UDK 517.938.5

M. K. Sparavalo, cand. tech. sci. (unit A 4000)

ATTRACTORS OF DYNAMICAL SYSTEMS WITH CONTROL: TOPOLOGY OF PURPOSEFUL FORMATION

АТТРАКТОРИ ДИНАМІЧНИХ СИСТЕМ З КЕРУВАННЯМ: ТОПОЛОГІЯ ЦІЛЕСПРЯМОВАНОГО ФОРМУВАННЯ

The definitions of homogeneous and mosaic attractors of codimensionality one are given. A topological method for their purposeful formation by using the feedback control laws of controlled dynamical systems is suggested.

Наводяться означення однорідних та мозаїчних аттракторів корозмірності один. Визначається топологічний метод їх цілеспрямованого формування за допомогою законів зворотного зв'язку для динамічних систем з керуванням.

At present, attractors (especially hyperbolic ones) draw the attention of many mathematicians. First, this interest was caused exclusively by the mathematical reasons (v., e.g., [1, 2]). However, as things turned out, the attractors as loci in the motion spaces of stable physical systems are typical phenomena. It is necessary to note that the attractors of these systems are often distinguished from the hyperbolic attractors. The first integral of a dynamical system generates the one-codimensional foliation in its motion space without singular points and limit cycles. The leaves of the foliation are level hypersurfaces of the first integral that can be the attractors under consideration. In particular, just these attractors have a directed bearing on some nonsteady-state aerodynamical manoeuvres of aircrafts.

There are two things to be taken into consideration in investigating the aforementioned attractors for the controlled dynamical systems. The first one deals with the purposeful formation of one-codimensional attractors in the motion spaces of dynamical systems by using the corresponding feedback control laws. The second one relates to the original existence of the attractors as losi in the motion spaces of dynamical systems or in the control processes spaces.

Assume that the motion of an original dynamical system (ODS) with control is described by the indicatrix of velocities $f(x, t, u)$, where $f = (f_1, \dots, f_n) \in C^s$ is a vector function, $x = (x_1, \dots, x_n)$ is a phase vector, $t \in T = [t_0; +\infty[$ is time, $u = (u_1, \dots, u_p)$ is a vector of controls, $u = \hat{u}(x, t) \in C^s$, $s \geq 0$.

Denote by

(a) $x(t, x_0)_u$ the phase trajectory of the ODS corresponding to an admissible control law $u = \hat{u}(x, t)$ starting at the point x_0 at time t_0, i.e., $x(t_0, x_0)_u == x_0$;

(b) $x_t = (x(t, x_0)_u, t)$ the integral curve of the ODS;

(c) $z_t = (x_t, \overline{u}(t))$ the control process of the ODS, where $\overline{u}(t) = \hat{u}(x(t, x_0)_u, t)$.

The basic class of considered attractors consists of the leaves of the above-mentioned foliations, i.e., one-codimensional manifolds each of which can be continuously deformed into a hyperplane. This allows us to give the following definition of two new types of attractors.

Definition. *A one-codimensional invariant manifold $M \in C^{s+1}$ of the motion space of the ODS with a control law $u = \hat{u}(x, t)$ is called a homogeneous ω-attractor if the following conditions are satisfied:*

(i) there exists a $n+1$-dimensional neighborhood V of M such that $x_t \subset V$, $\forall\, t = t_0$, immediately follows from $x_{t_0} \in V$;

(ii) $x_t^{\omega} \subset M$, where x_t^{ω} is the ω-limit curve of x_t;

(iii) $(d/dt)\|x_t - x_t^M\| < 0$ for any $t \in T$, where $x_t^M \subset M$ is the orthogonal (with respect to the hyperplane $\{x_k = 0\}$) projection of x_t onto the manifold M and $\|\cdot\|$ is the Euclidean vector norm. If (iii) holds at every time $t \in T$, then M is called a mosaic ω-attractor.

Denote by $[A^{\omega\text{-hom}}]$ and $[A^{\omega\text{-mos}}]$ the classes of homogeneous and mosaic attractors, respectively.

