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METHODS OF DERIVATION OF THE STOCHASTIC BOLTZMANN HIERARCHY

МЕТОДИ ВИВЕДЕННЯ СТОХАСТИЧНОЇ ІЄРАРХІЇ БОЛЬЦМАНА

Different methods of derivation of the stochastic Boltzmann hierarchy, associated with the stochastic dynamic that is the Boltzmann – Grad limit of the Hamiltonian dynamics of hard spheres, are considered. Solutions of the stochastic Boltzmann hierarchy are the Boltzmann – Grad limit of solutions of the BBGKY hierarchy of hard spheres in the entire phase space. A new conception of reduced distribution functions, associated with the stochastic dynamics are introduced. They takes into account contributions from hyperplanes of lower dimension where stochastic point particles interact. Solutions of the Boltzmann equation coincide with one-particle distribution functions of the stochastic Boltzmann hierarchy and they are represented by integrals over the hyperplanes where stochastic point particles interact.

Розглянуто різні методи виведення стохастичної ієрархії, що відповідає стохастичній динаміці, яка є границею Больцмана – Греда від гамільтонової динаміки пружних куль. Розв'язки стохастичної ієрархії є границею Больцмана – Греда розв'язків ієрархії ББГКІ для пружних куль у всьому фазовому просторі. Запропоновано нову концепцію редукованих функцій розподілу, що відповідають стохастичній динаміці. Нові функції розподілу враховують вклади від гіперплощин менших розмірностей, де взаємодіють стохастичні точкові частинки. Розв'язки рівняння Больцмана співпадають з одночастинковими функціями розподілу стохастичної ієрархії Больцмана і зображуються інтегралами по гіперповерхнях, де стохастичні точкові частки взаємодіють.

Introduction. Given paper is devoted to different methods of derivation of the stochastic Boltzmann hierarchy for hard spheres. We show that the usual Boltzmann hierarchy can be derived directly from the Boltzmann equation for the distribution functions that are product of solutions of the Boltzmann equation. Solutions of the usual Boltzmann hierarchy are the Boltzmann – Grad limit of solutions of the BBGKY hierarchy for hard spheres outside the hyperplanes where the difference of positions are parallel to the difference of momenta.

In series of papers [1, 2] we showed that the Boltzmann – Grad limit of the Hamiltonian dynamics of hard spheres defines the stochastic dynamics of point particles which interact if the differences of initial positions are parallel to the difference of initial momenta. We derived the stochastic Boltzmann hierarchy associated with the stochastic dynamics while the BBGKY hierarchy is associated with the Hamiltonian dynamics.

The stochastic Boltzmann hierarchy differs from the usual Boltzmann hierarchy by additional terms with δ-functions and some boundary conditions. Different equivalent representations of the stochastic Boltzmann hierarchy are given. These representations are connected with equivalent representations of the infinitesimal operator of the evolution operator of the stochastic dynamics.

We introduce a new conception of reduced distribution functions that takes into account contributions from the above mentioned hyperplanes of lower dimension where stochastic particles interact. These distribution functions satisfy the stochastic Boltzmann hierarchy and they coincide with the Boltzmann – Grad limit of solutions of the BBGKY hierarchy for hard spheres in the entire phase space.

We show that solutions of the Boltzmann equations coincide with the one-particle distribution function of the stochastic Boltzmann hierarchy with initial chaotic data. Iterations of the Boltzmann equation and the Boltzmann hierarchies are represented by integrals over the hyperplanes of lower dimension where the stochastic particles interact. It is a surprise that this property of solutions of the Boltzmann equation and the Boltzmann hierarchies have not been observed earlier.

Given paper is continuation of series of paper [1, 2] with the same denotation and it could be considered as detailed exposition of results announced in preprint [3].

I. The Boltzmann hierarchy and the stochastic Boltzmann hierarchy. 1. ***The Boltzmann hierarchy.*** There exist two methods of the derivation of the Boltzmann hierarchy for hard spheres. The first one – from the Boltzmann equation, the second one – from the BBGKY hierarchy for systems of hard spheres in the Boltzmann – Grad limit. Consider the first method.

We have the Boltzmann equation for hard spheres

$$\frac{\partial f(t, x_1)}{\partial t} = -p_1 \frac{\partial}{\partial q_1} f(t, x_1) + \int_{S_+^2} d\eta \int dp_2\, \eta\cdot(p_1 - p_2) \times$$

$$\times [f(t, q_1, p_1^*) f(t, q_1, p_2^*) - f(t, q_1, p_1) f(t, q_1, p_2)], \quad t \geq 0, \tag{1.1}$$

where

$$x = (q, p), \quad |\eta| = 1, \quad S_+^2(\eta | \eta\cdot(p_1 - p_2) \geq 0),$$

$$p_1^* = p_1 - \eta\,\eta\cdot(p_1 - p_2), \quad p_2^* = p_2 + \eta\,\eta\cdot(p_1 - p_2).$$

We consider the Cauchy problem to equation (1.1)

$$f(t, x_1)|_{t=0} = f(0, x_1) \tag{1.2}$$

in certain functional space and suppose that solution of (1.1), (1.2) exists.

Define the, associate with solution of the Boltzmann equation, sequence of s-particle distribution functions

$$F_s(t, x_1, \ldots, x_s) = f(t, x_1) f(t, x_2) \ldots f(t, x_s), \quad s \geq 2, \tag{1.3}$$

$$F_1(t, x_1) = f(t, x_1).$$

From (1.3) and (1.1), (1.2) it follows that sequence (1.3) satisfies the following chain of integro-differential equations

$$\frac{\partial F_s(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) +$$

$$+ \sum_{i=1}^{s} \int_{S_+^2} d\eta \int dp_{s+1}\, \eta\cdot(p_i - p_{s+1}) [F_{s+1}(t, x_1, \ldots, q_i, p_i^*, \ldots, x_s, q_i, p_{s+1}^*) -$$

$$- F_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, x_s, q_i, p_{s+1})], \quad s \geq 1, \quad t \geq 0 \tag{1.4}$$

with initial data

$$F_s(t, x_1, \ldots, x_s)|_{t=0} = F_s(0, x_1, \ldots, x_s) = F_1(0, x_1) \ldots F_1(0, x_s). \tag{1.5}$$

We say that initial data (1.5) satisfy the chaos property.

The chain of equations (1.4) is known us the *Boltzmann hierarchy*.

It is obvious that the Boltzmann hierarchy (1.4) with initial data (1.5) is equivalent to the Boltzmann equation (1.1) in the following meaning.

We have showed that the sequence of the s-particle distribution functions (1.3) are solutions of the Boltzmann hierarchy (1.4) with initial data (1.5). And, vice versa, solutions of the Boltzmann hierarchy (1.4) with initial data (1.5) have also the chaos property, i. e., the s-particle distribution functions are the products of the one-particle distribution functions

$$F_s(t, x_1, \ldots, x_s) = F_1(t, x_1) \ldots F_1(t, x_s),$$

where the one-particle distribution functions satisfies the Boltzmann equation (1.1).

The last assertion follows directly from (1.4). Indeed the Boltzmann hierarchy admits the separation of variables, because in the right-hand side of (1.4) we have the sum of s operators that act with respect to each s-variables x_i, $i = 1, \dots, s$.

2. ***The stochastic Boltzmann hierarchy.*** Now consider the second method.

We have the BBGKY hierarchy for systems of hard spheres with diameter a

$$\frac{\partial F_s^a(t, x_1, \dots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s^a(t, x_1, \dots, x_s) +$$

$$+ \sum_{i=1}^{s} a^2 \int_{S_+^2} d\eta \int dp_{s+1} \eta \cdot (p_i - p_{s+1}) [F_{s+1}^a(t, x_1, \dots, q_i, p_i^*, \dots, x_s, q_i - a\eta, p_{s+1}^*) -$$

$$- F_s^a(t, x_1, \dots, q_i, p_i, \dots, x_s, q_i + a\eta, p_{s+1})], \qquad t \geq 0 \tag{1.6}$$

with initial data

$$F_s(t, x_1, \dots, x_s)|_{t=0} = F_s(0, x_1, \dots, x_s).$$

The distribution functions $F_s^a(x_1, \dots, x_s)$ are symmetric and satisfy the following conditions $F_s^a(t, x_1, \dots, x_s) = 0$ if $|q_i - q_j| < a$ at least for one pair $(i, j) \subset (1, \dots$ $\dots, s)$. This means that the particles under consideration are hard spheres and the distances between their centers should be greater or equal a, i. e. hard spheres occupy the admissible configurations.

