Algebra and Discrete Mathematics RESEARCH ARTICLE
Number 3. **(2006).** pp. 119 – 126

The generalized path algebras over standardly stratified algebras

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Communicated by R. Wisbauer

Abstract. In this note, it is proved that the generalized path algebras over standardly stratified algebras are also standardly stratified, and the generalized path algebras over quasi-hereditary algebras are also quasi-hereditary. The $\Delta-$good module categories over these big quasi-hereditary algebras are determined in terms of those of the given algebras.

Quasi-hereditary algebras and their generalizations such as standardly stratified algebras were first introduced by Cline, Parshall and Scott [2] [3] in order to study highest weight categories in the representation theory of semi-simple Lie algebras and algebraic groups. Since then, these algebras have been studied by many authors (for example: Dlab-Ringel [5] [6], Ringel [11], and so on). Many algebras which arise rather naturally have been shown to be quasi-hereditary: the Schur algebras [10], the Auslander algebras [6], and the endomorphism algebras of direct sum of all indecomposable Δ-good modules for any $\mathcal{F}(\Delta)$-finite quasi-hereditary algebra [12], the smash product of a quasi-hereditary graded algebra graded by a finite group [13] [14]. In this note, we will prove that the generalized path algebras [4] over quasi-hereditary algebras are quasi-hereditary. In fact we first prove a more general result: the generalized path algebras over standardly stratified algebras are standardly stratified; and then as a corollary, we prove the mentioned result above for quasi-hereditary algebras. Given a quasi-hereditary algebra (A, Λ),

Supported in part the NSF of China (Grants 10471071).
2000 Mathematics Subject Classification: *16E10, 16G20, 18G20.*
Key words and phrases: *the generalized path algebras; quasi-hereditary algebras.*

of central importance are the modules filtered by (co-)standard modules (the precise meaning will be given later on) and characteristic tilting modules [11]. We will give the precise description of $\Delta-$good module category of the quasi-hereditary generalized path algebras. It follows that for any finitely many quasi-hereditary algebras A_i with $\Delta-$good modules categories $\mathcal{F}_{A_i}(\Delta)$, $i = 1, 2, \cdots, n$, we can construct a big algebra via generalized path algebras such that roughly speaking the extension of their $\Delta-$good module categories (the precise meaning will be given later) is the $\Delta-$good module category of the big algebra.

Throughout the paper, K will denote a fixed field. By an algebra A, we mean an associative finite dimensional K-algebra. By a module M, we mean finitely generated left A-module. Now we recall the notation of the generalized path algebras from [4][8]. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver with Γ_0 the set of vertices and Γ_1 the set of arrows. An arrow α from $s(\alpha)$ to $e(\alpha)$ is sometimes denoted by $s(\alpha) \xrightarrow{\alpha} e(\alpha)$, $s(\alpha)$(or $e(\alpha)$) is called the starting vertex (ending vertex, resp.) of α. A *path* in Γ is $(b|\alpha_t \cdots \alpha_1|a)$, where $\alpha_i \in \Gamma_1$, for $i = 1, \cdots, t$, and $s(\alpha_1) = a$, $e(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \cdots, t-1$, and $e(\alpha_t) = b$, the *length of a path* is the number of arrows in it. To each arrow α we can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(b|\overline{\alpha_t} \cdots \overline{\alpha_1}|a)$, where the $a \in \{s(\alpha_1),\, e(\alpha_1)\}$, $b \in \{s(\alpha_n),\, e(\alpha_n)\}$, and for each $i = 1, \cdots, t-1$,

$$\{s(\alpha_i), e(\alpha_i)\} \cap \{s(\alpha_{i+1}), e(\alpha_{i+1})\} \neq \emptyset.$$

A quiver is said to be *connected* if for each pair of vertices a and b , there exists a walk between them.

Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver and $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of $K-$algebras A_i with identity, indexed by the vertices of Γ. Unless otherwise stated, we shall indicate the identity of A_i as e_i , for $i \in \Gamma_0$. The elements of $\bigcup_{i\in\Gamma_0} A_i$ are called the $\mathcal{A}-$*paths of length zero*, and for each $t \geq 1$, an $\mathcal{A} -$ *path of length* t is given by $a_{t+1}\beta_t a_t \cdots \beta_2 a_2 \beta_1 a_1$, where $(e(\beta_t)|\beta_t, \cdots, \beta_1|s(\beta_1))$ is a path in Γ of length t, for each $i = 1, \cdots, t$, $a_i \in A_{s(\beta_i)}$, and $a_{t+1} \in A_{e(\beta_t)}$, Consider now the quotient Λ of the K-vector space with basis the set of all $\mathcal{A}-$paths of Γ by the subspace generated by all the elements of the form

$$(a_{t+1}\beta_t \cdots a_{j+1}\beta_j(a_j^1 + \cdots + a_j^m)\beta_{j-1} \cdots \beta_1 a_1)$$
$$-\sum_{l=1}^{m}(a_{t+1}\beta_t \cdots a_{j+1}\beta_j a_j^l \beta_{j-1} \cdots \beta_1 a_1)$$

where $(e(\beta_t)|\beta_t \cdots \beta_1|s(\beta_1))$ is a path in Γ of length t , for each $i = 1, \cdots, t$, $a_i \in A_{s(\beta_i)}$, $a_{t+1} \in A_{e(\beta_t)}$, and $a_j^l \in A_{s(\beta_j)}$ for $l = 1, \cdots, m$.

Define now in Λ the following multiplication. Given two elements

$$[a_{t+1}\beta_t\cdots\beta_1 a_1],\ [b_{m+1}\gamma_m\cdots\gamma_1 b_1]$$

we define

$$[a_{t+1}\beta_t\cdots\beta_1 a_1][b_{m+1}\gamma_m\cdots\gamma_1 b_1] =$$
$$= \begin{cases} [a_{t+1}\beta_t\cdots\beta_1 a_1 b_{m+1}\gamma_m\cdots\gamma_1 b_1], & \text{if } b_{m+1},\ a_1 \text{belong} \\ & \text{to the same } A_i \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that the above multiplication in Λ is well-defined and gives Λ the structure of a $K-$algebra. The algebra Λ defined above is called $\mathcal{A}-$path algebra of Γ and we denote it by $\Lambda = K(\Gamma, \mathcal{A})$. Λ is also called *generalized path algebra*.

Remark 1. $\Lambda = K(\Gamma, \mathcal{A})$ has identity if and only if Γ_0 is finite. Moreover if $\Gamma_0 = \{1, \cdots, n\}$, then $e = e_1 + \cdots + e_n$ is identity of Λ. In the following, we always assume $\Lambda = K(\Gamma, \mathcal{A})$ has identity.

Remark 2. The usual path algebra K can be embedded into the $\mathcal{A}-$path algebra $K(\Gamma, \mathcal{A})$, or say if $A_i = K$, for each $i \in \Gamma_0$, then $K(\Gamma, \mathcal{A}) = K\Gamma$.

Remark 3. The generalized path algebra $\Lambda = K(\Gamma, \mathcal{A})$ is of finite dimension over K if and only if $\dim K A_i =\!=\!= \infty$ for each $i \in \Gamma_0$, and Γ is a finite quiver without oriented cycles [4]. In the following, we always assume $\Lambda = K(\Gamma, \mathcal{A})$ is finite dimensional.

We can give an alternative definition for the generalized path algebra $\Lambda = K(\Gamma, \mathcal{A})$ as follows. Let ${}_jM_i$ be the free $A_j - A_i-$bimodule with free generators given by the arrows from i to j. If $A = \bigoplus_{i\in\Gamma_0} A_i$, the ${}_jM_i$ is also an $A-A-$bimodule by defining $A_k \cdot {}_jM_i = 0$, if $k \neq j$ and ${}_jM_i \cdot A_k = 0$ if $k \neq i$. Let $M = \bigoplus_{i\to j} {}_jM_i$, which is clearly an $A-A-$bimodule. It is easy to check that Λ is isomorphic to the algebra

$$A \oplus M \oplus (M \otimes_K M) \oplus (M \otimes_K M \otimes_K M) \oplus \cdots$$

with multiplication given by the tensor product.

