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RESEARCH ARTICLE

Arithmetic properties of exceptional lattice paths

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Abstract. For a fixed real number $\rho > 0$, let L be an affine line of slope ρ^{-1} in $\mathbb{R}^2$. We show that the closest approximation of L by a path P in $\mathbb{Z}^2$ is unique, except in one case, up to integral translation. We study this exceptional case. For irrational ρ, the projection of P to L yields two quasicrystallographic tilings in the sense of Lunnon and Pleasants [5]. If ρ satisfies an equation $x^2 = mx + 1$ with $m \in \mathbb{Z}$, both quasicrystals are mapped to each other by a substitution rule. For rational ρ, we characterize the periodic parts of P by geometric and arithmetic properties, and exhibit a relationship to the hereditary algebras $H_\rho(K)$ over a field K introduced in a recent proof of a conjecture of Roĭter.

Introduction

Let $\mathbb{Z}^2$ be the standard lattice in the real Euclidean space $\mathbb{R}^2$. We define a *lattice path* to be a subset $P := \{v_n \mid n \in \mathbb{Z}\}$ of $\mathbb{Z}^2$ such that each difference $v_{n+1} - v_n$ is one of the unit vectors $e_1 = \binom{1}{0}$ or $e_2 = \binom{0}{1}$.

For an affine line L in $\mathbb{R}^2$ of non-negative or infinite slope, let $\mathrm{d}(v, L)$ denote the distance between $v \in \mathbb{Z}^2$ and L. There is always a lattice path $P = \{v_n \mid n \in \mathbb{Z}\}$ such that the approximation of L by P cannot be improved. This means that each lattice path $\{v'_n \mid n \in \mathbb{Z}\}$ with $\mathrm{d}(v'_n, L) \leqslant \mathrm{d}(v_n, L)$ for all n satisfies $\mathrm{d}(v'_n, L) = \mathrm{d}(v_n, L)$ for all n. If P is unique, we call P the *best approximation* of L. Which lines L admit a best approximation? Of course, the answer will be the same if L is replaced by $L + v$ with a translation vector $v \in \mathbb{Z}^2$.

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Our first result (Proposition 1) states that for any $\rho \in [0, \infty]$, there is, up to an integral translation, exactly one line L of slope ρ^{-1} which does not admit a best approximation by a lattice path. It is this exceptional case on which we focus our attention in this paper. If L is a horizontal or vertical line, the exception clearly occurs when L has distance $\frac{1}{2}$ from $\mathbb{Z}^2$. So let us consider the (essentially unique) exceptional line L of positive slope $\rho^{-1} \in \mathbb{R}$. Although the distance between L and $\mathbb{Z}^2$ can be arbitrarily small in this case, there exists an integral translation that carries L to a line which passes through the point $O := \binom{1/2}{1/2}$.

It will be convenient to take the point O of L as a new origin, so that $\mathbb{Z}^2$ is shifted to the affine lattice E with coordinates of the form $n + \frac{1}{2}$, $n \in \mathbb{Z}$. The points $v \in E$ are the centers of the *cells* in $\mathbb{Z}^2$, i. e. the translates of the closed unit square. So the set C of cells inherits the structure of a two-dimensional lattice. Any sequence of cells in C corresponding to a finite piece of a lattice path will be called a *hook*.

Let ρ be rational. Then the integral points on L form a lattice line in $\mathbb{Z}^2$, and these are the only points where the infinitely many lattice paths in E which form a closest approximation of L are ambiguous. So if we remove the ambiguous points from these lattice paths, we get an infinite sequence of finite pieces P which are equal up to translation. The hook H_ρ associated to P will be called *rational*. For any hook H, let ∂H denote the boundary of the union $\bigcup H$ of its cells. We show that a hook H is rational if and only if the line that connects its extremal points intersects ∂H just in these two points (Theorem 1). If $\rho = \frac{a}{b}$ with relatively prime integers $a, b > 0$, the rational hook H_ρ fits into a rectangle of length a and height b.

Apart from this geometric description, we show that rational hooks can be characterized by rather nice arithmetic properties. First, we show that a hook H is rational if and only if its cells can be filled with positive numbers such that the row sums and the column sums are constant (Theorem 2). All such numberings are proportional, and the minimal row sum (column sum) is the length a (resp. the height b) of H. In the minimal numbering, the cells with value 1 are exactly those which can be removed so that the remaining cells make up one or two smaller rational hooks (Theorem 4). Second, we characterize a rational hook $H_{a/b}$ by the existence of a unique enumeration of its cells from 1 to n so that the difference between two sucessive cells in the same row (column) is b (resp. a) (Theorem 3).

If ρ is irrational, there are only two approximations of L by lattice paths P^+, P^- that cannot be improved. Furthermore, these lattice paths coincide except at one lattice point near O. The orthogonal projection of $P^{\pm}$ to L yields two quasicrystallographic tilings $X^{\pm}$ of L in the sense of [5], consisting of two tiles A and B. The tilings $X^{\pm}$ are of the form Y^*ABY respectively Y^*BAY, where Y and Y^* are limits of increasing filtrations of rational hooks. If ρ or ρ^{-1} is a quadratic unitary Pisot number, i. e. a solution $\beta > 1$ of an equation

$$x^2 = mx + d$$

with $m \in \mathbb{N}$ and $|d| = 1$, the tilings $X^{\pm}$ can be regarded as one-dimensional cut-and-project quasicrystals, generated by a generalized substitution rule in the sense of [7]. In the special case $\beta \in \{\frac{1}{2}(1 + \sqrt{5}), 1 + \sqrt{2}, 2 + \sqrt{3}\}$, the quasicrystals admit a unique quasiaddition [1, 8]. For $d = 1$, we show that X^+ and X^- are mapped into each other by the substitution rule $A \mapsto A^m B$, $B \mapsto A$.