The distribution functions $F_s^a(t, x_1, \dots, x_s)$ satisfy the following boundary conditions

$$F_s^a(t+0, x_1, \dots, x_i, \dots, x_j, \dots, x_s) = F_s(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_s)$$

if $q_i - q_j = a\eta_{ij}$, $\eta_{ij} \cdot (p_i - p_j) \geq 0$,

$$F_s^a(t+0, x_1, \dots, x_i, \dots, x_j, \dots, x_s) = F_s(t, x_1, \dots, x_i, \dots, x_j, \dots, x_s)$$

if $q_i - q_j = a\eta_{ij}$, $\eta_{ij} \cdot (p_i - p_j) \leq 0$,

$$x_i^* = (q_i, p_i^*), \qquad x_j^* = (q_j, p_j^*). \tag{1.7}$$

We have also the following boundary conditions in the Poisson bracket $-\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \dots, x_s)$. If $q_i - q_j = a\eta_{ij}$ and $\eta_{ij} \cdot (p_i - p_j) \geq 0$, i. e. $\eta_{ij} \in S_+^2$, then the momenta (p_i, p_j) should be replaced by (p_i^*, p_j^*). The momenta (p_i, p_j) do not change if $\eta_{ij} \cdot (p_i - p_j) \geq 0$, i. e. $\eta_{ij} \in S_-^2$.

Let us stress that the boundary conditions in the Poisson bracket are of great importance because these boundary conditions are associated with the dynamics of hard spheres. Namely they are responsible for the jumps of momenta $(p_i, p_j) \to$ $\to (p_i^*, p_j^*)$ when hard spheres touch each other $(q_i - q_j = a\eta_{ij})$ and elastically collide. The Poisson bracket together with the boundary conditions is the infinitesimal operator of the evolution operator of s hard spheres. (see [4, 5]).

Introduce the sequence of renormalized distribution functions

$$\tilde{F}_s^a(t, x_1, \dots, x_s) = a^{2s} F_s^a(t, x_1, \dots, x_s), \qquad s \geq 1 \tag{1.8}$$

and tend the diameter a to zero, $a \to 0$. We suppose that there exists the Boltzmann – Grad limit of the renormalized sequence (1.8) in some sense (see [5])

$$\lim_{a\to 0} \tilde{F}_s^a(t, x_1, \dots, x_s) = F_s(t, x_1, \dots, x_s), \quad s \geq 1. \tag{1.9}$$

Performing formally the limiting procedure (1.9) in (1.6) we obtain

$$\frac{\partial F_s(t, x_1, \dots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \dots, x_s) +$$

$$+ \sum_{i=1}^{s} \int_{S_+^2} d\eta \int dp_{s+1} \eta \cdot (p_i - p_{s+1}) [F_{s+1}(t, x_1, \dots, q_i, p_i^*, \dots, x_s, q_i, p_{s+1}^*) -$$

$$- F_{s+1}(t, x_1, \dots, q_i, p_i, \dots, x_s, q_i, p_{s+1})]. \tag{1.10}$$

At first sight hierarchy (1.10), that was obtained from the BBGKY hierarchy in the Boltzmann – Grad limits looks like the Boltzmann hierarchy (1.4) obtained from the Boltzmann equation (1.1). In fact, there exist principal differences between them.

Namely, the distribution functions $F_s(t, x_1, \dots, x_s)$ satisfy the following boundary conditions:

$$F_s(t+0, x_1, \dots, x_i, \dots, x_j, \dots, x_s) = F_s(t, x_1, \dots, x_i^*, \dots, x_j^*, \dots, x_s)$$

if $q_i = q_j$, $\eta_{ij} \cdot (p_i - p_j) \geq 0$,

$$F_s(t+0, x_1, \dots, x_i, \dots, x_j, \dots, x_s) = F_s(t, x_1, \dots, x_i, \dots, x_j, \dots, x_s)$$

if $q_i = q_j$, $\eta_{ij} \cdot (p_i - p_j) \leq 0$,

$$x_i^* = (q_i, p_i^*), \quad x_j^* = (q_j, p_j^*). \tag{1.7'}$$

In hierarchy (1.10) we have also the Poisson bracket $-\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \dots, x_s)$ with boundary conditions according to which one should replace (p_i, p_j) by (p_i^*, p_j^*) if $q_i = q_j$ and $\eta_{ij} \cdot (p_i - p_j) \geq 0$, i. e. $\eta_{ij} \in S_+^2$. The momenta (p_i, p_j) do not change if $\eta_{ij} \cdot (p_i - p_j) \leq 0$, i. e. $\eta_{ij} \in S_-^2$. These boundary conditions follow from the corresponding boundary conditions of the BBGKY hierarchy (1.4) for hard spheres. For hard spheres the boundary conditions are given on the spheres $q_i - q_j = a\eta_{ij}$ that are reduced to the hyperplanes $q_i - q_j = 0$ in the Boltzmann – Grad limit. In hierarchy (1.4) these boundary conditions are absent, because they are also absent in the Boltzmann equation (1.1).

The second principal difference consists in random character of the unit vector η in (1.10). In the BBGKY hierarchy (1.6) the vector η is directed between the centers of colliding spheres and defines a relative shift of the centers of the spheres. In the hierarchy (1.10) we have point-wise particles, and the vector η is random one with constant density of probability on the sphere S^2. It characterizes how the point-wise particles were obtained from the colliding hard spheres. The details of the mechanism of obtaining the stochastic point-wise particles from hard spheres have been described in papers [1 – 3].

Thus, we have derived two different hierarchies: (1.4) from the Boltzmann equation and (1.10) from the BBGKY hierarchy for hard spheres. The difference consists in the boundary conditions for the distribution functions and for the Poisson bracket (of the free particles) in hierarchy (1.10) obtained from the BBGKY hierarchy for hard spheres. These boundary conditions are absent for hierarchy (1.4) obtained from the Boltzmann equation.

More careful analysis of the derivation of the hierarchy (1.10) shows that there is equivalent to (1.10) hierarchy with one additional special term with δ-functions and finally the hierarchy for limit sequence (1.9) looks as follows

$$\frac{\partial F_s(t, x_1, \ldots, x_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, x_1, \ldots, x_s) + \sum_{i<j=1}^{s} \delta(q_i - q_j)\eta_{ij} \cdot (p_i - p_j) \times$$

$$\times\ \theta(\eta_{ij} \cdot (p_i - p_j))[F_s(t, x_1, \ldots, q_i, p_i^*, \ldots, q_j, p_i^*, \ldots, x_s) -$$

$$- F_s(t, x_1, \ldots, q_i, p_i, \ldots, q_j, p_j, \ldots, x_s)] + \sum_{i=1}^{s} \int_{S_+^2} d\eta \int dp_{s+1} \left|\eta \cdot (p_i - p_{s+1})\right| \times$$

$$\times\ [F_{s+1}(t, x_1, \ldots, q_i, p_i^*, \ldots, x_s, q_i, p_{s+1}^*) - F_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, x_s, q_i, p_{s+1})], \quad (1.11)$$

where θ is the Hewiside function. We will call hierarchy (1.11) as *the stochastic Boltzmann hierarchy.*

And again we have the above described boundary conditions for the distribution functions (1.7′) and in the Poisson bracket. Later we will show that hierarchies (1.10) and (1.11) are equivalent.

Thus, the Boltzmann hierarchy (1.4) is not the Boltzmann – Grad limit of the BBGKY hierarchy. The Boltzmann – Grad limit of the BBGKY hierarchy is the stochastic hierarchy (1.11) (or (1.10)) that differs from the Boltzmann hierarchy (1.4) by the boundary conditions for the distributions functions (1.7′) and in the Poisson bracket and additional term with δ-functions.

Nevertheless the Boltzmann hierarchy (1.4) is of great importance, because solutions of the Boltzmann hierarchy (1.4) are the Boltzmann – Grad limit of the solutions of the BBGKY hierarchy (1.6) for hard spheres in the following sense.

3. ***Solutions of the Boltzmann hierarchy and the Boltzmann – Grad limit of the solutions of the BBGKY hierarchy.*** We know that solutions of the Boltzmann hierarchy (1.4) and the BBGKY hierarchy (1.6) exist locally in time for initial data from the space E_ξ of sequences of functions bounded with respect to positions and exponentially decreasing with respect to squared momenta. Solutions of the both hierarchies exist globally in time for initial data from the space $X_{\xi,\beta}$ of the sequences of functions exponentially decreasing with respect to squared momenta and positions. These solutions can be represented by series of iterations [5].

