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A FUNCTIONAL ANALYTIC METHOD FOR THE ANALYSIS OF GENERAL PARTIAL DIFFERENTIAL EQUATIONS

Eszter Sikolya

Department of Applied Analysis, Eötvös Loránd University
P.O. Box 120, H-1518 Budapest, Hungary
E-Mail: seszter@cs.elte.hu

In this paper we give a short overview of operator semigroups. These objects are widely used for proving well-posedness of partial differential equations and for investigating qualitative properties of the solutions.
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1. Operator (semi)groups

Many physical phenomena can be described by so-called *dynamical systems*. Here we investigate the following model. The elements of the *state space* describe completely the temporal change of the system and they include all factors important for the observant. They also determine unambiguously the further motion of the system. The *time* is parameterized by $\mathbf{R}$ or $\mathbf{R}_+$ (it depends whether we want to handle the past or not). We assume that to each time $t \in \mathbf{R}(\mathbf{R}_+)$ belongs a state of the system $z(t) \in Z$ from the state space Z. We also assume that the motion is *deterministic*, that is, for every time instant t_0 and initial state z_0 there exists a unique motion

$$z_{t_0,z_0} : \mathbf{R} \to Z$$

such that

$$z_{t_0,z_0}(t_0) = z_0.$$

We further assume that the system is *autonomous* that means

$$z_{t_0,z_0}(t_0 + h) = z_{t_1,z_0}(t_1 + h)$$

holds for any $t_0, t_1, h \in \mathbf{R}$ and $z_0, z_1 \in Z$. This implies that the orbits of the motion do not intersect each other.

Using this model, we can define the operators $T(t) : Z \to Z$ for $t \in \mathbf{R}(\mathbf{R}_+)$ acting as

$$T(t)z := z_{t_0,z}(t_0 + t),$$

where t_0 can be chosen arbitrary since the system is autonomous. Then clearly

$$T(0)z = z$$

holds because

$$z_{t_0,z}(t_0) = z.$$

In this way we have defined a *one-parameter (semi)group* of operators satisfying

$$T(t+s) = T(t)T(s), \quad t,s \in \mathbf{R}\left(\mathbf{R}_+\right)$$

$$T(0) = Id_Z$$

since the system is deterministic.

Looking for the solutions of the Cauchy functional equation in $\mathbf{C}$

$$(CFE)\begin{cases} T(t+s) = T(t)\,T(s), \quad t,s \ge 0 \\ T(0) = 1 \end{cases}$$

we find that $T(t) = \mathrm{e}^{ta}$ is a solution for any $a \in \mathbf{C}$. It is easy to see that $T(t) = \mathrm{e}^{ta}$ satisfies the following differential equation, too.

$$(CDE)\begin{cases} \dfrac{d}{dt}T(t) = aT(t), \quad t \ge 0 \\ T(0) = 1. \end{cases}$$

If we suppose the solution $T(t)$ of (CFE) to be continuous, we obtain that it is unique (see Engel-Nagel, 2000).

Theorem—Assume that $T(\cdot):\mathbf{R}_+ \to \mathbf{C}$ is a continuous solution of (CFE). Then there exists a unique $a \in \mathbf{C}$ such that $T(t) = \mathrm{e}^{ta}$.

We now generalize the above result in an arbitrary Banach (complete normed) space X, e.g., in $X = \mathbf{C}^n$, $X = C[a,b]$, or $X = L^1(\mathbf{R})$. By $L(X)$ we denote the space of bounded linear operators on X. Let us look for solutions $T(\cdot):\mathbf{R}_+ \to L(X)$ of the following problem

$$(FE)\begin{cases} T(t+s) = T(t)\,T(s), \quad t,s \ge 0 \\ T(0) = Id_X \end{cases}$$

Definition—Let $T(\cdot):\mathbf{R}_+ \to L(X)$ be a solution of (FE) satisfying

$$\lim_{t\to 0+} T(t)x = x \quad \forall x \in X.$$

Then $(T(t))_{t\ge 0}$ is called a **strongly continuous (one-parameter) semigroup** (or C_0**-semigroup**). If these properties hold for $\mathbf{R}$ instead of $\mathbf{R}_+$, we call $(T(t))_{t\ge 0}$ a **strongly continuous (one-parameter) group** (or C_0**-group**) . For details see Engel-Nagel (2000) and Pazy (1983).