Rational hooks occurred in a quite different context. During the past decades, representations of partially ordered sets have played a vital rôle in the representation theory of finite dimensional algebras and orders over a Dedekind domain. For a finite poset Ω, Nazarova and Roĭter [9, 11] introduced a norm $\|\Omega\|$ such that Ω is representation-finite if and only if $\|\Omega\| > \frac{1}{4}$ and tame if and only if $\|\Omega\| = \frac{1}{4}$. In a recent paper [10], they define a P-*faithful* poset to be a non-empty poset Ω such that $\|\Omega'\| > \|\Omega\|$ holds for every proper subset Ω'. They prove that Kleĭner's celebrated list of critical posets [3] consists in the P-faithful posets of norm 1/4. Zeldich [14, 15, 16] and Sapelkin [13] proved Roĭter's conjecture [10] which explicitely describes the connected P-faithful posets, hence all P-faithful posets.

In [12] we generalize Roĭter's norm to arbitrary vector space K-categories over a field K and show that the above mentioned results hold in this broader context. Relating P-faithful posets to a class of hereditary algebras of type $\mathbb{A}_n$, we give a rather short proof of Roĭter's conjecture. In particular, we parametrize connected P-faithful posets by rational numbers $\rho > 1$. In the terminology of the present paper, the P-faithful posets are just the posets associated to rational hooks H_ρ. Thus every rational hook H_ρ corresponds to a hereditary K-algebra $H_\rho(K)$, and some of the arithmetic properties of H_ρ are reflected by the representation theory of $H_\rho(K)$ (see [12], §7).

For irrational $\rho > 0$, an infinite dimensional hereditary algebra $H_\rho(K)$ of type $\mathbb{A}_\infty^\infty$ can be defined analoguously. Then the finitely generated indecomposable preprojective $H_\rho(K)$-modules can be parametrized by the positive numbers in $\mathbb{Z} + \mathbb{Z}\rho$. For example, if $\rho = \tau := \frac{1}{2}(1 + \sqrt{5})$, the preprojective component of $H_\rho(K)$ looks as follows:

$$
\begin{tikzcd}[column sep=tiny, row sep=small]
 & & & \vdots & & \vdots & & \vdots & & \vdots & \\
 & & {[}3-\tau \arrow[rr, dash, dashed] \arrow[dr] & & 4 \arrow[rr, dash, dashed] \arrow[dr] & & \tau+5 \arrow[rr, dash, dashed] \arrow[dr] & & 2\tau+6 \arrow[dr] & & \cdots \\
 & {[}2-\tau \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & \tau+4 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau+5 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+6 & \\
 & & {[}2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & \tau+3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau+4 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+5 \arrow[ur] \arrow[dr] & & \cdots \\
 & {[}1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & \tau+2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau+3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+4 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 4\tau+5 & \\
 & & {[}\tau+1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau+2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 4\tau+4 \arrow[ur] \arrow[dr] & & \cdots \\
 & {[}\tau \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau+1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 4\tau+3 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 5\tau+4 & \\
{[}\tau-1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 2\tau \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 3\tau+1 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 4\tau+2 \arrow[rr, dash, dashed] \arrow[ur] \arrow[dr] & & 5\tau+3 \arrow[ur] \arrow[dr] & & \cdots \\
 & {[}2\tau-1 \arrow[rr, dash, dashed] \arrow[ur] & & 3\tau \arrow[rr, dash, dashed] \arrow[ur] & & 4\tau+1 \arrow[rr, dash, dashed] \arrow[ur] & & 5\tau+2 \arrow[rr, dash, dashed] \arrow[ur] & & 6\tau+3 & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{tikzcd}
$$

The indecomposable projective $H_\rho(K)$-modules, marked by a left bracket "[", are connected by two kinds of arrows, "$\nearrow$" or "$\searrow$", according to the tiling of the associated quasicrystal X^+. Note that the greater difference τ between the values of two adjacent projectives corresponds to a short tile, while the smaller difference 1 indicates a long tile of X^+. The mirror-image $X^-$of X^+ corresponds to the modified algebra $H_\rho^-(K)$, with a projective of value 0 instead of that with value $\tau+1$.

1. Approximation of affine lines

Let ρ, δ be real numbers with $\rho > 0$. By $L_{\rho\delta}$ we denote the affine line

$$\rho y - x = \delta \tag{1.1}$$

in the real vector space $\mathbb{R}^2$. We consider the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ generated by the unit vectors $e_1 := \binom{1}{0}$ and $e_2 := \binom{0}{1}$. A sequence $(v_n)_{n\in\mathbb{Z}}$ of vectors $v_n \in \mathbb{Z}^2$ will be called a *lattice path* if the difference $v_{n+1} - v_n$ of successive vectors is either e_1 or e_2. Up to a shift $n \mapsto n+k$, such a sequence is determined by its underlying set $P := \{v_n \mid n \in \mathbb{Z}\}$. Therefore, by a slight abuse, the subset $P \subset \mathbb{Z}^2$ will also be referred to as a *lattice path.* Now let us define a *minimal* lattice path to be one which provides a best approximation (in a sense to be made precise yet) of some line $L_{\rho\delta}$.