Consider a compact K_s in the space of positions $(q_1, \ldots, q_s)$ such that $|q_i - q_j| \geq a_0(a)$, $(i,j) \subset (1, \ldots, s)$, $s \geq 2$ and $a_0(a) \to 0$ as $a \to 0$ in such a way that $\lim_{a\to 0} \frac{a}{a_0(a)} = 0$. Consider also a compact in the momentum space $(p_1, \ldots, p_s)$, $\sqrt{p_1^2 + \ldots + p_s^2} < p_0$ and cones V_{ij} with respect to differences $p_i - p_j$, $(i,j) \subset (1, \ldots, s)$, $s \geq 2$ with axes parallel to the vectors $q_i - q_j$ with radius of the basis equal to $\frac{a}{a_0(a)}$, and height p_0.

Then

$$\lim_{a\to 0} (\tilde{F}_s^a(t, x_1, \ldots, x_s) - F_s(t, x_1, \ldots, x_s)) = 0, \quad s \geq 2 \qquad (1.12)$$

uniformly with respect $(q_1, \ldots, q_s) \subset K_s$ and with respect to momenta $(p_1, \ldots, p_s)$ outside the cones $\bigcup_{i<j=1}^{N} V_{ij}$. We suppose that the limits (1.12) exist for the initial distribution functions $\tilde{F}_s^a(0, x_1, \ldots, x_s)$, $F_s(0, x_1, \ldots, x_s)$ for $(x_1, \ldots, x_s)$ on arbitrary compacts of admissible configurations.

In other words renormalized solutions of the BBGKY hierarchy (1.6) *tends in the*

Boltzmann – Grad limit to the solutions of the Boltzmann hierarchy (1.4) *outside the hyperplanes* $q_i - q_j = 0$, *and hyperplanes with vectors* $p_i - p_j$ *parallel to the vector* $q_i - q_j$, $(i, j) \subset (1, \ldots, s)$. *Note that all vectors* $p_i - p_j$ *are parallel to vector* $q_i - q_j = 0$, $s \geq 2$.

For $s = 1$ we have

$$\lim_{a \to 0} (\tilde{F}_1^a(t, x_1) - F_1(t, x_1)) = 0$$

in the entire phase space of one particle. Note that in some papers it was made the assertions that formulae (1.12) hold outside the hyperplanes $q_i - q_j = 0$, $(i, j) \subset (1, \ldots, s)$. It is a rough mistake.

Thus, solutions of the Boltzmann hierarchy are the Boltzmann – Grad limit of the solutions of the BBGKY hierarchy for hard spheres outside the above described hyperplanes for $s \geq 2$ and in the entire phase space for $s = 1$.

Note that the term with δ-functions was not obtained in (1.10) because the Boltzmann – Grad limit was performed on the admissible configurations and outside the above described hyperplanes where δ-functions are different from zero. In the next section we will show that solutions of the hierarchy (1.11) are the Boltzmann – Grad limit of the solutions of the BBGKY hierarchy (1.6) *in the entire phase space*. We will show that the hierarchy (1.11) is associated with the stochastic dynamics introduced in series of papers [1–3] while the BBGKY hierarchy is associated with the Hamiltonian dynamics of hard spheres. For this reason we will call the hierarchy (1.11) as the *stochastic Boltzmann hierarchy* and hierarchy (1.4) as the *Boltzmann hierarchy*.

II. Derivation of the stochastic Boltzmann hierarchy from the evolution operator of the BBGKY hierarchy for hard spheres. 1. ***Solutions of the BBGKY hierarchy for hard spheres.*** Consider the solutions of the BBGKY hierarchy (1.6) represented by group of the evolution operators [4, 5]. Denote by $F^a(t)$ the sequence of distribution functions

$$F^a(t) = (F_1^a(t, x_1), \ldots, F_s^a(t, (x)_s), \ldots), \quad (x)_s = (x_1, \ldots, x_s). \tag{2.1}$$

The sequence $F^a(t)$ can be represented by the group of evolution operator $U^a(t)$ as follows

$$F^a(t) = U^a(t)F(0) = e^{\int dx} S^a(-t) e^{-\int dx} F(0). \tag{2.2}$$

The meaning of operators $\int dx$, $S^a(-t)$ can be found in [4, 5]. From (2.2) we obtain the following expression for

$$F_s^a(t, (x)_s) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!} \int d(x)_{s+n}^s S_{s+n-k}^a(-t, (x)_{s+n-k}) F_{s+n}^a(0, (x)_{s+n}),$$

$$s \geq 1, \quad d(x)_{s+n}^s = dx_{s+1} \ldots dx_{s+n}. \tag{2.3}$$

Here $S_{s+n-k}^a(-t, (x)_{s+n-k})$ is operator of evolution of $s + n - k$ particles with initial phase points $(x)_{s+n-k}$.

The group $U^a(t)$ is strongly continuous bounded operator in the space L of direct sum of integrable functions. The infinitesimal operator of the group $U^a(t)$ coincides on some everywhere dense set L_0 in L with the operator in the right-hand side of hierarchy (1.6).

The group $U^a(t)$ has also a meaning in the space E_ξ of sequences of functions bounded with respect to positions and exponentially decreasing with respect to squared momenta for finite interval of time and globally in time for sequences of locally perturbed equilibrium states (for details see [4, 5]).

Now use the group property of $U^a(t)$ and represent $F^a(t+\Delta t)$ through $F^a(t)$

$$F^a(t+\Delta t) = U^a(\Delta t)U^a(t)F^a(0) = U^a(\Delta t)F^a(t). \tag{2.4}$$

We will use the representation of the evolution operator $U^a(t)$ via series of iteration

$$U^a(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, S^a(-t)S^a(t_1)A^aS^a(-t_1)\ldots S^a(t_n)A^aS^a(-t_n), \tag{2.5}$$

where the operator A^a is defined by the second term of the right-hand side of the hierarchy (1.6) and the operator $S^a(-t)$ is direct sum of the evolution operators of s-particle subsystem $S^a_s(-t, (x)_s)$, $s \geq 1$. We restrict ourselves to $t > 0$, $\Delta t > 0$.

Consider an infinitesimal Δt and represent $U^a(\Delta t)$ by the following identical to (2.5) expression

$$U^a(\Delta t) = S^a(-\Delta t) + \int_0^{\Delta t} d\tau\, S^a(\tau - \Delta t)A^aS^a(-\tau). \tag{2.6}$$

Substituting (2.6) into (2.4) we obtain

$$F^a(t+\Delta t) = S^a(-\Delta t)F^a(t) + \int_0^{\Delta t} d\tau\, S^a(\tau-\Delta t)A^aS^a(-\tau)F^a(t) \tag{2.7}$$

or componentwise

$$\begin{aligned} F^a(t+\Delta t, (x)_s) &= S^a_s(-\Delta t, (x)_s)F^a_s(t, (x)_s) + \\ &+ \int_0^{\Delta t} d\tau\, S^a_s(\tau - \Delta t, (x)_s) \sum_{i=1}^{s} \int_{S^2_+} d\eta \int dp_{s+1}\, \eta\cdot(p_i - p_{s+1}) \times \\ &\times [S^a_{s+1}(-\tau, (x^*)_{s+1})F^a_{s+1}(t, (x)^*_{s+1})|_{q_{s+1}=q_i - a\eta} - \\ &- S^a_{s+1}(-\tau, (x)_{s+1})F^a_{s+1}(t, (x)_{s+1})|_{q_{s+1}=q_i+a\eta}], \end{aligned} \tag{2.8}$$

where $(x)^*_{s+1} = (x_1, \ldots, q_i, p^*_i, \ldots, x_s, q_{s+1}, p^*_{s+1})$ in term with number i.