We now introduce the definition of quasi-hereditary algebras and standardly stratified algebras [2][3][7][10]. Quasi-hereditary algebras and standardly stratified algebras depend heavily on an ordering of the simple modules. For a finite dimensional algebra Λ over K, We fix an ordering on the simple $\Lambda-$modules: $E(1), E(2), \cdots, E(n)$. Let $P(i)$ be the projective cover of $E(i)$. We denote by $\Delta(i)$ the maximal factor of $P(i)$ with composition factors of the form $E(j)$, where $j =\!=\!= i$. Let

$\Delta = \{\Delta(1), \cdots, \Delta(n)\}$. We denote by $\mathcal{F}(\Theta)$ the full subcategory of ModΛ consisting of modules which have a filtration with factors in Θ, where ModΛ denotes the category of $f.g.$ left modules over Λ and Θ is a set of modules. These modules are said to be Θ-good. The algebra (Λ, E) is called *standardly stratified* with respect to the ordering of simple modules if $P(i) \in \mathcal{F}(\Delta)$, for all $i = 1, \cdots, n$ [7]; if in addition, End$\Lambda(\Delta(i))$ is a division ring, for all $i = 1, \cdots, n$, then (Λ, E) is *quasi-hereditary* [11].Standardly stratified algebras as a generalization of quasi-hereditary algebras have been studied recently by some authors in various aspects [1] [7] [9] [15].

Before stating our main result, we need to fix an ordering on the set of simple modules over the generalized path algebra. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of quasi-hereditary algebras A_i w.r.t. the ordering on the set E_i of simple A_i-modules. Since Γ is a finite quiver and connected and without oriented cycles, we can assume that $\Gamma_0 = \{1, 2, \cdots, n\}$, such that i is a source of the quiver obtained from Γ by deleting the vertices $1, 2, \cdots, i-1$. For any $i = 1, 2, \cdots, n$. Let $E_i = (E_i(1), \cdots, E_i(s_i))$ be the ordering on simple A_i-modules. Thus there is a complete set of orthogonal primitive idempotents $\underline{e}_i = (e_{i1}, \cdots, e_{is_i})$ of A_i corresponding to the ordered index set E_i of simple A_i-modules. Let $P_i(j) = A_i e_{ij}$. Then $P_i = (P_i(1), \cdots, P_i(S_i))$ is the corresponding set of indecomposable projective A_i-modules, and $\frac{P_i(j)}{radP_i(j)} \cong E_i(j),\ i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, s_i$. By identifying A_i with the subalgebra of $\Lambda = K(\Gamma, \mathcal{A})$ generated by paths of length 0 at the vertex i, $\underline{e} = (e_{11}, \cdots, e_{1s_1}, \cdots, e_{n1}, \cdots, e_{ns_n})$ is a complete set orthogonal primitive idempotents of $\Lambda = K(\Gamma, \mathcal{A})$. This index set is endowed with the ordering :$(i, j) < (k, h) \Leftrightarrow i < k$, or $i = k$ and $j < h$. With this notation , we have the following:

Theorem 1. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of standardly stratified algebras $(A_i, E_i),\ i \in \Gamma_0$. Then $\Lambda = K(\Gamma, \mathcal{A})$ is a standardly stratified algebra algebra w.r.t. the ordering index set $\underline{e}$.

Proof. Let $\underline{e}_i = (e_{i1}, \cdots, e_{is_i})$ be the complete set of primitive idempotents of A_i and $P_i(j) = A_i e_{ij}$ the projective indecomposable A_i-modules. Then the standard A_i-modules, by definition, are

$$\triangle_i = (\triangle_i(1), \cdots, \triangle_i(s_i)),$$

where

$$\begin{aligned}\triangle_i(j) &= \tfrac{P_i(j)}{U_i(j)},\\ U_i(j) &= A_i(\textstyle\sum_{j<h} e_{ih})A_ie_{ij}\\ &= A_i(\textstyle\sum_{k=j+1}^{s_i} e_{ik})A_ie_{ij}\\ &= \textstyle\sum_{\varphi\in\cup_{k=j+1}^{s_i} Hom(A_ie_{ik},A_ie_{ij})} Im\varphi\end{aligned}$$

