\infty)$ can be defined as the matrices $U_{a,b}$ which have theirs coefficients satisfying $(2_{\mathrm{i}})-(2_{\mathrm{iv}})$.

Theorem 1. *Denote*

$$\mathcal{Z} := \{ z \in \mathcal{L}(H^-; H^+);\ z^T = z,\ \|z\|_2 < \infty \} \tag{3_{i}}$$

and denote $u(H^+) = \{ y \in \mathrm{End}(H^+);\ y^\dagger + y = 0 \}$. *Then the Lie algebra of the symplectic group* $\mathcal{G} \simeq u(H^+) \oplus \mathcal{Z}$, *the product being given by*

$$[(y, z), (y_1, z_1)] = ([y, y_1] + z\overline{z}_1 - z_1\overline{z},\ (y z_1 + z\overline{y}_1 - y_1 z - z_1 \overline{y})). \tag{3_{ii}}$$

Proof. By linearizing the equations (2_{i}), (2_{ii}) at the neighbourhood of $e = U_{e,0}$ we get

$$y^T + \overline{y} = 0, \qquad \overline{z} - z^\dagger = 0.$$

By conjugating the first equation becomes $y^\dagger + y = 0$ and the second becomes $z^T - - z = 0$. By bilinearity it is sufficient to consider several special cases of (3_{ii}).

The case $z = z_1 = 0$ is trivial as reduce to the unitary group.

The case $y_1 = 0$, $z = 0$ comes from the computation

$$[U_{y,0}, U_{0,z_1}] = U_{0, uz - z\overline{u}}\,.$$

The case $y = y_1 = 0$ comes from the computation

$$[U_{0,z}, U_{0,z_1}] = U_{z\overline{z}_1 - z_1\overline{z}}\,. \quad \bullet$$

We define the *infinite dimensional Siegel disk*

$$D_\infty := \{ Z \in \mathcal{Z};\ 1 - Z^\dagger Z > 0 \}. \tag{4_{i}}$$

Theorem 2. *Considering* $Z \in \mathrm{End}(H)$ *through the matrix* $\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$, *the group* $\mathrm{Sp}(\infty)$ *operates on* $\mathcal{D}_\infty$ *by*

$$Z \mapsto Y = (aZ + b)(\overline{b}\, Z + \overline{a})^{-1}. \tag{4_{ii}}$$

***Remark* 1.** In the above formula a, b are identified $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in \in \mathrm{End}(H)$.

Proof. We have firstly to show that $Y \in \mathcal{L}(H^-; H^+)$: we have $\overline{a} \in \mathrm{End}(H^-)$, $\overline{b}\, Z \in \mathrm{End}(H^-)$; therefore $(\overline{b}\, Z + \overline{a})^{-1} \in \mathrm{End}(H^-)$.

We have secondly to show that $Y^T = Y$:

$$Y^T = (Z b^\dagger + a^\dagger)^{-1} (Z a^T + b^T),$$

therefore the identity $Y^T = Y$ is equivalent to

$$0 = (Z a^T + b^T)(\overline{b}\, Z + \overline{a}) - (Z b^\dagger + a^\dagger)(a Z + b) =$$
$$= Z(a^T \overline{b} - b^\dagger a) Z + (b^T \overline{b} - a^\dagger a) Z + Z(a^T \overline{a} - b^\dagger b) + b^T \overline{a} - a^\dagger b,$$

the first coefficient vanishes accordingly (2_{ii}); by conjugating (2_{ii}) we obtain the vanishing of the fourth coefficient; using (2_i) and its conjugation we obtain

$$= \pi^- Z + Z\pi^+$$

these two terms are zero according the fact that Z is matrix $\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}$. •