If we regard $\mathbb{R}^2$ as a Euclidean space with orthonormal basis $\{e_1, e_2\}$, the vector $\binom{-1}{\rho}$ is orthogonal to $L_{\rho\delta}$. Therefore, the distance $\mathrm{d}(v, L_{\rho\delta})$ between any $v = \binom{x}{y} \in \mathbb{Z}^2$ and $L_{\rho\delta}$ satisfies

$$\sqrt{1+\rho^2} \cdot \mathrm{d}(v, L_{\rho\delta}) = |\rho y - x - \delta|. \tag{1.2}$$

Now let $v - e_1, v, v + e_2$ be three successive vectors of a lattice path $P \subset \mathbb{Z}^2$. If we replace $v = \binom{x}{y}$ by $v' := v - e_1 + e_2 = \binom{x-1}{y+1}$, we get another lattice path P' with $\rho(y+1)-(x-1)-\delta = (\rho y - x - \delta)+(1+\rho)$. Thus P' cannot approximate $L_{\rho\delta}$ better that P unless $\rho y - x - \delta < 0$. If $\rho y - x - \delta < 0$, then $\mathrm{d}(v, L_{\rho\delta}) < \mathrm{d}(v', L_{\rho\delta})$ holds if and only if v belongs to

$$P_{\rho\delta} := \left\{ \tbinom{x}{y} \in \mathbb{Z}^2 \;\middle|\; |\rho y - x - \delta| < \tfrac{1+\rho}{2} \right\}. \tag{1.3}$$

A similar statement applies when e_1 and e_2 are permuted. Therefore, we define a *best approximation* of $L_{\rho\delta}$ to be a (necessarily unique) lattice path $P \subset P_{\rho\delta}$.

Proposition 1.. Let $\rho, \delta \in \mathbb{R}$ with $\rho > 0$ be given. A best approximation P of $L_{\rho\delta}$ exists if and only if

$$\tfrac{\rho-1}{2} - \delta \notin \mathbb{Z} + \rho\mathbb{Z}. \tag{1.4}$$

If the best approximation exists, it coincides with $P_{\rho\delta}$.

Proof. By Eq. (1.3), we have

$$\tbinom{x}{y} \in P_{\rho\delta} \;\Leftrightarrow\; \rho y - \delta - \tfrac{1+\rho}{2} < x < \rho y - \delta + \tfrac{1+\rho}{2}. \tag{1.5}$$

So there is an open interval I_y of length $1+\rho$ such that $\binom{x}{y} \in P_{\rho\delta} \Leftrightarrow x \in I_y$. Hence $\binom{x+1}{y} \in P_{\rho\delta} \Leftrightarrow x+1 \in I_y$ and $\binom{x}{y+1} \in P_{\rho\delta} \Leftrightarrow x - \rho \in I_y$. Thus if $v := \binom{x}{y} \in P_{\rho\delta}$ and $x + 1 \neq \rho y - \delta + \frac{1+\rho}{2}$, exactly one of the numbers $x - \rho$ and $x + 1$ belongs to I_y, i. e. either $v + e_1 \in P_{\rho\delta}$ or $v + e_2 \in P_{\rho\delta}$. This shows that $P_{\rho\delta}$ is a best approximation of $L_{\rho\delta}$ if the equation

$$x + 1 = \rho y - \delta + \tfrac{1+\rho}{2} \tag{1.6}$$

has no solution $\binom{x}{y} \in \mathbb{Z}^2$, and there is no best approximation if Eq. (1.6) is solvable in $\mathbb{Z}^2$. Now the Diophantine equation (1.6) is not solvable if and only if (1.4) holds. This completes the proof. □

Remark. By a translation

$$x \mapsto x - x_0\,; \quad y \mapsto y - y_0 \tag{1.7}$$

with $x_0, y_0 \in \mathbb{Z}$, Eq. (1.1) becomes $\rho y - x = \delta + (\rho y_0 - x_0)$. Therefore, the problem to approximate $L_{\rho\delta}$ by a lattice path is turned into an equivalent one if an arbitrary element of $\mathbb{Z} + \rho\mathbb{Z}$ is added to δ. So Proposition 1 shows that up to translation (1.7), there is just one exceptional value of δ for which the best approximation of $L_{\rho\delta}$ does not exist, namely,

$$\delta = \tfrac{\rho-1}{2}. \tag{1.8}$$

Assume that (1.4) is satisfied. Then the orthogonal projection of the lattice path $P_{\rho\delta} = \{v_n \mid n \in \mathbb{Z}\}$ to the line $L_{\rho\delta}$ yields a two-way infinite sequence $(v'_n)_{n\in\mathbb{Z}}$ of points $v'_n \in L_{\rho\delta}$ with

$$|v'_{n+1} - v'_n| = \begin{cases} \frac{\rho}{\sqrt{1+\rho^2}} & \text{if } v_{n+1} - v_n = e_1 \\ \frac{1}{\sqrt{1+\rho^2}} & \text{if } v_{n+1} - v_n = e_2. \end{cases}$$

So we get a one-dimensional quasicrystallographic tiling with two tiles in the sense of [5], Definition 2. By [5], §4, two such quasicrystals X, X' with the same length ratio ρ of their tiles are in the same *species*, i. e. every finite sequence of successive tiles in X has infinitely many copies in X' which are asymptotically equally distributed. (In particular, this property holds for $X' = X$.) The two-tile sequences $(T_n)_{n\in\mathbb{Z}}$ with $T \in \{L, S\}$ arising in this way (L = long, S = short) can also be characterized by the property that the number of L's in a word of length l can take only two values [6].

2. The exceptional case

In the Euclidean space $\mathbb{R}^2$, the unit square Q is a fundamental domain for the group of integral translations (1.7). The translates of the closed unit square will be called *cells* of $\mathbb{Z}^2$.