(1). By construction, $\Lambda = \bigoplus_{i=1}^n A_i \bigoplus \sum_{t=1}^\infty M^{\otimes^t}$. We can view A_i- module $\Delta_i(j)$ as a Λ-module via the algebra quotient $\Lambda \to A_i$, actually by defining

$$\lambda x = \lambda_i x, \forall \lambda = \lambda_1 + \cdots + \lambda_i + \cdots + \lambda_t + \cdots \in \Lambda,\ x \in \Delta_i(j).$$

We will prove the standard $\Lambda-$module $\Delta(i,j)$ is isomorphic to $\Delta_i(j)$ for any i,j. By definition, we have $\Delta(i,j) = \frac{P(i,j)}{U(i,j)}$, where

$$\begin{aligned}P(i,j) &= \Lambda e_{ij}\\ &= (A_1 \oplus \cdots \oplus A_n \oplus \textstyle\sum_{t=1}^\infty M^{\otimes^t})e_{ij}\\ &= A_ie_{ij} \oplus \textstyle\sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t} {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}, (i=i_0)\\ U(i,j) &= \textstyle\sum_{\varphi\in Hom(\Lambda e_{kh},\Lambda e_{ij}),(i,j)<(k,h)} Im\varphi\\ &= \textstyle\sum_{\varphi\in\cup_{k=j+1}^{s_i} Hom(A_ie_{ik},A_ie_{ij})} Im\varphi + \sum_{\varphi\in\cup_{i<k} Hom(\Lambda e_k,\Lambda e_{ij})} Im\varphi.\end{aligned}$$

For any $0 \neq m = m_t \otimes \cdots \otimes m_1 e_{ij} \in {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \subseteq M^{\otimes^t}$ where $(i=)i_0 \to i_1 \to \cdots \to i_t$ is the path of length t in Γ, then

$$0 \neq me_{ij} = (m_t \otimes \cdots \otimes m_1)e_{ij} \in {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}.$$

We define

$$\varphi_m : \Lambda e_{i_t} \to \Lambda e_{ij}; ae_{i_t} \mapsto ae_{i_t}me_{ij}, (i<k),$$

then $\varphi_m \in Hom(\Lambda e_k, \Lambda e_{ij})$. Thus

$$\sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t} {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}$$

$$\subseteq \sum_{\varphi\in\cup_{i<k} Hom(\Lambda e_k,\Lambda e_{ij})} Im\varphi.$$

Hence

$$\Delta(i,j) = \frac{\Lambda e_{ij}}{U(i,j)} \cong \frac{A_ie_{ij}}{U_i(j)} = \Delta_i(j).$$

(2). Now we prove $P(i,j)$ belong to $\mathcal{F}_\Lambda(\Delta)$. By definition

$$P(i,j) = \Lambda e_{ij}$$

$$= Aie_{ij} \oplus \sum_{i=1}^\infty \bigoplus_{i_0\to i_1\to\cdots\to i_t} {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}, (i=i_0).$$

Since A_i is Standardly stratified , we have that $A_ie_{ij} = P_i(j) \in \mathcal{F}_{A_i}(\Delta_i) \subset \mathcal{F}_\Lambda(\Delta)$. We show that ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_2}M_{i_1} \otimes {}_{i_1}M_{i_0}$ is a free left $A_{i_t}-$module. Since ${}_{i_1}M_{i_0}$ is free $A_{i_1} - A_{i_0}-$ bimodule, ${}_{i_1}M_{i_0}$ is free left $A_{i_1} \otimes A_{i_0}-$module. We can assume free rank of ${}_{i_1}M_{i_0}$ is l, then

$$ {}_{i_1}M_{i_0} \cong (A_{i_1} \otimes_k A_{i_0})^l, \text{ for } l \in \mathbf{N}. $$