We have to check that $Y^\dagger Y - \pi^- < 0$; denote

$$\Delta_Y := \pi^- - Y^\dagger Y = \pi^- - \overline{Y} Y, \quad D := (\overline{b} Z + \overline{a}) \tag{4_{iii}}$$

then

$$D^\dagger \Delta_Y D = -(Z^\dagger a^\dagger + b^\dagger)(aZ + b) + (Z^\dagger b^T + a^T)\pi^-(\overline{b} Z + \overline{a}) =$$

$$= -Z^\dagger (a^\dagger a - b^T \overline{b}) Z - Z^\dagger (a^\dagger b - b^T \overline{a}) - (b^\dagger a - a^T \overline{b}) Z - (b^\dagger b - a^T \overline{a}) =$$

$$= \Delta_Z,$$

equality obtained by using (2_{ii}) and remarking that the first term can be written as $-Z^\dagger \pi^+ Z = -Z^\dagger Z$. Finally we have proved that

$$\Delta_Y = (D^{-1})^\dagger \Delta_Z D^{-1} > 0. \ \bullet \tag{4_{iv}}$$

We have

$$\|(aZ + b)(\overline{b} Z + \overline{a})^{-1}\|_2 < \|(\overline{b} Z + \overline{a})^{-1}\|_\infty (\|Z\|_2 + \|b\|_2) < \infty. \ \bullet$$

Finally we have to check that the action is a group homomorphism: we make in (4_{ii}) the substitution $Z \mapsto (a_1 Y + b_1)(\overline{a}_1 + \overline{b}_1 Y)$ and we get

$$(a(a_1 Y + b_1) + b(\overline{a}_1 + \overline{b}_1 Y))(\overline{a}(\overline{a}_1 + \overline{b}_1 Y) + \overline{b}(a_1 Y + b_1))^{-1} =$$

$$= (a_2 Y + b_2)(\overline{a}_2 + \overline{b}_2 Y)^{-1}$$

where $a_2 = a a_1 + b \overline{b}_1$, $b_2 = a b_1 + b \overline{a}_1$. •

Proposition 1. *The orbit through* $\mathrm{Sp}(\infty)$ *of the matrix* $Z_0 = 0$ *is the set of matrices of the form*

$$Z = b(\overline{a}^{-1}). \tag{5_i}$$

As $(a, 0) \in \mathrm{Sp}(\infty)$ *iff* $a \in U(H^+)$ *the unitary group of* H, *then*

$$\text{the orbit of } Z_0 \simeq \mathrm{Sp}(\infty) / U(H^+); \tag{5_{ii}}$$

$$\text{the orbit of } Z_0 \text{ contains a neighbourhood of } e \in \mathrm{Sp}(\infty). \tag{5_{iii}}$$

Proof. We consider the map $\Phi : \mathrm{Sp}(\infty) \mapsto \mathcal{D}(\infty)$ defined by $(a, b) \mapsto b\overline{a}^{-1}$. Its derivative $\Phi'((I, 0)) : \mathcal{G} \mapsto \mathcal{Z}$ and it can be identified to the projection of $\mathcal{G}$ on its second component it is therefore surjective. The implicit function theorem gives the conclusion. •

The *Kähler potential* on $\mathcal{D}_\infty$ is defined as

$$K(Z) = -\log\det(1 - Z^\dagger Z) = -\operatorname{trace}\log(1 - Z^\dagger Z) = -\operatorname{trace}\log\Delta_Z \tag{6_i}$$

these equalities being intrinsic can be proved using a basis diagonalizing $Z^\dagger Z$.