For $\rho > 0$ and $\delta \in \mathbb{R}$ as in Proposition 1, assume that there is no best approximation of $L_{\rho\delta}$. By the preceding remark, we can assume, without loss of generality, that δ satisfies Eq. (1.8). Therefore, Eq. (1.6) takes the simple form

$$x = \rho y. \tag{2.9}$$

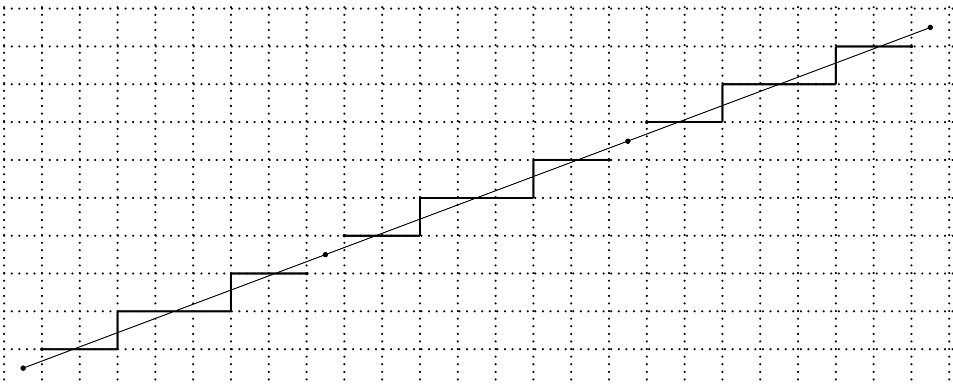
For each solution $v = \binom{x}{y} \in \mathbb{Z}^2$ of Eq. (2.9), there is a cell

$$\begin{tikzcd} v + e_2 \arrow[r, dash] & v + e_1 + e_2 \\ v \arrow[u, dash] \arrow[r, dash] & v + e_1 \arrow[u, dash] \end{tikzcd}$$

in $\mathbb{Z}^2$, where v and $v + e_1 + e_2$ belong to $P_{\rho\delta}$, while $v + e_1$ and $v + e_2$ are equally distant from $L_{\rho\delta}$. So there exist approximations of $L_{\rho\delta}$ by lattice paths that cannot be improved, but which are ambiguous at each solution of Eq. (2.9).

If ρ is irrational, Eq. (2.9) has no integral solution except the trivial one. Accordingly, there are two equally good approximations of $L_{\rho\delta}$, differing only at the origin. If the slope ρ^{-1} of $L_{\rho\delta}$ is the golden ratio $\tau := \frac{1+\sqrt{5}}{2}$, we get the example considered in [5], Fig. 4, a cut-and-project quasicrystal of Fibonacci type, with acceptance window either $[0,1)$ or $(0,1]$, according to the ambiguity at the origin.

In case ρ is rational, say, $\rho = \frac{a}{b}$ with relatively prime integers $a, b > 0$, the integral solutions of Eq. (2.9) form the lattice line $\mathbb{Z}\binom{a}{b}$. So there are infinitely many lattice paths giving approximations of $L_{\rho\delta}$ which cannot be improved, but which are ambiguous at each $v \in \mathbb{Z}\binom{a}{b}$. The situation is illustrated by the following example with $\rho = \frac{8}{3}$.



Note that in the exceptional case (1.8), the vector $\binom{1/2}{1/2}$ satisfies Eq. (1.1). So it is convenient to shift the origin to $\binom{1/2}{1/2}$. Then the line $L_{\rho\delta}$ is replaced by the subspace

$$L_\rho := \mathbb{R}\binom{\rho}{1}, \tag{2.10}$$

and the lattice $\mathbb{Z}^2$ is changed into the affine lattice

$$E := \left\{ \binom{x}{y} \in \mathbb{R}^2 \;\middle|\; x + \tfrac{1}{2}, y + \tfrac{1}{2} \in \mathbb{Z} \right\}. \tag{2.11}$$

Accordingly, we set

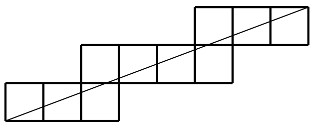
$$P_\rho := \left\{ \binom{x}{y} \in E \;\middle|\; \binom{x+\frac{1}{2}}{y+\frac{1}{2}} \in P_{\rho\delta} \right\} = \left\{ \binom{x}{y} \in E \;\middle|\; |\rho y - x| < \tfrac{1+\rho}{2} \right\}. \tag{2.12}$$

The elements of E are the centers v of the cells Q of $\mathbb{Z}^2$. Hence, via the correspondence $Q \leftrightarrow v$, the set C of cells of $\mathbb{Z}^2$ becomes a lattice which can be identified with E.

Proposition 2.. A vector $v \in E$ belongs to P_ρ if and only if L_ρ intersects the interior of the corresponding cell Q.

Proof. L_ρ intersects the interior of Q if and only if $v - \frac{1}{2}e_1 + \frac{1}{2}e_2$ and $v + \frac{1}{2}e_1 - \frac{1}{2}e_2$ are on different sides of the line L_ρ, i. e. if and only if $\rho(y+\frac{1}{2}) - (x - \frac{1}{2})$ and $\rho(y - \frac{1}{2}) - (x + \frac{1}{2})$ have different sign. The latter condition is equivalent to the inequality $|\rho y - x| < \frac{1+\rho}{2}$. □

In the sequel, we consider the exceptional case (1.8) with $\rho = \frac{a}{b} \in \mathbb{Q}$, where the integers $a, b > 0$ are relatively prime. Then the solutions of Eq. (2.9) coincide with the integral points $n\binom{a}{b}$ on L_ρ. At these points, two cells of C touch the line L_ρ, which causes the mentioned ambiguity of the approximation. So P_ρ consists of a sequence $(F_n)_{n\in\mathbb{Z}}$ of finite pieces F_n of a lattice path P in E, such that the translation $v \mapsto v + \binom{a}{b}$ carries F_n to F_{n+1}. Moreover, the F_n are separated by the integral points $n\binom{a}{b} \in L_\rho \cap \mathbb{Z}^2$, and at each $n\binom{a}{b}$, there are two choices to connect the pieces F_n to a lattice path P. Up to translation, there is a unique subset H_ρ of C corresponding to $F_n \subset E$. We call H_ρ a *rational hook*. For example, the rational hook $H_{8/3}$, together with the line $L_{8/3}$, looks as follows:



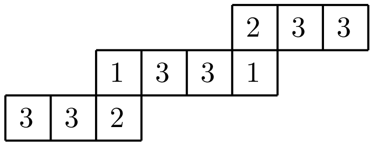
(2.13)

Remark. Proposition 2 shows that rational hooks can be obtained by an extremely simple geometric construction. For any rational $\rho > 0$ with reduced representation $\rho = \frac{a}{b}$, there is a path of $a + b - 1$ cells which intersect a lattice line of slope ρ^{-1} properly, and these cells constitute the rational hook H_ρ.