It is necessary to find the feedback control law $u = \hat{u}(x, t)$ for the ODS that would ensure the validity of the following condition: $x_t^{\omega} \subset M$ for any integral curve x_t. This is one of the typical problems of terminal control. This problem can be settled by the method of purposeful formation of a local topological structure of one-codimensional foliations [3]. The essence of this method is that a given one-codimensional manifold M is transformed to a mosaic or homogeneous ω-attractor with the help of the corresponding feedback control law $u = \hat{u}(x, t)$ for the ODS. The step-by-step realization of this method is presented below.

Step 1 (statement of the problem). Let the manifold M be defined as follows:

$$M = \{\omega_k(x, t) = 0 \Rightarrow x_k = \sigma_k(x^k, t)\},$$

where

$$x^k = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n),\ \sigma_k(x^k, t) \in C^{s+1}.$$

It is necessary to find a feedback control law $u = \hat{u}(x, t)$ that guarantees that the condition $M \in [A^{\omega\text{-hom}}]([A^{\omega\text{-mos}}])$ holds.

Step 2 (change of variables). Consider a one-parameter function $\eta_k(\nu_k, x, t)$, where $\eta_k(0, x, t) = \omega_k(x, t)$ and ν_k is a real parameter. By assuming that the equation $\eta_k(\nu_k, x, t) = 0$ is solvable for x_k and ν_k, we obtain

$$\eta_k(\nu_k, x, t) = 0 \Rightarrow \begin{cases} x_k = \varphi_k(\nu_k, x^k, t) \in C^{s+1}, \\ \nu_k = \varphi_k^{-1}(x, t) \in C^{s+1}, \end{cases}$$

where

$$\varphi_k(0, x^k, t) \equiv \sigma_k(x^k, t), \quad \frac{\partial \varphi_k(\nu_k, x^k, t)}{\partial \nu_k} \neq 0 \quad \forall\, (\nu_k, x^k, t) \in R^{n+1}.$$

Let us substitute $y = (y_1, \dots, y_n)$ for x, i.e.,

$$x^k = y^k, \quad x_k = \varphi_k(y, t), \tag{1}$$

where $y^k = (y_1, \dots, y_{k-1}, y_{k+1}, \dots, y_n)$. Substitution (1) transforms the ODS to a new dynamical system (NDS) with the indicatrix of velocities $g(y, t, u)$, where

$$g^k(y, t, u) = f^k(y^k, \varphi_k(y, t), t, u),$$

$$g_k(y, t, u) =$$

$$= \left\{ f_k(y^k, \varphi_k(y,t), t, u) - \frac{\partial \varphi_k(y,t)}{\partial t} - \frac{\partial \varphi_k(y,t)}{\partial y^k} f^k(y^k, \varphi_k(y,t), t, u) \right\} \left\{ \frac{\partial \varphi_k(y,t)}{\partial y_k} \right\}^{-1}.$$

Relation (1) enables us to conclude that

(a) M corresponds to the manifold $N = \{ y_k = 0 \}$;

(b) a phase trajectory $x(t, x_0)_u$ of the ODS corresponds to the phase trajectory $y(t, y_0)_u$ of the NDS with the same control

$$u = \bar{u}(t) = \hat{u}(x(t, x_0)_u, t) = \hat{u}(y^k(t, y_0)_u, \varphi_k(y(t, y_0)_u, t), t),$$

where

$$y(t_0, y_0)_u = y_0,$$

$$x^k(t, x_0)_u = y^k(t, y_0)_u,$$

$$x_k(t, x_0)_u = \varphi_k(y(t, y_0)_u, t),$$

$$y_k(t, y_0)_u = \varphi_k^{-1}(x(t, x_0)_u, t).$$

Step 3 (formation of the right-hand side for a new differential equation). Consider a smooth function $p_k(y, t)$ that generates a smooth map

$$\omega_k : R_{y,t}^{n+1} \to S_k = \{ q_k = p_k(y, t), (y, t) \in R_{y,t}^{n+1} \}$$

with a 0-level submanifold $S_k^0 = \{ y_k = 0 \}$, where ω_k belongs to the class $[\omega_k]^A$. We have

$$\omega_k : \{\{ y_k < 0 \} \to S_k^1, \; S_k^0 \to S_k^0, \; \{ y_k > 0 \} \to S_k^2 \},$$

where $S_k^1 \in [S_k^+]$, $S_k^2 \in [S_k^-]$; $[S_k^+]$ $([S_k^-])$ is the strata class of a manifold S_k satisfying the inequality $q_k > 0$ $(q_k < 0)$.