2. Generator

If $A \in L(X)$ – e.g. $A \in M_n(\mathbf{C}), X = \mathbf{C}^n$ – then using the exponential series we can define $\mathrm{e}^{tA} \in L(X)$. It is easy to see that the operator family $T(t) := \mathrm{e}^{tA}, t \ge 0$ forms a C_0-semigroup satisfying (FE). Furthermore, $T(t)$ is a solution of the following differential equation:

$$(DE)\begin{cases} \dfrac{d}{dt}T(t) = AT(t), \quad t \ge 0 \\ T(0) = Id_X \end{cases}$$

In this case

$$A = \frac{d}{dt}T(t)\Big|_{t=0}$$

and A is called the *generator* of the semigroup.

In general, we can define the generator of a strongly continuous semigroup as follows (see Engel-Nagel, 2000 and Pazy, 1983).

Definition—Let $(T(t))_{t\ge 0}$ be a strongly continuous semigroup. The linear (but not necessarily bounded) operator

$$D(A) := \left\{ x \in X : \exists \lim_{t\to 0+} \frac{T(t)x - x}{t} \in X \right\}$$

$$Ax := \lim_{t\to 0+} \frac{T(t)x - x}{t} = \frac{d}{dt}(t \mapsto T(t)x)\Big|_{t=0}$$

is called the **generator** of $(T(t))_{t\ge 0}$.

Since $(A, D(A))$ is defined as the derivative of the orbits of the semigroup in 0, $T(t)$ is in some ways the generalization of the exponential function of A. Of course, in this case e^{tA} can not be defined by the exponential series because $(A, D(A))$ is not bounded and the series not necessarily converges in norm. But one can prove that $D(A)$ is always *dense* in X and $(A, D(A))$ is *closed*.

3. Abstract Cauchy problems

Up to now it is not clear how operator semigroups can be used for solving problems in the applications. The clue is the *abstract Cauchy problem*. It is well-known that many physical phenomena can be formulated mathematically as a system of partial differential equations, see e.g. the air pollution transport model in the next section. These systems can often be rewritten as an abstract Cauchy problem, that is

$$(ACP)\begin{cases} \dot{u}(t) = Au(t), \quad t \ge 0 \\ u(0) = u_0 \end{cases}$$

The operator A on the right-hand side is usually an (unbounded) differential operator on a function (Banach) space X, $x(t)\in X, t\geq 0$. One can prove the following (see e.g. in Engel-Nagel, 2000).

Theorem—Let $(A, D(A))$ be a closed, densely defined linear operator on X and let (ACP) be the associated abstract Cauchy problem defined as above. Then the following assertions are equivalent.

a) For every $x_0 \in D(A)$ there exists a unique solution of (ACP) depending continuously on the initial data x_0.

b) $(A, D(A))$ is the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on X.

In this case the solution is $x(t) = T(t)x_0, \quad t\geq 0.$

Hence, to prove well-posedness of a problem written in the form of an abstract Cauchy problem one has to verify that the operator on the right-hand side is the generator of a C_0-semigroup. In general it is not easy, but in many important cases it is possible.

4. Examples

The next examples can be found in Engel-Nagel (2000).

Diffusion semigroup

Let us take a look at the one-dimensional heat conduction equation with Neumann boundary conditions:

$$\frac{\partial}{\partial t}u(t,s) = \frac{\partial^2}{\partial s^2}u(t,s), \quad t\geq 0,\ s\in(0,1)$$
$$u(0,s) = f(s), \quad s\in[0,1]$$
$$\frac{\partial}{\partial s}u(t,0) = \frac{\partial}{\partial s}u(t,1) = 0, \quad t\geq 0.$$

We can rewrite it as

$$(ACP)\begin{cases}\dot{x}(t) = Ax(t), & t\geq 0\\ x(0) = x_0\end{cases}$$

with

$$Af := f''$$
$$D(A) := \{f\in C^2[0,1] : f'(0) = f'(1) = 0\}.$$

Here the Banach space is $X = C[0,1]$ and $x(t) = u(t,\cdot)$. Observe that the boundary conditions appear in the *domain* of A hence the operator becomes unbounded – but still it is closed and densely defined in X.