Let us define a *hook* to be a finite subset H of C, consisting of a non-empty sequence $Q_1, \dots, Q_n$ of cells with the following property. If v_i denotes the center of Q_i, then $v_{i+1} - v_i \in \{e_1, e_2\}$ for all $i \in \{1, \dots, n-1\}$. Thus a hook is just a finite version of a lattice path in C. Clearly, a hook H is determined by the union $\bigcup H$ of its cells. The boundary ∂H of $\bigcup H$ will be called the *boundary* of the hook H. Furthermore, we define the *interior line* of H to be the open line segment which connects the lower left corner of H with the upper right corner of H (see (2.13)). As an immediate consequence of Proposition 2, we get

Theorem 1.. A hook H is rational if and only if its interior line does not intersect the boundary ∂H.

Our next aim is to show that rational hooks can be obtained by an equally simple arithmetic construction. If the cells of a hook H can be filled with positive integers such that the row sums as well as the column sums are constant, we call H a *magic hook*. For example, the rational hook $H_{8/3}$ is a magic one:



Here the row sums are 8, and the column sums are 3. In what follows, the function which attaches a number to each cell of a magic hook will be denoted by v.

Proposition 3.. For any pair a, b of positive integers, there is exactly one magic hook with row sum a and column sum b.

Proof. We construct a magic hook H with row sum a and column sum b, and show that such a hook must be unique. Let $Q_1, \dots, Q_n$ be the sequence of cells of H, ordered from the lower left to the upper right corner of H. If $a = b$, there can be only one cell, and we are done. If $a > b$, then $n \geqslant 2$, and the cells Q_1 and Q_2 cannot be in the same column. Similarly, $a < b$ implies that Q_1 and Q_2 cannot be in the same row. Thus let us assume, without loss of generality, that $a > b$. Then $v(Q_1) = b$. Starting with $r_0 := 0$, we define the sequences $(r_i)_{i \in \mathbb{N}}$ and $(q_i)_{i \in \mathbb{N}}$ of integers recursively by the formula

$$a + r_i = q_i b + r_{i+1}, \tag{2.14}$$

where $0 \leqslant r_i < b$ for all i. Thus $a = q_0 b + r_1$. If $r_1 = 0$, we get a hook of q_0 cells in one row with $v(Q_i) = b$ for all i. Otherwise, the cells $Q_1, \dots, Q_{q_0+1}$ are in one row, and $v(Q_i) = b$ for $i \leqslant q_0$. As the row sum is a, we obtain $v(Q_{q_0+1}) = r_1 > 0$. Since the column sum is b, the cell Q_{q_0+2} above Q_{q_0+1} satisfies $v(Q_{q_0+2}) = b - r_1$. Now Eq. (2.14) gives $a + r_1 = q_1 b + r_2$, that is, $(b - r_1) + (q_1 - 1)b + r_2 = a$. Therefore, the cells $Q_{q_0+2}, \dots, Q_{q_0+q_1+1}$ are in the second row, with $v(Q_i) = b$ for $i \in \{q_0 + 3, \dots, q_0 + q_1 + 1\}$. If $r_2 = 0$, the magic hook is finished. Otherwise, there is another cell $Q_{q_0+q_1+2}$ in the second row, with $v(Q_{q_0+q_1+2}) = r_2$, and the procedure has to be continued as before.

It remains to be shown that the recursion (2.14) eventually stops. If we add Eqs. (2.14) for $i \in \{0, \dots, i-1\}$, we get

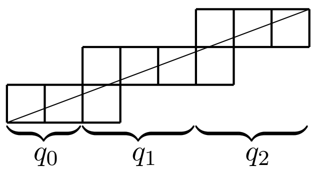
$$ia = (q_0 + \cdots + q_{i-1})b + r_i. \tag{2.15}$$

Hence $r_i \equiv i \cdot a \pmod{b}$, which proves that $r_b = 0$. □

If the numbering of a magic hook is multiplied by a fixed positive integer, another magic numbering is obtained. Therefore, the magic hooks of Proposition 3 only depend on the rational number $\rho := \frac{a}{b}$. Our next result shows that all magic hooks are rational, and conversely, that every rational hook admits a magic numbering.

Theorem 2.. A hook is rational if and only if it admits a magic numbering.

Proof. Let $\rho = \frac{a}{b}$ be a rational number with $a, b > 0$ relatively prime. We show that the magic hook H'_ρ with row sum a and column sum b, constructed via (2.14), coincides with the rational hook H_ρ. By symmetry, we can assume that $a \geqslant b$. If $r_1 = 0$, then H'_ρ consists of a single row of q_0 cells, and $H'_\rho = H_\rho$. Otherwise, both H'_ρ and H_ρ start with a horizontal sequence of $q_0 + 1$ cells, followed by one step upwards. The line $\mathbb{Q}\binom{a}{b}$ contains the point $\binom{2\rho}{2}$, and Eq. (2.15) implies that $2\rho = q_0 + q_1 + \frac{r_2}{b}$. Hence, if $r_2 = 0$, the upward step is followed by q_1 horizontal steps, and the resulting magic hook is rational. If $r_2 > 0$, another horizontal step has to be added in H'_ρ and also in H_ρ, again followed by one step upwards.