One can now write an equation

$$g_k(y, t, u) = p_k(y, t). \tag{2}$$

Step 4 (finding the required feedback control law). Assume that there exists a unique solution of equation (2) for u_α, $\alpha \in (1, \dots, p)$. Solving this equation with respect to u_α, we obtain $u_\alpha = \hat{b}_\alpha(y, t, u^\alpha)$, where $u^\alpha = (u_1, \dots, u_{\alpha-1}, u_{\alpha+1}, \dots, u_p)$. The other components of the vector of controls can be found by indicating certain continuous functions

$$b^\alpha(y, t) = (b_1(y, t), \dots, b_{\alpha-1}(y, t), b_{\alpha+1}(y, t), \dots, b_p(y, t)).$$

Thus, we get the following feedback control law for the NDS $u = b(y, t)$, where

$$b_\alpha(y, t) = \hat{b}_\alpha(y, t, b^\alpha(y, t)), \quad b = (b_1, \dots, b_p).$$

Finally, after returning to the phase vector x by the inverse transformation $y^k = x^k$, $y_k = \varphi_k^{-1}(x, t)$, the required law for the ODS can be written in the form $u = h(x, t)$, where $h(x, t) = b(x^k, \varphi_k^{-1}(x, t), t)$ and $h = (h_1, \dots, h_p)$.

According to [4], the following theorem is true.

Theorem 1. *Let* $\omega_k \in [\omega_k]^A$. *Then* $N \in [A^{\omega\text{-hom}}]$.

However, there arises the question. To what class does the manifold M belong or what conditions must be imposed on the diffeomorphism (1) to guarantee that it preserve local topological structures of one-codimensional ω-attractors?

Two theorems given below answer this question [3].

Theorem 2. *Assume that diffeomorphism* (1) *is given by an implicit function* $\eta_k(\nu_k, z_k) = 0$ *satisfying the following conditions*:

(i) $\eta_k(\nu_k, x, t) = \eta_k(\nu_k, x_k - \sigma_k(x^k, t)) = 0$;

(ii) the implicit function $\eta_k(\nu_k, z_k) = 0$ *defines a strictly increasing or decreasing explicit function* $\nu_k = \varphi_k^{-1}(z_k) \in C^{s+1}$, *where* $\varphi_k^{-1}(0) = 0$ *and* $\eta_k(\varphi_k^{-1}(z_k), z_k) \equiv 0$. *Then diffeomorphism* (1) *preserves the local topological structures of the one-codimensional homogeneous* ω*-attractors, i.e.,*

$$N \in [A^{\omega\text{-hom}}] \Rightarrow M \in [A^{\omega\text{-hom}}].$$

Theorem 3. *Assume that diffeomorphism* (1) *is given by an equation* $\eta_k(\nu_k, x, t) = 0$ *that defines the function*

$$x_k = \varphi_k(\nu_k, x^k, t)$$

uniformly convergent to the function

$$x_k = \sigma_k(x^k, t) = \varphi_k(0, x^k, t)$$

with respect to the parameter ν_k. *Then* $N \in [A^{\omega\text{-hom}}] \Rightarrow M \in [A^{\omega\text{-hom}}]$ *or* $M \in [A^{\omega\text{-mos}}]$.

If there is a constraint on the vector of controls in the form $\bar{u}(t) \in U$ *for any* $t \in T$, $\dim U = p$, then one should frame the functions $p_k(y, t)$, $b^\alpha(y, t)$ so that the constraint will not be violated in the moving ODS. It is not a very difficult task.

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Received 06.04.94