2. ***The Boltzmann – Grad limit.*** Now multiply relation (2.9) by a^{2s}, use the renormalized functions

$$\tilde{F}^a_s(t, (x)_s) = a^{2s}F^a_s(t, (x)_s), \quad s \geq 1$$

and tend diameter a to zero. This procedure is known as the Boltzmann – Grad limit. Taking into account that

$$\lim_{a\to 0} S^a_s(-\Delta t, (x)_s) = S_s(-\Delta t, (x)_s), \tag{2.9}$$

where $S_s(-\Delta t, (x)_s)$ is the evolution operator corresponding to the stochastic dynamics [1 – 3] and supposing existence of the Boltzmann – Grad limit

$$\lim_{a\to 0} \tilde{F}^a_s(t, (x)_s) = F_s(t, (x)_s), \quad s \geq 1 \tag{2.10}$$

we obtain from (2.8) the following relation *in the entire phase space* $(x)_s$

$$F_s(t+\Delta t,(x)_s) = S_s(-\Delta t,(x)_s)F_s(t,(x)_s) +$$

$$+\int_0^{\Delta t} d\tau S_s(\tau-\Delta t,(x)_s)\sum_{i=1}^{s}\int_{S_+^2} d\eta \int dp_{s+1}\,\eta\cdot(p_i-p_{s+1})\times$$

$$\times[S_{s+1}(-\tau,(x^*)_{s+1})F_{s+1}(t,(x)^*_{s+1})|_{q_{s+1}=q_i} +$$

$$+ S_{s+1}(-\tau,(x)_{s+1})F_{s+1}(t,(x)_{s+1})|_{q_{s+1}=q_i}] = S_s(-\Delta t,(x)_s)F_s(t,(x)_s) +$$

$$+\sum_{i=1}^{s}\int dx_{s+1}\int_0^{\Delta t} d\tau \int_{S_+^2} d\eta_{is+1}\eta_{is+1}\cdot(p_i-p_{s+1})\delta(q_i-p_i\tau-q_{s+1}+p_{s+1}\tau)\times$$

$$\times[F_{s+1}(t,q_1-p_1\Delta t,p_1,\ldots,q_i-p_i\tau-p_i^*(\Delta t-\tau),\ p_i^*,\ldots,q_{s+1}-p_{s+1}\tau-$$

$$-p^*_{s+1}(\Delta t-\tau),\ p^*_{s+1})-F_{s+1}(t,q_1-p_1\Delta t,p_1,\ldots,q_i-p_i\Delta t,p_i,\ldots,q_{s+1}-$$

$$-p_{s+1}\Delta t,p_{s+1})] = S_s(-\Delta t,(x)_s)F_s(t,(x)_s)+I_s,\qquad s\geq 1. \tag{2.11}$$

Obtaining the final expression for I_s we replace the operators $S_s(\tau-\Delta t,(x)_s)$, $S_{s+1}(-\tau,(x^*)_{s+1})$, $S_{s+1}(-\tau,(x)_{s+1})$ of the stochastic evolution by the corresponding operators $S^0(\tau-\Delta t,(x)_s)$, $S^0_{s+1}(-\tau,(x^*)_{s+1})$, $S^0_{s+1}(-\tau,(x)_{s+1})$ of the free evolution and neglect terms of higher order with respect to Δt (see derivation of formula (4.3)). It is obvious that we can also replace all these operators by unit operators under integral sign $\int_0^{\Delta t} d\tau$.

As known [1 – 3]

$$S_s(-\Delta t,(x)_s)F_s(t,(x)_s) = F_s(t,q_1-p_1\Delta t,p_1,\ldots,q_i-p_i\tau-$$

$$-p_i^*(\Delta t-\tau),\ p_i^*,\ldots,q_j-p_j\tau-p_j^*(\Delta t-\tau),\ p_j^*,\ldots,q_s-p_s\Delta t,p_s)$$

if $q_i-q_j=(p_i-p_j)\tau$, for some $0\leq\tau\leq\Delta t$, and some pair $(i,j)\subset(1,\ldots,s)$, $\eta_{ij}\in$ $\in S_+$,

$$S_s(-\Delta t,(x)_s)F_s(t,(x)_s) = F_s(t,q_1-p_1\Delta t,p_1,\ldots,q_i-$$

$$-p_i\Delta t,p_i,\ldots,q_j-p_j\Delta t,p_j,\ldots,q_s-p_s\Delta t,p_s)$$

if $q_i-q_j\neq(p_i-p_j)\tau$ for all pairs $(i,j)\subset(1,\ldots,s)$ and for all $0\leq\tau\leq\Delta t$, and we suppose that Δt is infinitesimal.

Denote by $D_{-\Delta t}$ the set $\bigcup_{(i,j)}\ \bigcup_{0\leq\tau\leq\Delta t}(q_i-q_j=(p_i-p_j)\tau)$. Then the operator $S_s(-\Delta t,(x)_s)$ is equal to $S^0_s(-\Delta t,(x)_s)$ outside of the set $D_{-\Delta t}$.

From (2.11) we see that the function $F_s(t+\Delta t,(x)_s)$ depends on the random vectors η_{ij} created on the interval $(t,t+\Delta t)$ for $(x)_s\subset D_{-\Delta t}$ only through the term $S_s(-\Delta t,(x)_s)F_s(t,(x)_s)$, the second term I_s does not depend on these random vectors.

3. *The functional* $(F_s(t+\Delta t),\varphi_s)$. Now define the functional $(F_s(t+\Delta t),\varphi_s)$ that is the average of the observable $\varphi_s((x)_s)$ over the state $F_s(t+\Delta t,(x)_s)$ with respect to the random vectors η_{is}, $1\leq i<j\leq s$ corresponding to the stochastic dynamics described by the operator $S_s(-\Delta t,(x)_s)$ for infinitesimal Δt. In doing this

we take into account the contribution from the set $D_{-\Delta t}$. According to the definition [1 – 3], the functional $(F_s(t+\Delta t), \varphi_s)$ is equal to the two terms. One of them is defined by the average with respect to the random vectors η_{ij} of integral over the set $D_{-\Delta t}$ with integrand

$$[S_s(-\Delta t, (x)_s) - S_s^0(-\Delta t, (x)_s)]\, F_s(t, (x)_s)\, \varphi_s((x)_s).$$

The second one is defined by integral over the entire phase space of s-particle system with integrand

$$[\, S_s^0(-\Delta t, (x)_s)\, F_s(t, (x)_s) + I_s]\, \varphi_s((x)_s).$$

Finally we obtain the functional

$$\begin{aligned}
(F_s(t+\Delta t), \varphi_s) = \int d(x)_s \Bigg\{ \sum_{i<j=1}^{s} \int_0^{\Delta t} d\tau \int_{S_+^2} d\eta_{ij}\eta_{ij}\cdot(p_i - p_j)\delta(q_i - p_i\tau - q_j + p_j\tau) \times \\
\times [F_s(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau),\ p_i^*, \ldots, q_j - p_j\tau - \\
- p_j^*(\Delta t - \tau),\ p_j^*, \ldots, q_s - p_s\Delta t, p_s) - F_s(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, \\
p_i, \ldots, q_j - p_j\Delta t, p_j, \ldots, q_s - p_s\Delta t, p_s)\,]\,\varphi_s((x)_s) \Bigg\} + \\
+ \int d(x)_s [S_s^0(-\Delta t, (x)_s)] F_s(t, (x)_s)\varphi_s((x)_s) + \\
+ \int \Bigg\{ \sum_{i=1}^{s} \int d(x)_{s+1} \int_0^{\Delta t} d\tau \int_{S_+^2} d\eta_{is+1}\eta_{is+1}\cdot(p_i - p_{s+1})\delta(q_i - p_i\tau - q_{s+1} + p_{s+1}\tau) \times \\
\times [F_{s+1}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau),\ p_i^*, \ldots, q_s - p_s\Delta t, p_s, \\
q_{s+1} - p_{s+1}\tau - p_{s+1}^*(\Delta t - \tau),\ p_{s+1}^*) - F_{s+1}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, \\
p_i, \ldots, q_s - p_s\Delta t, p_s, q_{s+1} - p_{s+1}\Delta t, p_{s+1})]\,\varphi_s((x)_s) \Bigg\}. \qquad (2.12)
\end{aligned}$$