In general, the equivalence of the geometric construction of H_ρ and the recursive construction of H'_ρ via (2.14) results from Eq. (2.15), divided by b:

$$i \cdot \rho = (q_0 + \cdots + q_{i-1}) + \tfrac{r_i}{b}.$$

Note that $\frac{r_i}{b} < 1$. Since $\binom{i\rho}{i} \in \mathbb{Q}\binom{a}{b}$, we get $H'_\rho = H_\rho$. □

Corollary. *Every rational hook is centrally symmetric.*

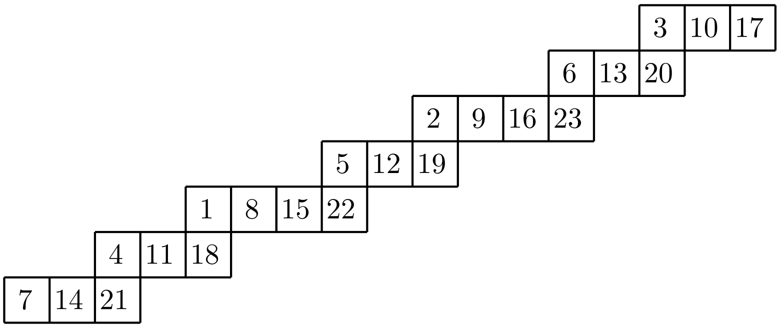
Proof. If we apply a rotation with angle π to a rational hook $H_{a/b}$ with $a, b > 0$ relatively prime, we get another magic hook H with row sum a and column sum b. Hence $H = H_{a/b}$. □

3. Uniform enumeration

In this section, we give a third characterization of rational hooks. Let H be a hook with n cells. An enumeration of the cells from 1 to n defines a bijection

$$f : H \longrightarrow \{1, \ldots, n\}.$$

W call f a *uniform enumeration* if there are integers $a, b > 0$ such that each pair of adjacent cells Q, Q' satisfies $f(Q') - f(Q) = a$ if Q' is below Q and $f(Q') - f(Q) = b$ if Q' is on the right of Q. The following example displays a uniform enumeration of the rational hook $H_{17/7}$.



Theorem 3..

(a) A hook is rational if and only if it admits a uniform enumeration.

(b) A rational hook has exactly one uniform enumeration.

(c) If Q is the lower left cell and Q' the upper right cell of $H_{a/b}$, where $a, b > 0$ are relatively prime, the uniform enumeration f of $H_{a/b}$ satisfies $f(Q) = b$ and $f(Q') = a$.

Proof. We make the affine lattice E into a partially ordered set, defining

$$\binom{x}{y} \leqslant \binom{x'}{y'} \;:\Leftrightarrow\; x \leqslant x' \text{ and } y \geqslant y'. \tag{3.16}$$

Every hook H can also be regarded as a partially ordered set. For a pair of cells $Q, Q' \in H$, we write $Q < Q'$ if either Q and Q' are in the same row with Q' on the right of Q, or they are in the same column with Q' below Q. If the cells of H are replaced by their centers, we get a finite subposet Ω of E, unique up to translation. Then a magic numbering of H is tantamount to a uniform vector $v > 0$ of Ω in the sense of [12], §5. Similarly, a uniform enumeration of H is equivalent to a uniform enumeration [12] of Ω. Therefore, part (a) and (b) of the theorem follow by [12], Proposition 9. For te reader's convenience, the one-to-one correspondence between a magic numbering v and a uniform enumeration f of H is given in Eqs. (3.17) below. For a cell $Q \in H$, we denote the set of lower neighbours of Q by Q^-.

$$v(Q) = f(Q) - \sum_{Q' \in Q^-} f(Q')\,; \quad f(Q) = \sum_{Q' \leqslant Q} v(Q'). \tag{3.17}$$

To prove (c), note first that the rational hook $H_{a/b}$ has $a+b-1$ cells. Let $Q_1, \ldots, Q_{a+b-1}$ be the sequence of cells of $H_{a/b}$, ordered from the lower left to the upper right of $H_{a/b}$. If Q_i and Q_{i+1} are in the same row, then

$f(Q_{i+1}) = f(Q_i)+b$; if they are in the same column, $f(Q_{i+1}) = f(Q_i)-a$. Hence

$$f(Q_{i+1}) \equiv f(Q_i) + b \pmod{a+b}.$$

Since b is invertible modulo $a+b$, and $f(Q_i)$ runs through all the residue classes modulo $a+b$ except 0, we infer that $f(Q_1)-b \equiv 0 \equiv f(Q_{a+b-1})+b \pmod{a+b}$, which proves the claim. □

In the sequel, the partial ordering of a hook introduced in the above proof will be assumed without further notice.

Corollary. *Let v be a magic numbering of a hook H with at least two rows and columns. Assume that the values of v are relatively prime. Then there are exactly two cells Q of H with $v(Q) = 1$. These cells are related by central symmetry.*

Proof. By assumption, there are relatively prime integers $a, b \geqslant 2$ with $H = H_{a/b}$. Therefore, a cell Q of H with $v(Q) = 1$ must be either minimal or maximal. (Since H has more than one cell, the sets of minimal respectively maximal cells are disjoint.) Let f denote the uniform enumeration of H. By Eqs. (3.17), a minimal cell Q satisfies $v(Q) = 1$ if and only if $f(Q) = 1$. Therefore, the corollary of Theorem 1 implies that there are exactly two, centrally symmetric, cells Q of H which satisfy $v(Q) = 1$. □

Next we give an explicit formula for the minimal magic numbering v of a rational hook H_ρ. Assume that $\rho = \frac{a}{b}$ with $a > b > 0$ relatively prime. For a minimal cell Q (with respect to the partial ordering of H_ρ), there is exactly one vertex $\binom{x_Q}{y_Q} \in \mathbb{Z}^2$ strictly below the line L_ρ, and this property characterizes the minimal cells. Similarly, the maximal cells Q have exactly one vertex strictly above L_ρ (see (2.13)). If f denotes the uniform enumeration of H_ρ, we have $v(Q) = f(Q)$ if and only if Q is minimal. Thus, by symmetry, the minimal magic numbering is given by the values $f(Q)$ for minimal $Q \in H_\rho$. Note that $v(Q) = b$ if Q is neither minimal nor maximal.