Theorem 3. *The Kähler potential is invariant under the action of* $U(H^+) \subset \subset \mathrm{Sp}(\infty)$. *We have*

$$K(b\overline{a}^{-1}) = \operatorname{trace}\log(1 + b^\dagger b). \tag{6_{ii}}$$

Proof. We remark that $a \in U(H^+)$ implies $\overline{a}^{-1} = \overline{a^\dagger} = a^T$; therefore the action of $U(H^+)$ can be described by

$$Z \mapsto aZa^T \quad \text{and} \quad \overline{Z}Z \mapsto c\overline{Z}Zc^\dagger$$

with $c = \overline{a}$. Therefore $\det(1 - Z^\dagger Z)$ is invariant under the action of $U(H^+)$. •

We have on the orbit of Z_0

$$Z = b\overline{a}^{-1}, \quad Z^\dagger Z = (a^T)^{-1} b^\dagger b \overline{a}.$$

Therefore

$$-\log\det(1 - Z^\dagger Z) = -\log\det(1 - (a^T)^{-1} b^\dagger b \overline{a}^{-1}) =$$
$$= \log\det((a^T)^{-1}(a^T\overline{a} - b^\dagger b)\overline{a}^{-1})$$

using (2_{i}) we get

$$= \log\det(\overline{a}a^T)$$

then the conclusion results from (2_{iv}). •

Theorem 4. *The mixed Hessian, which is the* (1, 1) *differential form* $\partial\overline{\partial}K$, *is invariant under the action of symplectic group; at* Z_0 *it is equal to the symplectic form* $\overline{\omega}$; *therefore* $\mathcal{D}(\infty)$ *becomes an homogeneous infinite dimensional Kähler manifold.*

Proof. We remark that (4_{ii}) implies that

$$K((aZ + b)(\overline{b}Z + \overline{a})^{-1}) = K(Z) - 2\Re(\operatorname{trace}\log(\overline{b}Z + \overline{a}))$$

the last term is the real part of a holomorphic function in Z and has therefore a vanishing (1, 1) Hessian: we have obtained the invariance of the Hessian under the symplectic action. •

We take as tangent plane at Z_0 the element of the form $(0, \varepsilon z)$ where $\varepsilon \in C$. Then (6_{ii}) gives

$$K(\varepsilon z) = \operatorname{trace}(\overline{\varepsilon}\varepsilon\overline{z}z) + o(\varepsilon^2). \quad •$$

2. Regularized Brownian motion on the Siegel disk. We shall follow the methodology of the *horizontal Laplacian* and of the *horizontal diffusion* which has the advantage to realize the Brownian motion on $\mathcal{D}(\infty)$ on the symplectic group itself, where the Maurer – Cartan differential form gives a natural coordinate system.

In order to construct an orthonormal C-basis of $\mathcal{Z}$ we shall implement a coordinate system adapted the content of Section (1). We choose a C-basis e_k of H^+. Then $\overline{e}_k := e_{\overline{k}}$ is a basis of H^-. We define $v_{j,k} \in \mathcal{L}(H^-; H^+)$ by

$$v_{j,k} = e_j \otimes e_k \text{ which means } v_{j,k}(h^-) = \langle h^-, e_j\rangle e_k \quad \forall\, h^- \in H^-.$$

In the same way we define

$$v_{\overline{j},k} = e_{\overline{j}} \otimes e_k \in \mathcal{L}(H^+; H^+),$$
$$v_{j,\overline{k}} = e_j \otimes e_{\overline{k}} \in \mathcal{L}(H^-; H^-),$$
$$v_{\overline{j},\overline{k}} = \overline{e}_j \otimes \overline{e}_k \in \mathcal{L}(H^+; H^-).$$

Denote $\alpha = j$ or $\alpha = \overline{j}$ and in the same way β denotes either k either $\overline{k}$. Then the following formulas holds true:

$$v_{\alpha,\beta} = e_\alpha \otimes e_\beta, \quad \overline{v}_{\alpha,\beta} = v_{\overline{\alpha},\overline{\beta}}, \quad v^T_{\alpha,\beta} = v_{\beta,\alpha}, \quad v^\dagger_{\alpha,\beta} = v_{\overline{\beta},\overline{\alpha}}.$$

We define

$$u_{j,k} = \frac{1}{\sqrt{2}}(v_{j,k} + v_{k,j}), \quad j \neq k; \quad u_{j,j} = v_{j,j}. \tag{7_{i}}$$

Then for $1 \leq j \leq k$ the $u_{j,k}$ constitutes a C-orthonormal basis of $\mathcal{Z}$. We deduce the following R-basis $u^\delta_{j,k} = (\sqrt{-1})^\delta u_{j,k}$, where δ takes the values $0, 1$.