From (2.12) we obtain

$$\begin{aligned}
F_s(t+\Delta t, (x)_s) = S_s^0(-\Delta t, (x)_s) F_s(t, (x)_s) + \\
+ \sum_{i<j=1}^{s} \int_0^{\Delta t} d\tau \int_{S_+^2} d\eta_{ij}\eta_{ij}\cdot(p_i - p_j)\delta(q_i - p_i\tau - q_j + p_j\tau) \times \\
\times [F_s(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau),\ p_i^*, \ldots, q_j - p_j\tau - \\
- p_j^*(\Delta t - \tau),\ p_j^*, \ldots, q_s - p_s\Delta t, p_s) - F_s(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\Delta t, \\
p_i, \ldots, q_j - p_j\Delta t, \ldots, q_s - p_s\Delta t, p_s)] + \\
+ \sum_{i=1}^{s} \int dx_{s+1} \int_0^{\Delta t} d\tau \int_{S_+^2} d\eta_{is+1}\eta_{is+1}\cdot(p_i - p_{s+1})\delta(q_i - p_i\tau - q_{s+1} + p_{s+1}\tau) \times \\
\times [F_{s+1}(t, q_1 - p_1\Delta t, p_1, \ldots, q_i - p_i\tau - p_i^*(\Delta t - \tau),\ p_i^*, \ldots, q_s - p_s\Delta t, p_s,
\end{aligned}$$

$$q_{s+1}-p_{s+1}\tau-p_{s+1}^{*}(\Delta t-\tau),\ p_{s+1}^{*})-F_{s+1}(t,q_1-p_1\Delta t,p_1,\dots,q_i-p_i\Delta t,$$
$$p_i,\dots,q_s-p_s\Delta t,p_s,q_{s+1}-p_{s+1}\Delta t,p_{s+1})]. \qquad (2.12')$$

In the distribution functions (2.12′) the averaging procedure is performed with respect to the random vectors η_{ij} in points $q_i-q_j=\tau(p_i-p_j)$, $0\le\tau\le\Delta t$, $1\le i<j\le\le s$, where stochastic particles interact. We preserve for them the same denotation $F_s(t+\Delta t,x_s)$ as for these defined by (2.11). In expression (2.12) the contributions from the hyperplanes $D_{-\Delta t}$ of lower dimension, where the stochastic particles interact, are taken into account. Usually in statistical mechanics the hyperplanes of lower dimension are neglected. We will show later (Section IV) that solutions of the Boltzmann equation and hierarchy are expressed in terms of introduced above distribution functions that takes into account the contributions from the hyperplanes where the stochastic particles interact.

Note that the average of the observable $\varphi_s((x)_s)$ over the state $F_s(t+\Delta t,(x)_s)$ (2.12′) is equal to the following integral

$$(F_s(t+\Delta t),\varphi_s) = \int F_s(t+\Delta t,(x)_s)\varphi_s((x)_s)d(x)_s$$

analogously to the usual statistical mechanics.

4. ***The stochastic Boltzmann hierarchy.*** It follows directly from (2.12) that. $F_s(t,(x)_s)$, $s\ge1$ satisfy the following equations in the weak form

$$\left(\frac{\partial F_s(t)}{\partial t},\varphi_s\right) = \int d(x)_s\left[-\sum_{i=1}^{s}p_i\frac{\partial}{\partial q_i}F_s(t,(x)_s)\right]\varphi_s((x)_s)+$$
$$+\int d(x)_s\left\{\sum_{i<j=1}^{s}\int_{S_+^2}d\eta_{ij}\eta_{ij}\cdot(p_i-p_j)\delta(q_i-q_j)\times\right.$$
$$\times[F_s(t,q_1,p_1,\dots,q_i,p_i^*,\dots,q_j,p_j^*,\dots,q_s,p_s)-$$
$$\left.-F_s(t,q_1,p_1,\dots,q_i,p_i,\dots,q_j,p_j,\dots,q_s,p_s)]\right\}\varphi_s((x)_s)+$$
$$+\int\left\{\sum_{i=1}^{s}\int d(x)_{s+1}\int_{S_+^2}d\eta_{is+1}\eta_{is+1}\cdot(p_i-p_j)\delta(q_i-q_{s+1})\times\right.$$
$$\times[F_{s+1}(t,q_1,p_1,\dots,q_i,p_i^*,\dots,q_s,p_s,q_{s+1},p_{s+1}^*)-$$
$$\left.-F_{s+1}(t,q_1,p_1,\dots,q_i,p_i,\dots,q_s,p_s,q_{s+1}p_{s+1})]\right\}\varphi_s((x)_s),\quad s\ge1. \qquad (2.13)$$

It follows from (2.13) that

$$\frac{\partial F_s(t,(x)_s)}{\partial t} = -\sum_{i=1}^{s}p_i\frac{\partial}{\partial q_i}F_s(t,(x)_s)+$$
$$+\sum_{i<j=1}^{s}\int_{S_+^2}d\eta_{ij}\eta_{ij}\cdot(p_i-p_j)\delta(q_i-q_j)\times$$
$$\times[F_s(t,q_1,p_1,\dots,q_i,p_i^*,\dots,q_j,p_j^*,\dots,q_s,p_s)-$$

$$- F_s(t, q_1, p_1, \dots, q_i, p_i, \dots, q_j, p_j, \dots, q_s, p_s)] +$$

$$+ \sum_{i=1}^{s} \int d(x)_{s+1} \int_{S_+^2} d\eta_{is+1} \eta_{is+1} \cdot (p_i - p_j) \delta(q_i - q_{s+1}) \times$$

$$\times [F_{s+1}(t, q_1, p_1, \dots, q_i, p_i^*, \dots, q_s, p_s, q_{s+1}, p_{s+1}^*) -$$

$$- F_{s+1}(t, q_1, p_1, \dots, q_i, p_i, \dots, q_s, p_s, q_{s+1} \; p_{s+1})], \quad s \geq 1. \tag{2.14}$$

The distribution functions (2.12′) that satisfy (2.14) do not depend on any random vectors, because the the infinitesimal operator in (2.14) does not depend on any random vectors, they are averaged with respect to the all random vectors.

From (2.14) we derive equations for distribution functions that depend on the random vectors η_{ij}, $1 \leq i, j \leq s$. They are given by formula (2.11). We use for these functions again the same denotation $F_s(t, (x)_s)$. For arbitrary fixed η_{ij} we have

$$\frac{\partial F_s(t, (x)_s)}{\partial t} = - \sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t, (x)_s) +$$

$$+ \sum_{i<j=1}^{s} \theta(\eta_{ij} \cdot (p_i - p_j)) \eta_{ij} \cdot (p_i - p_j) \delta(q_i - q_j) \times$$

$$\times [F_s(t, q_1, p_1, \dots, q_i, p_i^*, \dots, q_j, p_j^*, \dots, q_s, p_s) -$$

$$- F_s(t, q_1, p_1, \dots, q_i, p_i, \dots, q_j, p_j, \dots, q_s, p_s)] +$$

$$+ \sum_{i=1}^{s} \int d(x)_{s+1} \int_{S_+^2} d\eta_{is+1} \eta_{is+1} \cdot (p_i - p_{s+1}) \delta(q_i - q_{s+1}) \times$$

$$\times [F_{s+1}(t, q_1, p_1, \dots, q_i, p_i^*, \dots, q_s, p_s, q_{s+1}, p_{s+1}^*) -$$

$$- F_{s+1}(t, q_1, p_1, \dots, q_i, p_i, \dots, q_s, p_s, q_{s+1}, p_{s+1})]. \tag{2.15}$$

The last term in the right-hand side of (2.15) depends on the same random vectors as $F_s(t, (x)_s)$ because $s + 1$-th particle can interact with the rest s particles only for momenta p_{s+1} of the set of lower dimension that does not contribute to integral with respect to p_{s+1}.

Formulae (2.12), (2.12′) represent the exact expression of the semigroup of the evolution operator $U(\Delta t)$ associated with the stochastic Boltzmann hierarchy (2.14)

$$F(t + \Delta t) = U(\Delta t) F(t), \tag{2.16}$$

where $F(t)$ is the sequence of distribution function $F_s(t, (x)_s)$, $s \geq 1$ and $F_s(t + \Delta t, (x)_s) = (U(\Delta t) F(t))_s((x)_s)$.

Semigroup $U(t)$ with orbitrary t is defined by formula

$$U(t) = \lim_{n \to \infty} \prod_{i=1}^{n} U(\Delta t_i), \qquad \sum_{i=1}^{n} \Delta t_i = t. \tag{2.17}$$

Corresponding semigroup $U(\Delta t)$ for hierarchy (2.15) is defined by formula (2.11). For this semigroup formulae (2.16), (2.17) also hold.

III. Equivalence of different forms of the stochastic Boltzmann hierarchy. 1. ***Different representations of the infinitesimal operator of the semigroup*** $S_s(-t, (x)_s)$. We have obtained the stochastic Boltzmann hierarchy (2.15) in the weak

$$\left.\frac{\partial S_s(-\Delta t,(x)_s)}{\partial \Delta t}\right|_{\Delta t=0} F_s(t,(x)_s) = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,(x)_s) +$$

$$+ \sum_{i<j=1}^{s} \theta(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)[F_s(t,(x)_s^*) - F_s(t,(x)_s)]. \qquad (3.5)$$

Substituting the last expression into hierarchy (3.1) we obtain the stochastic Boltzmann hierarchy (1.11).