Then $A_{i_1} \otimes_k A_{i_0} = A_{i_1} \otimes K^h \cong (A_{i_1} \otimes K)^h \cong {A_{i_1}}^h$ where h is the dimension of A_{i_0}. It follows that

$$ {}_{i_1}M_{i_0} \cong (A_{i_1})^{hl} $$

is a free left $A_{i_1}-$module. Also

$$ {}_{i_2}M_{i_1} \otimes_{A_{i_2}} {}_{i_1}M_{i_0} \cong {}_{i_2}M_{i_1} \otimes_{A_{i_1}} (A_{i_1})^{hl} $$
$$ \cong ({}_{i_2}M_{i_1} \otimes_{A_{i_1}} A_{i_1})^{hl} \cong ({}_{i_2}M_{i_1})^{hl}. $$

Similarly ${}_{i_2}M_{i_1}$ is free left $A_{i_2}-$module, therefore ${}_{i_2}M_{i_1} \otimes_{A_{i_2}} {}_{i_1}M_{i_0}$ is free left $A_{i_2}-$module. An easy induction shows that ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}$ is a free left $A_{i_t}-$module. We may suppose that free rank of ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}$ is f. then ${}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij}$ is finite rank and free left $A_{i_t}-$module. Hence

$$ {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \cong A_{it}^f \in \mathcal{F}_{A_{i_t}}(\Delta) \subseteq \mathcal{F}_\Lambda(\Delta). $$

Then

$$ P(i,j) = A_ie_{ij} \oplus \sum_{t=1}^{\infty} \bigoplus {}_{i_t}M_{i_{t-1}} \otimes \cdots \otimes {}_{i_1}M_{i_0}e_{ij} \in \mathcal{F}_\Lambda(\Delta) $$

for $i = 1, \cdots, n$, $j = 1, \cdots, s_i$.

Combining (1) and (2), we have $\Lambda = K(\Gamma, \mathcal{A})$ is a Standardly stratified. The proof is finished.

Since quasi-hereditary algebras are standardly stratified algebras with endomorphism rings of $\Delta(i)$ being division rings, as a corollary, we have that if all algebras A_i are quasi-hereditary, then the generalized algebra $\Lambda = K(\Gamma, \mathcal{A})$ is also quasi-hereditary. Moreover, we can get a description of $\Delta-$good modules over this quasi-hereditary algebra by the terms of $\Delta-$good modules of these given quasi-hereditary algebras.

Corollary 2. Let $\Gamma = (\Gamma_0, \Gamma_1)$ be a finite connected quiver without oriented cycles. Let $\mathcal{A} = \{A_i | i \in \Gamma_0\}$ be a family of quasi-hereditary algebra (A_i, E_i), $i \in \Gamma_0$. Then $\Lambda = K(\Gamma, \mathcal{A})$ is a quasi-hereditary algebra w.r.t. the ordering index set $\underline{e}$. Moreover

$$ \mathcal{F}_\Lambda(\Delta) = \mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int F_{A_n}(\Delta), $$

which is by definition, the class of $\Lambda-$modules M with a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_{n-1} \supseteq M_0 = 0$, such that $M_{i-1}/M_i \in \mathcal{F}_{A_i}(\Delta)$.

Proof. In the following, we keep the notations in proof of Theorem 1. By Theorem 1 and its proof, we know that the standard $\Lambda-$module $\Delta(i,j)$ is isomorphic to standard A_i-module $\Delta_i(j)$ for any i,j, and $P(i,j)$ belong to $\mathcal{F}_\Lambda(\Delta)$. It follows that $\mathrm{End}_\Lambda(\Delta(i,j)) \cong \mathrm{End}_{A_i}(\Delta_i(j))$, and then by the quasi-heredity of A_i, $\mathrm{End}_\Lambda(\Delta(i,j))$ is a division ring. Therefore $\Lambda = K(\Gamma, \mathcal{A})$ is a quasi-hereditary algebra w.r.t. the ordering index set $\underline{e}$. Now we prove the second conclusion. From (1) in the proof of Theorem 1, we know that $\Lambda-$module $\Delta(i,j) \simeq \Delta_i(j)$ for any i,j. It follows that

$$\mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int \mathcal{F}_{A_n}(\Delta) \subseteq \mathcal{F}_\Lambda(\Delta).$$