We define the *horizontal Laplacian* on $G := \mathrm{Sp}(\infty)$ as

$$(\Delta_G \Phi)(g) = \frac{1}{2} \sum_\delta \sum_{1 \leq j \leq k} \frac{d^2}{d\varepsilon^2}\bigg|_{\varepsilon = 0} \Phi(g \exp(\varepsilon u^\delta_{j,k})). \tag{7_{ii}}$$

Proposition 2. *The differential operator Δ_G is invariant under the left action of G.*

It is right invariant under the action of the unitary group $U(H^+)$. (7_{iii})

Proof. The left invariance is obvious. The infinitesimal action of $U_{(\exp(\varepsilon y), 0)}$ on $\mathcal{Z}$ is computed in (3_{ii}) as $\mathcal{A}_y : z \mapsto yz - z\overline{y}$; then $\mathcal{A}_y$ is antihermitian for the Hilbertian metric $\mathrm{trace}(\overline{z}z)$:

$$\mathrm{trace}((\overline{y}\,\overline{z} - \overline{z}y)z + \overline{z}(yz - z\overline{y})) = \mathrm{trace}(\overline{y}(\overline{z}z) - (\overline{z}z)\overline{y}) = 0. \quad \bullet$$

Corollary. *Denote $\Delta_{\mathcal{D}}$ the Laplace–Beltrami operator on the Kählerian maniforld constituted by the Siegel disk $\mathcal{D}$; consider the projection $p : G \mapsto$ $\mapsto G/U(H^+) \simeq \mathcal{D}$; then for any smooth functional φ defined on $\mathcal{D}$, we have*

$$(\Delta_{\mathcal{D}}\varphi) \circ p = \Delta_G(\varphi \circ p). \tag{7_{iv}}$$

The regularized Brownian motion on $\mathcal{Z}$. To the Hilbert space $\mathcal{Z}$, Irving Segal construction associates a canonic probability space, let $\mathrm{Seg}(\mathcal{Z})$ which carries the canonic Gaussian cylindrical measure associated to $\mathcal{Z}$. In the same way a canonic probability space which carries the canonic cylindrical Brownian motion on $\mathcal{Z}$ is defined. A realization of this probability space can be made by *choosing* an orthonormal C-basis $u_{j,k}$ of $\mathcal{Z}$ and by defining

$$x(t) := \sum_{1 \leq j \leq k}^{\infty} u_{j,k}\, x^{j,k}(t)$$

where $x^{j,k}(*)$ are independent C-valued Brownian motions. All the content of the Segal construction is to decipher the properties of $x(*)$ which do not depend of the choice of a basis.

We will exponentiate $x(*)$ to G by solving an SDE. To be able to apply the scheme it is needed to regularize $x(*)$ by introducing as in [1] a regularization parameter $r \in\ ]0, 1[$ and considering for r fixed the *regularized Brownian motion* on $\mathcal{Z}$ defined by

$$x_r(t) := \sum_{1 \leq j \leq k} u_{j,k}\, r^{j+k}\, x^{j,k}(t). \tag{8_{i}}$$

The *horizontal diffusion* is defined by the solution of the following Stratonovitch SDE

$$g_x(t) = g_x(t)\begin{pmatrix} 0 & \circ\, dx_r(t) \\ \circ\, d\overline{x}_r(t) & 0 \end{pmatrix}, \quad g_x(0) = e. \tag{8$_{\text{ii}}$}$$

Then $t \mapsto p(g_x(t))$ describes the regularized Brownian motion on $\mathcal{D}$.