Note that we have the boundary condition in the Poisson bracket according to which the momenta (p_i, p_j) should be replaced by (p_i^*, p_j^*) if $q_i = q_j$ and $\eta_{ij}\cdot(p_i - p_j) \geq 0$.

We stress that expressions (3.2) or (3.3) and (3.4) are identical in the following sense. Namely, numerically $\left.\frac{\partial S_s(-\Delta t,(x)_s)}{\partial \Delta t}\right|_{\Delta t=0} F_s(t,(x)_s)$ is given by expressions (3.2), (3.3), for calculation of functionals (averages) one should use expression (3.4).

Now we want to explain from physical point of view in what sense expressions (3.2), (3.3) and (3.4) are identical. For this aim we consider the following simple example. In the three-dimensional space we have a mass m distributed along the first axis with density $m(q^1)$, $m = \int m(q^1)\,dq^1$. The distribution $m(q^1)$ considered as in the three-dimensional space is concentrated on the first axis

$$m(q) = m(q^1)|_{q^2=0,q^3=0}, \quad m(q) = 0 \quad \text{if} \quad q^2 \neq 0 \quad \text{or} \quad q^3 \neq 0. \qquad (3.6)$$

This distribution considered as the generalized function in three-dimensional space is identical to the following function

$$m(q) = m(q^1)\delta(q^2)\delta(q^3) \qquad (3.7)$$

and corresponding mass is equal to

$$m = \int m(q)\,dq = \int m(q^1)\delta(q^2)\delta(q^3)\,dq^1\,dq^2\,dq^3.$$

The functional of distribution (3.6) with test functions $\varphi(q)$ defined on the three-dimensional space should be calculate with help the distribution (3.7) as follows

$$(m,\varphi) = \int m(q^1)\delta(q^2)\delta(q^3)\varphi(q)\,dq = \int m(q^1)\,\varphi(q^1,0,0)\,dq^1. \qquad (3.8)$$

Thus, numerically distribution of mass are given by expression (3.6), for calculation of functionals one should use expression (3.7). In this sense expressions (3.6) and (3.7) are identical.

In expressions (3.2) and (3.3) we have one-dimensional δ function $\delta(\tau_{ij}) = \delta(q_i^1 - q_j^1)(p_i^1 - p_j^2)$. As known one-dimensional δ-function is equivalent to the following boundary condition for $q_i = q_j$; $\eta_{ij}\cdot(p_i - p_j) \geq 0$.

$$F_s(t+0, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s) = F_s(t, x_1, \ldots, q_i, p_i^*, \ldots, q_j, p_j^*, \ldots, x_s),$$
$$F_s(t-0, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s) = F_s(t, x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s). \qquad (3.9)$$

For $q_i = q_j$, $\eta_{ij}\cdot(p_i - p_j) \leq 0$ we have

$$F_s(t-0,(x)_s) = F_s(t+0,(x)_s). \qquad (3.10)$$

2. ***Different forms of the stochastic Boltzmann hierarchy.*** Thus, the stochastic Boltzmann hierarchy (1.11) is equivalent to the usual Boltzmann hierarchy (1.4) without δ-function sense from relation (2.12), (2.12′). Now we derive it by differentiating relation (2.11) using point by point convergence and show that the stochastic Boltzmann hierarchy (2.15) is equivalent to hierarchy (1.10) with boundary conditions in the Poisson bracket and in the functions $F_s(t, (x)_s)$.

We have

$$\frac{\partial F_s(t,(x)_s)}{\partial t} = \frac{\partial S_s(-\Delta t,(x)_s)}{\partial(\Delta t)}\bigg|_{\Delta t=0} F_s(t,(x)_s) +$$

$$+\sum_{i=1}^{s}\int dx_{s+1}\int_{S_+^2} d\eta_{is+1}\,\eta_{is+1}\cdot(p_i-p_{s+1})\delta(q_i-q_{s+1})$$

$$\times[F_{s+1}(t,q_1,p_1,\ldots,q_i,p_i^*,\ldots,q_{s+1},p_{s+1}^*)-$$

$$-F_{s+1}(t,q_1,p_1,\ldots,q_i,p_i,\ldots,q_{s+1},p_{s+1})]. \qquad (3.1)$$

For the infinitesimal operator of the semigroup $S_s(-t, (x)_s)$ we obtained the following expression [1]

$$\frac{\partial S_s(-\Delta t,(x)_s)}{\partial \Delta t} F_s(t,(x)_s) = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,(x)_s) +$$

$$+\sum_{i<j=1}^{s}\theta(\eta_{ij}\cdot(p_i-p_j))\delta(\Delta t-\tau_{ij})[F_s(t,(x)_s^*)-F_s(t,(x)_s)]|_{q_i=q_j}, \quad \Delta t = 0, \qquad (3.2)$$

$$(x)_s^* = (x_1,\ldots,q_i,p_i^*,\ldots,q_j,p_j^*,\ldots,x_s).$$

Consider the function $\delta(\Delta t-\tau_{ij})|_{\Delta t=0}$. In the coordinate system where the first component of the vector (q_i-q_j) is directed along the vector η_{ij} the time of collision τ_{ij} is defined as follows

$$\tau_{ij} = \frac{q_i^1-q_j^1}{p_j^1-p_i^1}.$$

The (i, j)-th term in (3.2) can be expressed as follows

$$\theta(p_i^1-p_j^1)\delta(q_i^1-q_j^1)\cdot(p_i^1-p_j^1)[F_s(t,(x)_s^*)-F_s(t,(x)_s)]|_{q_i^2-q_j^2=0,\,q_i^3-q_j^3=0},$$

$$\eta_{ij}\cdot(p_i-p_j) = (p_i^1-p_j^1). \qquad (3.3)$$

This term is different from zero on the first axis $q_i^1-q_j^1$ (with respect to the vector q_i-q_j, i. e. for $q_i^2-q_j^2=0$, $q_i^3-q_j^3=0$) and considered as a generalized function in the three-dimensional space is equal to

$$\theta(p_i^1-p_j^1)\delta(q_i^1-q_j^1)\delta(q_i^2-q_j^2)\delta(q_i^3-q_j^3)\cdot(p_i^1-p_j^1)[F_s(t,(x)_s^*)-F_s(t,(x)_s)] =$$

$$= \theta(\eta_{ij}\cdot(p_i-p_j))\delta(q_i-q_j)\eta_{ij}\cdot(p_i-p_j)[F_s(t,(x)_s^*)-F_s(t,(x)_s)]. \qquad (3.4)$$

(For analogous calculation see [6, p. 48].) Obtained expression does not depend on choice of coordinate system because $\delta(q_i-q_j)$ and $\eta_{ij}\cdot(p_i-p_j)$ are invariant under rotation.

Substituting expression (3.4) into (3.3) we obtain finally [1 – 3]

$$\frac{\partial F_s(t,(x)_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,(x)_s) +$$

$$+\sum_{i=1}^{s} \int dx_{s+1} \int_{S_+^2} d\eta_{is+1}\, \eta_{is+1}\cdot(p_i - p_{s+1})\delta(q_i - q_{s+1}) \times$$

$$\times [F_{s+1}(t, x_1, \ldots, q_i, p_i^*, \ldots, q_{s+1}, p_{s+1}^*) -$$

$$- F_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_{s+1}, p_{s+1})], \qquad s \geq 1$$

but with the boundary conditions (3.9), (3.10) and the boundary conditions in the Poisson bracket according to which for $q_i = q_j$, $\eta_{ij}\cdot(p_i - p_j) \geq 0$ the momenta (p_i, p_j) should be replaced by (p_i^*, p_j^*) in it.