Now for any $X \in \mathcal{F}_\Lambda(\Delta)$, X admits a $\Delta-$filtration:

$$X = X_{1,0} \supseteq X_{1,1} \cdots \supseteq X_{1,s_1} = X_{2,0} \supseteq X_{2,1} \cdots X_{n,0} \supseteq \cdots X_{n,s_n} = 0,$$

with $X_{i,j-1}/X_{i,j} \simeq \Delta(i,j)^{t_{ij}}$. Then we can get a filtration of $X:\ X = X_{1,0} \supseteq X_{2,0} \cdots X_{n,0} \supseteq \cdots X_{n,s_n} = 0$ with $X(i-1,0)/X(i,0) \in \mathcal{F}_{A_i}(\Delta)$. It follows that $X \in \mathcal{F}_\Lambda(\Delta)$. Then

$$\mathcal{F}_{A_1}(\Delta) \int \mathcal{F}_{A_2}(\Delta) \int \cdots \int F_{A_n}(\Delta) \supseteq \mathcal{F}_\Lambda(\Delta).$$

The proof is finished.

If we take the quiver Γ is $\overrightarrow{A_2}$. we get one of main results in [14].

Corollary 3(Zhu)[14]. Let $\Lambda = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be the triangular matrix algebra, where ${}_AM$ is free left $A-$module, A and B is quasi-hereditary algebras. Then Λ is quasi-hereditary algebras, $\mathcal{F}_\Lambda(\Delta) = \{(X,Y,f) | X \in \mathcal{F}_A(\Delta),\ Y \in \mathcal{F}_B(\Delta)\}$.

The author would like to take this opportunity to thank Professor B.Zhu for his guidance to this paper. She is grateful to the referee for his or her careful suggestions to improve the manuscript.

References

[1] Ágoston, I.; Happel, D.; Lukács, E.; Unger,L., Standardly stratified algebras and tilting, *J. Algebra* **226** (2000), 144–160.

[2] Cline, E.; Parshall, B.; Scott, L., Finite dimensional algebras and highest weight categories,J. Reine Angew. Math., *Mem. Amer. Math. Soc.*, **391** (1988),85-99.

[3] Cline, E.; Parshall, B.; Scott, L., Stratified endomorphism algebras, *Mem. Amer. Math. Soc.*, **591** (1996).

[4] Coelho,F.u.; Liu,S.X., Generalized path algebras,Interactions between ring theory and representations of algebras (Murcia),53-66, Lecture Note in Pure and Appl.Math.210,Dekker,New York,2000.

[5] Dlab, V.; Ringel, C.M. Quasi-hereditary algebras, *Illinois J. Math.*, **33(2)** (1989),280–291.

[6] Dlab, V.; Ringel, C. M. Auslander algebras as quasi-hereditary algebras. *J.London Math.Soc.*, **39(2)**, (1989),457-466.

[7] Dlab, V., Properly stratified algebras, *C. R. Acad. Sci. Paris*, Série Math, **330** (2000), 1–6.

[8] Li, F., Characterization of finite dimensional algebras through generalized path algebras. Preprint (2003).

[9] Mazorchuk, V.; Ovsienko,S., Finitistic dimension of properly stratified algebras. *Adv. Math.*, **186(1)** (2004),251–265

[10] Parshall, B. J. Finite-dimensional algebras and algebraic groups. Classical groups and related topics (Beijing, 1987), 97–114, *Contemp. Math.*, **82**, Amer. Math. Soc., Providence.

[11] Ringel, C. M. The category of modules with a good filtration over a quasi-hereditary algebra has almost split sequences. *Math. Z.* **208**, (1991), 209-223.

[12] Xi, C.C., Endomorphism algebras of $\mathcal{F}(\Delta)$ over quasi-hereditary algebras.*J. Algebra*, **175(3)** (1995), 966–978.

[13] Zhu,B., Smash products of quasi-hereditary graded algebras, *Arch.Math.*, **72** (1999),433–437.

[14] Zhu,B., Triangular matrix algebras over quasi-hereditary algebras, *Tsukuba,J.Math.*, **25(1)** (2001), 1–11.

[15] Zhu, B. Caenepeel, S., On good filtration dimensions for standardly stratified algebras. *Commun in Algebra*, **32(4)** (2004), 1603-1614.

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Received by the editors: 28.04.2005
and in final form 28.11.2006.