Using the eigenvalues $-\pi^2 n^2$ and eigenfunctions $1, \sqrt{2}\cos\pi ns, n\geq 2$ of A and the theory of linear ordinary differential equations, one can prove the following.

Theorem—The operator $(A, D(A))$ defined above generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = C[0,1]$ with

$$(T(t)f)(s) = \int_0^1 k_t(s,r)f(r)dr, \quad f\in C[0,1], s\in[0,1]$$
$$k_t(s,r) := 1 + 2\sum_{n=0}^{+\infty} e^{-\pi^2 n^2 t}\cos\pi ns\cdot\cos\pi nr.$$

This semigroup is called the *one-dimensional diffusion semigroup.*

In $\mathbf{R}^n$ one can prove the following.

Theorem—Consider the *closure* of the Laplace operator

$$\Delta f(s_1, s_2, \ldots, s_n) = \sum_{j=1}^{n}\frac{\partial^2}{\partial s_j^2} f(s_1, s_2, \ldots, s_n),$$

defined for every f from the Schwartz space of rapidly decreasing, infinitely many times differentiable functions on $\mathbf{R}^n$. It generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = L^1(\mathbf{R}^n)$ with

$$(T(t)f)(\mathbf{s}) = \frac{1}{\sqrt{4\pi t}}\int_{\mathbf{R}^n} e^{-\frac{|\mathbf{s}-\mathbf{r}|^2}{4t}} f(\mathbf{r})d\mathbf{r}, \quad t>0, \mathbf{s}\in\mathbf{R}^n$$
$$T(0) = Id.$$

This semigroup is called the *n-dimensional diffusion semigroup.*

Translation semigroup
Let us investigate the closure of the following first order differential operator

$$Af := \nabla f$$

$$D(A) := C_c^1(\mathbf{R}^n).$$

Here $C_c^1(\mathbf{R}^n)$ denotes the space of continuously differentiable functions having compact support in $\mathbf{R}^n$. One can easily prove that $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $X = C_0(\mathbf{R}^n)$ (the space of continuous functions vanishing at infinity on $\mathbf{R}^n$) with

$$(T(t)f)(\mathbf{s}) = f(t\cdot\mathbf{1}+\mathbf{s}), \quad \mathbf{s}\in\mathbf{R}^n,$$

called the *translation semigroup* on $\mathbf{R}^n$.

Multiplication semigroup
Let $q:\mathbf{R}^n \to \mathbf{C}$ be a continuous function. We can define the following closed, densely defined linear operator on $X = C_0(\mathbf{R}^n)$.

$$\mathrm{M}_q\ f := qf$$

$$D(\mathrm{M}_q) := \{f\in C_0(\mathbf{R}^n) : qf \in C_0(\mathbf{R}^n)\}.$$

If

$$\sup_{s\in\mathbf{R}^n} \ \mathrm{Re}\, q(s) < \infty$$

then

$$T_q(t)f := \mathrm{e}^{tq}\ f, \quad t\geq 0,\ f\in C_0(\mathbf{R}^n)$$

defines the strongly continuous *multiplication semigroup*, generated by $(\mathrm{M}_q, D(\mathrm{M}_q))$.

Air pollution transport model
We now turn to a concrete problem that is treated in details in Csomós, Faragó (2005),
Dimov, Faragó, Havasi, Zlatev (2001, 2006). Air pollution transport can be modeled by the following partial differential equation.

$$(APM)\begin{cases} \dfrac{\partial c}{\partial t} = -\nabla(\mathbf{u}c) + \Delta c + E - \sigma c + R(c), \quad t\in(0,T] \\ c(\mathbf{x},0) = c_0(\mathbf{x}), \quad \mathbf{x}\in\mathbf{R}^n. \end{cases}$$

Here $c = c(\mathbf{x},t)$ denotes the concentration of the air pollutant, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$ describes the wind velocity, $E = E(\mathbf{x},t)$ is the emission function, $\sigma = \sigma(\mathbf{x},t)$ the deposition and $R(c)$ the chemistry operator. For the sake of simplicity we assumed the diffusion coefficient to be 1. If we look at the right-hand side of (APM) we find that all the operators acting on c are of type discussed above, hence generate strongly continuous semigroups on appropriate spaces. Using the perturbation theory of semigroups (see Engel-Nagel, 2000) we obtain well-posedness for (APM).