The stochastic Boltzmann hierarchy with three-dimensional δ-function (1.11)

$$\frac{\partial F_s(t,(x)_s)}{\partial t} = -\sum_{i=1}^{s} p_i \frac{\partial}{\partial q_i} F_s(t,(x)_s) +$$

$$+ \sum_{i<j=1}^{s} \theta(\eta_{ij}\cdot(p_i - p_j))\delta(q_i - q_j)[F_s(t,(x)_s^*) - F_s(t,(x)_s)] +$$

$$+\sum_{i=1}^{s} \int dx_{s+1} \int_{S_+^2} d\eta_{is+1}\, \eta_{is+1}\cdot(p_i - p_{s+1})\delta(q_i - q_{s+1}) \times$$

$$\times [F_{s+1}(t, x_1, \ldots, q_i, p_i^*, \ldots, q_{s+1}, p_{s+1}^*) -$$

$$- F_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_{s+1}, p_{s+1})], \qquad s \geq 1$$

has the same structure as the BBGKY hierarchy for smooth potentials, because the integral term is obtained from the infinitesimal operator with δ-functions, that describe interaction of the stochastic particles, by integrating over x_{s+1} and averaging with respect to η_{is+1}. We have, of course, the boundary conditions in the Poisson bracket and for distribution functions. In the both form of the stochastic Boltzmann hierarchy (1.4) and (1.11) we use different equivalent representation (3.2) and (3.5) of the infinitesimal operator $\mathcal{H}_s$ of the stochastic evolution operator $S_s(-t,(x)_s)$.

We can represent the stochastic Boltzmann hierarchies (1.4) and (1.11) in the following form independent from representation of the operator $\mathcal{H}_s$:

$$\frac{\partial F_s(t,(x)_s)}{\partial t} = \mathcal{H}_s F_s(t,(x)_s) +$$

$$+\sum_{i=1}^{s} \int dx_{s+1} \int_{S_+^2} d\eta_{is+1}\, \eta_{is+1}\cdot(p_i - p_{s+1})\delta(q_i - q_j) \times$$

$$\times [F_{s+1}(t, x_1, \ldots, q_i, p_i^*, \ldots, q_{s+1}, p_{s+1}^*) -$$

$$- F_{s+1}(t, x_1, \ldots, q_i, p_i, \ldots, q_{s+1}, p_{s+1})], \qquad s \geq 1. \tag{3.11}$$

IV. Boltzmann equation and its solutions in terms of the stochastic dynamics.
1. *Iterations of the Boltzmann equation.* In this section we show that the Boltzmann equation and its solutions can be represented in terms of the stochastic dynamics and functional associated with it. For the sake of simplicity, we start with solution of the

Boltzmann equation (1.1) in the second approximation.

It is obvious that the Boltzmann equation (1.1) with the initial data (1.2) can be represented as the following integral equation

$$f(t,x_1) = f(0,q_1-p_1t,p_1) + e^{p_1\nabla q_1(\tau-t)}\int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, \eta\cdot(p_1-p_2)\times$$

$$\times[f(\tau,q_1,p_1^*)f(\tau,q_1,p_2^*) - f(\tau,q_1,p_1)f(\tau,q_1,p_2)]. \quad (4.1)$$

The first approximation of solutions of (4.1) is equal to

$$f^{(1)}(t,x_1) = f(0,q_1-p_1t,p_1). \quad (4.2)$$

Substituting (4.2) into (4.1) we obtain the second approximation of solutions of (4.1)

$$f^{(2)}(t,x_1) = f(0,q_1-p_1t,p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, \eta\cdot(p_1-p_2)e^{p_1\nabla q_1(\tau-t)}\times$$

$$\times[f(0,q_1-p_1^*\tau,p_1^*)f(0,q_1-p_2^*\tau,p_2^*) - f(0,q_1-p_1\tau,p_1)f(0,q_1-p_2\tau,p_2)] =$$

$$= f(0,q_1-p_1t,p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, \eta\cdot(p_1-p_2)\,[f(0,q_1-p_1(t-\tau)-p_1^*\tau,p_1^*)\times$$

$$\times f(0,q_1-p_1(t-\tau)-p_2^*\tau,p_2^*) - f(0,q_1-p_1t,p_1)f(0,q_1-p_2\tau-p_1(t-\tau),p_2)] =$$

$$= f(0,q_1-p_1t,p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, \eta\cdot(p_1-p_2)\times$$

$$\times f(0,q_1-p_1\tau-p_1^*(t-\tau),p_1^*)f(0,q_1-p_1\tau-p_2^*(t-\tau),p_2^*) -$$

$$-f(0,q_1-p_1t,p_1)f(0,q_1-p_1\tau-p_2(t-\tau),p_2)] =$$

$$= f(0,q_1-p_1t,p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\,dq_2\, \eta\cdot(p_1-p_2)\delta(q_1-p_1\tau-q_2+p_2\tau)\times$$

$$\times[f(0,q_1-p_1\tau-p_1^*(t-\tau),p_1^*)f(0,q_2-p_2\tau-p_2^*(t-\tau),p_2^*) -$$

$$-f(0,q_1-p_1t,p_1)f(0,q_2-p_2t,p_2)]. \quad (4.3)$$

Now consider the two-particle stochastic system with initial distribution function

$$F_2(0,x_1,x_2) = f(0,x_1)f(0,x_2), \qquad \int dx\, f(0,x) = 1. \quad (4.4)$$

It is easy to see that the second approximation (4.3) can be identically represented as follows

$$f^{(2)}(t,x_1) = F_1(t,x_1) = \int S_2^0(-t,x_1,x_2)F_2(0,x_1,x_2)\,dx_2 +$$

$$+\int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\,dq_2\, \eta\cdot(p_1-p_2)\delta(q_1-p_1\tau-q_2+p_2\tau)\times$$

$$\times[S_2(-t,x_1,x_2)F_2(0,x_1,x_2) - S_2^0(-t,x_1,x_2)F_2(0,x_1,x_2)], \quad (4.5)$$

where $F_1(t,x_1)$ coincides with the one-particle distribution function of the given two-particle stochastic system.

Multiply the both sides of (4.5) by test function $\varphi_1(x_1)$ and integrate over the variable x_1. We obtain

$$\int F_1(t,x_1)\varphi_1(x_1)\,dx_1 = \int S_2^0(-t,x_1,x_2)F_2(0,x_1,x_2)\varphi_1(x_1)\,dx_1\,dx_2 +$$

$$+\int_0^t d\tau \int_{S_+^2} d\eta \int dx_1\,dx_2\,\eta\cdot(p_1-p_2)\delta(q_1-p_1\tau-q_2+p_2\tau)\times$$

$$\times[S_2(-t,x_1,x_2)F_2(0,x_1,x_2)-S_2^0(-t,x_1,x_2)F_2(0,x_1,x_2)]\varphi_1(x_1) =$$

$$= \frac{1}{2}(S_2(-t)F_2(0),\varphi_2), \tag{4.6}$$

where $(S_2(-t)F_2(0),\varphi_2)$ is the functional [1 – 3] that represents the average of the one-particle observable

$$\varphi_2(x_1,x_2) = \varphi_1(x_1)+\varphi_1(x_2)$$

over the state $S_2(-t,x_1,x_2)F_2(0,x_1,x_2)$.

Thus, we have shown that the *second approximation of the solution of the Boltzmann equation coincides with the one-particle distribution function of the two particle stochastic system with chaotic initial distribution function* (4.4).

Note that in definition of the one-particle distribution function the contribution of the two particle distribution function on the hyperplanes of lower dimension $q_1-p_1\tau-$ $-q_2+p_2\tau=0$ is taken into account. Usually in statistical mechanics the sets of lower dimension are neglected.

Now proceed to general case.