We explicit this SDE by writing $g_x(t) = U_{a_x(t), b_x(t)}$ and we get the system:

$$da_x(t) = b_x(t) \circ d\overline{x}_r(t), \quad db_x(t) = a_x(t) \circ dx_r(t);$$
$$a_x(0) = I, \quad b_x(0) = 0. \tag{8$_{\text{iii}}$}$$

To get estimates it is necessary, as usual, to work in Itô formalism; we must therefore compute stochastic contractions.

Proposition 3. *The Itô contractions of* (8_{iii}) *are given by*

$$d\overline{x}(t) * dx(t) = \mathcal{A}_r\, dt, \quad \textit{where} \quad \mathcal{A}_r = \sum_{j=1}^{\infty}\left(\frac{r^{2j}}{1-r^2} + r^{4j}\right) e_{\bar{j}} \otimes e_j, \tag{8$_{\text{iv}}$}$$

$$dx(t) * d\overline{x}(t) = \mathcal{A}_r^T\, dt. \tag{8$_{\text{v}}$}$$

Remark 2. When $r \to 1$ the positive operator $\mathcal{A}_r$ converges towards $+\infty$, fact which shows the necessity of some renormalization.

Proof. We have

$$d\overline{x} = \sum_{1 \le j} r^{2j} v_{\bar{j},\bar{j}} + \sum_{1 \le j < k} \frac{r^{j+k}}{\sqrt{2}}(v_{\bar{j},\bar{k}} + v_{\bar{k},\bar{j}})\, dx^{j,k}.$$

The contraction $dx(t) * d\overline{x}(t)$ gives rise to three series: the first series

$$\sum_{1 \le j} r^{4j} v_{j,j} v_{\bar{j},\bar{j}} = \sum_{1 \le j} r^{4j} e_{\bar{j}} \otimes e_j;$$

the second series is composed of terms

$$v_{j,k} v_{\bar{j},\bar{k}} = 0, \quad \text{as} \quad j \ne k;$$

the third series is obtained by permuting above j and k leaving fixed $\bar{j}$ and $\bar{k}$:

$$\sum_{1 \le j \le k} r^{2(j+k)} (\langle e_j, e_{\bar{j}}\rangle e_{\bar{k}} \otimes e_k + \langle e_k, e_{\bar{k}}\rangle e_{\bar{j}} \otimes e_j) = \frac{1}{1-r^2}\sum_{s=1}^{\infty} r^{2s} e_{\bar{s}} \otimes e_s. \quad \bullet$$

Theorem 5. *The regularized horizontal diffusion on* G *satisfies the system of Itô equations*

$$da_x(t) = b_x(t)\, d\overline{x}_r(t) + \frac{1}{2} a_x(t)\, \mathcal{A}_r\, dt,$$
$$db_x(t) = a_x(t)\, dx_r(t) + \frac{1}{2} b_x(t)\, \mathcal{A}_r^T\, dt. \tag{9}$$

It can be shown that for r fixed (9) has a solution. Using in the next section another system of coordinates we shall see the persistence of this need for renormalization.

3. Using holomorphic kernels as space of coordinates. The Hilbertian structure associated to the classical Wiener measure is the space H^1 of functions defined on $[0, 1]$ having their first derivative in L^2. We want to interpret the Kählerian metric on the Siegel disk in a similar manner.

In our setting we shall consider the space $\mathcal{H}^+$ of holomorphic functions inside the unit disk D (resp. $\mathcal{H}^-$ will be the space of antiholomorphic functions on D). In this setting $a \simeq A(\overline{\zeta}_1, \zeta_2) \in \mathcal{H}^- \otimes \mathcal{H}^+$ and $b \simeq B(\zeta_1, \zeta_2) \in \mathcal{H} \otimes \mathcal{H}$.