5. Qualitative behaviour

The importance of the operator semigroup theory is revealed especially in proving *qualitative properties* of solutions of partial differential equations (abstract Cauchy problems, resp.). A rich theory for qualitative properties of C_0 –semigroups has been developed in the last 50 years that can be useful also in the applications.

Here we mention only one example. Let us recall the famous Liapunov Stability Theorem for matrices (1892).

Theorem—Let $A\in M_n(\mathbf{C})$ an be an $n\times n$ matrix. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| \mathrm{e}^{tA} \| = 0$

b) All eigenvalues of A have negative real part, i.e., $\mathrm{Re}\,\lambda < 0$ for all $\lambda\in\sigma(A)$.

This result can be generalized for the asymptotic of semigroups having bounded generator (see Engel-Nagel, 2000 and Pazy, 1983).

Theorem—Let $A\in L(X)$ on some Banach space X and $T(t) := \mathrm{e}^{tA}, t\geq 0$ the strongly continuous semigroup generated by A. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| T(t) \| = 0$

b) $\operatorname{Re}\lambda < 0$ for all $\lambda \in \sigma(A)$.

If the semigroup is regular enough, we also can characterize stability with the spectrum of the unbounded generator A .

Theorem—Let A be the generator of an eventually norm-continuous semigroup $(T(t))_{t\geq 0}$ on X , that is, there exists $t_0 \geq 0$ such that the function $t \mapsto T(t)$ is norm continuous from (t_0, ∞) into $L(X)$. Then the following assertions are equivalent.

a) $\lim_{t\to\infty} \| T(t) \| = 0$

b) $\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\} < 0$.

The same holds if the semigroup is *positive* on a function space, that is, it maps positive (i.e. greater or equal to zero) functions into positive functions. This is the case in many important applications such as heat diffusion etc.

Hence, to prove that the solutions of an abstract Cauchy problem converge to 0 if $t \to \infty$ it is enough to investigate the spectrum of the operator on the right-hand side.

6. A few words about applications in numerical analysis

In the numerical solution of (complicated) partial differential equations the *operator splitting method* is often used. Here we divide the spatial differential operator of the system into simpler operators and solve the corresponding problems one after the other, by connecting them through their initial conditions (see e.g. Faragó, 2005). To use this method one has to assume that the sub-problems are well-posed which in practice is often hard to prove. We also have to know the error caused by the operator splitting and by the use of numerical methods. Applying operator semigroup techniques helps a lot to answer these questions.

1. *Csomós, P.* and *Faragó I.,* 2005: Error analysis of the numerical solution obtained by applying operator splitting. Preprint.
2. *Csomós, P., Faragó, I.* and *Havasi, Á.*, 2005: Weighted sequential splittings and their analysis. *Comput. Math. Appl. 50*, 1017–1031.
3. *Dimov, I., Faragó, I., Havasi, Á.* and *Zlatev, Z.,* 2001: L–commutativity of the operators in splitting methods for air pollution models. *Annales Univ. Sci. Sec. Math. 44*, 127–148.
4. *Dimov, I., Faragó, I., Havasi, Á.* and *Zlatev, Z.,* 2006: Different splitting techniques with application to air pollution models, Int. *J. Environmental Pollution* (to appear).
5. *Engel, K.-J.* and *Nagel, R.*, 2000: *One-parameter Semigroups for Linear Evolution Equations*, Springer, New York.
6. *Faragó, I.*, 2005: Splitting methods for abstract Cauchy problems, *Lect. Notes Comp. Sci. 3401*, Springer Verlag, Berlin, 35–45.
7. *Pazy, A., 1983: Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York.*

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