Introduce the following function

$$\tilde f(\tau,q_1-p_1\tau,p_1) = f(\tau,q_1,p_1), \quad f(\tau,q_1,p_1) = e^{-p_1\nabla_{q_1}\tau}\tilde f(\tau,q_1,p_1),$$

$$\tilde f(\tau,q_1,p_1) = e^{p_1\nabla_{q_1}\tau} f(\tau,q_1,p_1) = f(\tau,q_1+p_1\tau,p_1) \tag{4.7}$$

and represent equation (4.1) as follows

$$\tilde f(t,q_1-p_1t,p_1) = f(t,x_1) = f(0,q_1-p_1t,p_1)+\int_0^t d\tau\int_{S_+^2} d\eta\,\eta\cdot(p_1-p_2)\times$$

$$\times[\tilde f(\tau,q_1-p_1(t-\tau)-p_1^*\tau,p_1^*)\,\tilde f(\tau,q_1-p_1(t-\tau)-$$

$$-p_2^*\tau,p_2^*)-\tilde f(\tau,q_1-p_1t,p_1)\,\tilde f(\tau,q_1-p_2\tau-p_1(t-\tau),p_2)] =$$

$$= f(0,q_1-p_1t,p_1)+\int_0^t d\tau\int_{S_+^2} d\eta\,\eta\cdot(p_1-p_2)\times$$

$$\times[\tilde f(t-\tau,q_1-p_1\tau-p_1^*(t-\tau),p_1^*)\,\tilde f(t-\tau,q_1-p_1\tau-$$

$$-p_2^*(t-\tau),p_2^*)-\tilde f(t-\tau,q_1-p_1t,p_1)\,\tilde f(t-\tau,q_1-p_2(t-\tau)-p_1\tau,p_2)] =$$

$$= f(0,q_1-p_1t,p_1)+\int_0^t d\tau\int_{S_+^2} d\eta\int dp_2\,dq_2\,\eta\cdot(p_1-p_2)\delta(q_1-p_1\tau-q_2+p_2\tau)\times$$

$$\times[\tilde f(t-\tau,q_1-p_1\tau-p_1^*(t-\tau),p_1^*)\,\tilde f(t-\tau,q_2-p_2\tau-$$

$$-p_2^*(t-\tau),p_2^*)-\tilde f(t-\tau,q_1-p_1t,p_1)\,\tilde f(t-\tau,q_2-p_2t,p_2)] =$$

$$= f(0, q_1 - p_1 t, p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, dq_2\, \eta\cdot(p_1 - p_2)\delta(q_1 - p_1\tau - q_2 + p_2\tau) \times$$
$$\times [\, S_2(-t, x_1, x_2)\{\, \tilde{f}(t-\tau, q_1, p_1)\, \tilde{f}(t-\tau, q_2, p_2)\} -$$
$$- S_2^0(-t, x_1, x_2)\{\, \tilde{f}(t-\tau, q_1, p_1)\, \tilde{f}(t-\tau, q_2, p_2)\}]. \qquad (4.8)$$

We represent the Boltzmann equation in terms of the operator of evolution of two stochastic particles. In a weak form we have

$$\int f(t, q_1 - p_1 t, p_1)\, \varphi_1(q_1, p_1)\, dq_1\, dp_1 =$$
$$= \int \{\, S_2^0(-t, x_1, x_2) f(0, q_1, p_1) f(0, q_2, p_2)\}\, \varphi_1(x_1)\, dx_1\, dx_2 +$$
$$+ \int_0^t d\tau \int_{S_+^2} d\eta\, \eta\cdot(p_1 - p_2)\delta(q_1 - p_1\tau - q_2 + p_2\tau) \times$$
$$\times\ [\, S_2(-t, x_1, x_2)\{\, \tilde{f}(t-\tau, x_1) f(t-\tau, x_2)\} -$$
$$- S_2^0(-t, x_1, x_2)\{\, \tilde{f}(t-\tau, x_1) f(t-\tau, x_2)\}\, \varphi_1(x_1)\, dx_1\, dx_2. \qquad (4.9)$$

Note that in (4.8), (4.9) the functions $S_2(-t, x_1, x_2)\, \tilde{f}(t-\tau, q_1, p_1)\, \tilde{f}(t-\tau, q_2, p_0)$ and $S_2^0(-t, x_1, x_2)\, \tilde{f}(t-\tau, q_1, p_1) f(t-\tau, q_1, p_1)$ are integrated over the hyperplane $q_1 - p_1 - q_2 + p_2\tau = 0$, $0 \le \tau \le t$. From (4.8) we obtain the representation of the n-th approximation of solution $f^{(n)}(t, x_1)$ through the $n-1$-th approximation

$$f^{(n)}(t, x_1) = \tilde{f}^{(n)}(t, q_1 - p_1 t, p_1) = f(0, q_1 - p_1 t, p_1) +$$
$$+ \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, dq_2\, \eta\cdot(p_1 - p_2)\delta(q_1 - p_1\tau - q_2 + p_2\tau) \times$$
$$\times [\, \tilde{f}^{(n-1)}(t-\tau, q_1 - p_1\tau - p_1^*(t-\tau),\ p_1^*)\, \tilde{f}^{(n-1)}(t-\tau, q_2 - p_2\tau -$$
$$- p_2^*(t-\tau),\ p_2^*) - \tilde{f}^{(n-1)}(t-\tau, q_1 - p_1 t, p_1)\, \tilde{f}^{(n-1)}(t-\tau, q_2 - p_2 t, p_2)] =$$
$$= f(0, q_1 - p_1 t, p_1) + \int_0^t d\tau \int_{S_+^2} d\eta \int dp_2\, dq_2\, \eta\cdot(p_1 - p_2)\delta(q_1 - p_1\tau - q_2 + p_2\tau) \times$$
$$\times [\, S_2(-t, x_1, x_2)\{\, \tilde{f}^{(n-1)}(t-\tau, q_1, p_1)\, \tilde{f}^{(n-1)}(t-\tau, q_2, p_2)\} -$$
$$- S_2^0(-t, x_1, x_2)\{\, \tilde{f}^{(n-1)}(t-\tau, q_1, p_1)\, \tilde{f}^{(n-1)}(t-\tau, q_2, p_2)\}]. \qquad (4.10)$$

From (4.10) we see that every n-th approximation takes into account one new collision according to δ-function $\delta(q_1 - p_1 - q_2 + p_2\tau)$. The previous collisions were taken into account by $\tilde{f}^{(n-1)}$.

2. ***Iterations of the Boltzmann hierarchies.*** Solutions of the Boltzmann hierarchy (1.4) can be represented by series of iterations

$$F(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, S^0(-t) S^0(t_1) A S^0(-t_1) \ldots S^0(t_n) A S^0(-t_1) F(0), \qquad (4.11)$$

where operator A is defined by the second term of the right-hand side of (1.4) and

$S^0(t)$ is direct sum of the evolution operators $S_s^0(t, (x)_s)$ of the free systems. We consider intial data $F(0)$ such that series (4.11) is convergent [5].

Projection of (4.11) onto s-particle space is equal to s-particle distribution function

$$F_s(t,(x)_s) = \sum_{n=0}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{n-1}} dt_n\, S_s^0(-t,(x)_s) S_s^0(t_1,(x)_s) \sum_{i=1}^{s} \int dx_{s+1}\, \delta(q_i - q_{s+1}) \times$$
$$\times \int_{S_2^+} d\eta_{is+1}\, \eta_{is+1} \cdot (p_i - p_{s+1}) [S_{s+1}^0(-t_1,(x)_{s+1}^*) - S_{s+1}^0(-t,(x)_{s+1})] \ldots$$
$$\ldots S_{s+n+1}^0(t_{n-1},(x)_{s+n-1}) \sum_{i=1}^{s+n-1} \int dx_{s+n}\, \delta(q_i - q_{s+n}) \times$$
$$\times \int_{S_2^+} d\eta_{is+1}\, \eta_{is+n} \cdot (p_i - p_{s+n}) [S_{s+n}^0(-t_n,(x)_{s+n}^*) - S_{s+n}^0(-t_n,(x)_{s+n})] F_{s+n}(0,(x)_{s+n}). \tag{4.12}$$

From representation (4.12) it is easy to see that integral with respect to $x_{s+1}, \ldots$ $\ldots, x_{s+n}$ is taken over the hypersurfaces of lower dimension than phase space $(x_{s+1}, \ldots, x_{s+n})$. Really, initial positions of i-th and $s+j$-th particles $(1 \le j \le n,\ 1 \le$ $\le i \le s+j-1)$ coincide $q_i = q_{s+j}$ and after action of the operators $S_{s+j}^0(-t_j,(x)_{s+j}^*)$, $S_{s+j}^0(-t_j,(x)_{s+j})$ positions of these particles are situated on hyperplanes of lower dimension such that vector of difference of their positions are parallel to the vector of difference of their momenta.

Analogical result holds for the stochactic Boltzmann hierarchy (1.11). Its solutions are also represented by series (4.11), (4.12) but instead of the operator $S^0(t)$ one should put the operator $S(t)$ of the stochactic evolution. Results of action of the operators of stochactic evolution $S_{s+j}(-t_j,(x)_{s+j}^*)$, $S_{s+j}(-t_j,(x)_{s+j})$ differ of that of the operators of free evolution $S_{s+j}^0(-t_j,(x)_{s+j}^*)$, $S_{s+j}^0(-t_j,(x)_{s+j})$ only on hyperplanes of lower dimension with respect to p_{s+j} and we negled this difference in intervals with respect to p_{s+j} (integration with respect to q_{s+j} is performed by using δ-function).

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