Theorem 6. *The kernels K_z associated to $z \in \mathcal{Z}$ are characterized by the following relations*:

$$\| z \|^2 = \frac{1}{2} \int_{D^2} \left| \frac{\partial^2 K_z}{\partial \zeta_1 \partial \zeta_2} \right|^2 d\zeta_1 d\zeta_2 < \infty, \quad K_z(\zeta_1, \zeta_2) = K_z(\zeta_2, \zeta_1), \tag{10$_{\rm i}$}$$

$$b(\overline{h}) \simeq \int_D \frac{\partial B}{\partial \zeta_2}(\zeta_1, \zeta_2) \frac{\partial \overline{h}}{\partial \overline{\zeta}_2} d\zeta_2 . \tag{10$_{\rm ii}$}$$

Proof. For $f \in \mathcal{H}^+$ we have the identity

$$\|f\|^2_{\mathcal{H}_1^+} := \int_D |f'(\zeta)|^2 d\zeta = \sum_{n=1}^{\infty} n |c_n|^2, \tag{10$_{\rm iii}$}$$

where

$$f(\zeta) = \sum_{n=1}^{\infty} c_n \zeta^n,$$

identity which together $(1_{\rm ii})$ implies the theorem. •

Reproducing kernels. Given $\zeta_0 \in D$ the linear form $l_{\zeta_0}\colon \mathcal{H}_1^+ \mapsto C$ defined by $h \mapsto h(\zeta_0)$ is continuous. Therefore there exists $R_{\zeta_0}(\overline{\zeta}) \in \mathcal{H}_1^-$ such that

$$h(\zeta_0) = \int_D \frac{\partial}{\partial \overline{\zeta}} R_{\zeta_0}(\overline{\zeta}) \frac{\partial h}{\partial \zeta}(\zeta) d\zeta, \tag{11$_{\rm i}$}$$

$$R_{\zeta_0}(\overline{\zeta}) = \sum_{n=1}^{\infty} \frac{1}{n^2} (\zeta_0 \overline{\zeta})^n = \Phi(\zeta_0 \overline{\zeta}) \tag{11$_{\rm ii}$}$$

where

$$\Phi(\lambda) = (\lambda - 1) \log(1 - \lambda) + \lambda.$$

Then

$$\frac{\partial^2}{\partial \zeta_0 \partial \overline{\zeta}} R_{\zeta_0}(\zeta) = \Phi(\lambda), \quad \text{where} \quad \Phi(\lambda) = \frac{1}{1 - \lambda}. \tag{11$_{\rm iii}$}$$

Brownian motion on $\mathcal{Z}$. The $x^{j,k}$ being defined as in $(8_{\rm i})$ we define

$$X_t(\zeta_1, \zeta_2) = \sum_{1 \le j} \frac{1}{j^2} (\zeta_1 \zeta_2)^j x^{j,j}(t) + \frac{1}{\sqrt{2}} \sum_{1 \le j \le k} \frac{1}{jk} (\zeta_1^j \zeta_2^k + \zeta_2^j \zeta_1^k) x^{j,k}(t). \tag{12$_{\rm i}$}$$

Then the stochastic contraction becomes

$$(d\overline{X} * dX)(\overline{\eta}_1, \eta_2) = \Bigg(\sum_{1 \le j} \frac{1}{j^2} (\overline{\eta}_1 \eta_2)^j + \\ + \frac{1}{\sqrt{2}} \sum_{1 \le j \le k} \frac{1}{k^2} (\overline{\eta}_1 \eta_2)^k + \frac{1}{j^2} (\overline{\eta}_1 \eta_2)^j \Bigg) dt = \infty \times \Phi(\overline{\eta}_1 \eta_2). \tag{12$_{\rm ii}$}$$

Again a regularization procedure is needed as in Section 2.

1. *Malliavin P.* The canonic diffusion above the diffeomorphism group of the circle // C. r. Acad. sci. – 1999. – 329. – P. 325–329.

Received 08.06.2000