Proposition 4.. Let $H_{a/b}$ be a rational hook with $a, b > 0$ relatively prime, and let f be its uniform enumeration. Then

$$f(Q) = bx_Q - ay_Q \tag{3.18}$$

holds for the minimal cells Q of $H_{a/b}$.

Proof. We use Eqs. (2.14) to determine the minimal cells $Q_1, \ldots, Q_b$ of $H_{a/b}$, ordered from left to right. We have

$$x_{Q_i} = 1 + \sum_{j=0}^{i-2} q_j\,; \quad y_{Q_i} = i - 1 \tag{3.19}$$

for $i \in \{1, \dots, b\}$. For the maximal cells Q'_i below Q_i (for $i > 1$), the proof of Proposition 3 shows that $v(Q'_i) = r_{i-1}$. Hence $v(Q_i) = b - r_{i-1}$, which holds for all $i \in \{1, \dots, b\}$. So we get

$$bx_{Q_i} - ay_{Q_i} = b + \sum_{j=0}^{i-2} bq_j - a(i-1) =$$

$$= b + \sum_{j=0}^{i-2}(a + r_j - r_{j+1}) - a(i-1) = b - r_{i-1} = v(Q_i) = f(Q_i).$$

□

4. Splitting of rational hooks and quasicrystals

In all what follows, for any rational hook H, we denote the minimal magic numbering by v and the uniform enumeration by f. Let us call a cell Q of H *removable* if $H \smallsetminus \{Q\}$ is a disjoint union of (one or two) rational hooks. The following theorem adds a converse to [12], Proposition 10, which states that the cell $Q \in H$ with $f(Q) = 1$ is removable. Using Proposition 4, we give a simple geometric proof of this property.

Theorem 4. A cell Q of a rational hook H is removable if and only if $v(Q) = 1$.

Proof. Assume that $H = H_{a/b}$ with $a, b > 0$ relatively prime. If H has only one row or column, every cell Q is removable and satisfies $v(Q) = 1$. Thus let us assume, without loss of generality, that $a > b \geqslant 2$. The proof of Proposition 3 shows that H is a union $R_1 \cup \cdots \cup R_b$ of hooks R_i, each with one row. By Eqs. (2.14), the hooks R_i with $i < b$ consist of $q_i + 1$ cells, while R_b has q_b cells. Therefore, each R_i consists of either $q_0 + 1$ or $q_0 + 2$ cells. Since every rational hook is centrally symmetric by the corollary of Theorem 1, a removable cell Q cannot be one of the two ends of H and must be either maximal or minimal with respect to the partial ordering of H, i. e. $v(Q) < b$. Now the right-hand side of Eq. (3.18) can be written as a scalar product $\binom{b}{-a} \cdot \binom{x_Q}{y_Q}$, where the vector $\binom{b}{-a}$ is orthogonal to the line $L_{a/b}$. Therefore, $bx_Q - ay_Q$ measures the distance between $v_Q := \binom{x_Q}{y_Q}$ and $L_{a/b}$. By Theorem 1, a rational hook H_ρ is characterized by the geometric property that the interior line does not intersect ∂H_ρ. If a cell Q of H is removed, this property remains valid for the two hooks of $H \smallsetminus \{Q\}$ if the distance between v_Q and $L_{a/b}$ is minimal, i. e. $bx_Q - ay_Q = \pm 1$. By Proposition 4, the latter means that $v(Q) = 1$. To prove the converse, assume that $bx_Q - ay_Q = 1$, and that $Q' \in H$ is removable, too. Assume, without loss of generality, that $d := bx_{Q'} - ay_{Q'}$ is positive. Thus $H \smallsetminus \{Q'\}$ splits into two rational hooks $H_{\rho'}$ and $H_{\rho''}$. Assume that Q belongs to $H_{\rho'}$. If $v_{Q'} = d \cdot v_Q$, then $v_Q \in L_{\rho'}$, a contradiction. Hence $|v_{Q'}| > d \cdot |v_Q|$. Since $d \cdot v_Q = \binom{dx_Q}{dy_Q}$ satisfies $b \cdot dx_Q - a \cdot dy_Q = d$, this is impossible. □

Corollary. *Let $H = H_{a/b}$ be a rational hook with $a, b \geqslant 2$ relatively prime. There are exactly two removable cells of H. If one of these cells is removed, we obtain two rational hooks $H_{p/q}$ and $H_{r/s}$ with reduced fractions $\frac{p}{q} > \frac{r}{s}$. These hooks $H_{p/q}$ and $H_{r/s}$ are uniquely determined by the determinantal relation*

$$\begin{vmatrix} p & r \\ q & s \end{vmatrix} = 1. \tag{4.20}$$

Proof. By Theorem 4 and the corollary of Theorem 3, there are two centrally symmetric removable cells. Let Q be the minimal one. Then $bx_Q - ay_Q = 1$ by Proposition 4. Hence Eq. (4.20) holds with $p = x_Q, q = y_Q, r = a - x_Q$, and $s = b - y_Q$, which yields a splitting of H into $H_{p/q}$ and $H_{r/s}$. Conversely, since $p + r = a$ and $q + s = b$, Eq. (4.20) is equivalent to $\begin{vmatrix} p & a \\ q & b \end{vmatrix} = 1$, which has exactly one solution $\binom{p}{q} \in \mathbb{Z}^2$ with $0 < p < a$ and $0 < q < b$. □

Let H be any rational hook. If we successively remove the cells Q with $f(Q) = 1$, we get a unique decomposition of H into hooks of one cell. Next we will show that this decomposition of a rational hook H_ρ with $\rho > 1$ can be obtained from the continued fraction expansion of ρ. Recall first that H_ρ corresponds to a finite sequence $v_0, \dots, v_n$ in E, such that the differences $v_i - v_{i-1}$ are either e_1 or e_2. Therefore, the orthogonal projection of $\{v_0 \dots, v_n\}$ to the interior line of H_ρ yields a sequence W_ρ of n tiles L or S, corresponding to the differences e_1 and e_2, respectively. We call W_ρ the *tiling* of H_ρ. Note that the restriction $\rho > 1$ is not essential. For the symmetric case $0 < \rho < 1$, however, the "long" tile L would be shorter than S.

By the corollary of Theorem 4, every hook H_ρ with more than one row and column has two removable cells Q such that $H_\rho \smallsetminus \{Q\}$ splits into two rational hooks $H_{\rho'}$ and $H_{\rho''}$ with $\rho' < \rho''$. Accordingly, the tiling W_ρ has two decompositions

$$W_\rho = W_{\rho''} T^+ W_{\rho'} = W_{\rho'} T^- W_{\rho''}, \tag{4.21}$$

where

$$T^+ := SL, \quad T^- := LS. \tag{4.22}$$

In fact, if Q is minimal, Eq. (3.18) shows that

$$\rho'' = \frac{x_Q}{y_Q} > \rho' = \frac{a - x_Q}{b - y_Q}, \tag{4.23}$$

which yields the first equation in (4.21). The other case is dual.

For any $\rho > 1$, there is a unique continued fraction expansion

$$\rho = [a_0, a_1, a_2, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} \tag{4.24}$$

with integers $a_n \geqslant 1$. (For rational ρ, the expansion is finite, and we assume that the last a_n is $\geqslant 2$.) The partial quotients

$$[a_0, \ldots, a_{n-1}] = \frac{p_n}{q_n} \tag{4.25}$$

with relatively prime $p_n, q_n > 0$ are given by the recursion (cf. [2], chap. X)

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}} \tag{4.26}$$

with $p_{-1} = q_0 = 0$ and $p_0 = q_{-1} = 1$. Hence

$$\begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} = (-1)^n \tag{4.27}$$

holds for all $n \in \mathbb{N}$.

Proposition 5.. Let $\rho > 1$ be a rational number with continued fraction expansion $\rho = [a_0, \ldots, a_m]$ and partial quotients (4.25). With $w_n := W_{p_n/q_n}$, the tiling $W_\rho = w_{m+1}$ of H_ρ satisfies the reduction formula

$$w_{n+1} = \begin{cases} (w_n T^+)^{a_n} w_{n-1} & \text{for } n \text{ even} \\ (w_n T^-)^{a_n} w_{n-1} & \text{for } n \text{ odd.} \end{cases} \tag{4.28}$$

Proof. We set $v_n := \binom{p_n}{q_n}$. Assume first that n is even. Then Eq. (4.27) implies that $\det(v_n, iv_n + v_{n-1}) = \det(v_n, v_{n-1}) = 1$ for all $i \in \mathbb{N}$. Hence (4.28) follows by Eq. (4.26) and an iterated application of Eq. (4.21). For odd n, we have $\det(iv_n + v_{n-1}, v_n) = \det(v_{n-1}, v_n) = 1$ for all $i \in \mathbb{N}$. In this case, we get the same result, with T^- instead of T^+. □

Remark. If we introduce the improper tilings

$$w_{-1} := L^{-1}, \quad w_0 := S^{-1} \tag{4.29}$$

of length -1, the reduction formula (4.28) yields the correct values

$$w_1 = W_{a_0} = L^{a_0 - 1}, \quad w_2 = W_{a_0 a_1 + 1/a_1} = L^{a_0} S L^{a_0} S \cdots S L^{a_0}, \tag{4.30}$$

where in the second equation, L^{a_0} occurs a_1 times.

The composition formula (4.21) and its repetition (4.28) with the connecting tiles $T^{\pm}$ shed some light upon the quasicrystallographic case. Let $\rho > 1$ be an irrational number. The orthogonal projection of P_ρ, augmented by one of the points $\pm\frac{1}{2}(e_1 - e_2) \in E$, to the line L_ρ yields two quasicrystallographic tilings

$$\cdots T_2 T_1 T_0 T^{\pm} T_0 T_1 T_2 \cdots \tag{4.31}$$

of L_ρ with $T_i \in \{L, S\}$, where the connecting tiles T^+ or T^- arise at the point of ambiguity. The explicit form of the tiling (4.31) can be obtained from Proposition 5 with ρ irrational (cf. [4]). Note that each w_{n+1} starts with w_n, so that w_n converges to $T_0 T_1 T_2 \cdots$.

If we extend the concept of hook to infinite sequences of cells, there are infinite hooks of three types:

type ω:	H has a lower left cell
type ω^*:	H has an upper right cell
type $\omega^* + \omega$:	H is a two-way infinite hook.

Thus (4.31) represents a hook of type $\omega^* + \omega$ with a splitting into hooks of type ω^* and ω, respectively. Like in the finite case (4.21), the connecting piece is T^+ or T^-. For a finite hook H_ρ, however, the two composites $W_{\rho''} T^- W_{\rho'}$ and $W_{\rho'} T^+ W_{\rho''}$ in (4.21) are equal, while there are two different composites in the infinite case (4.31). On the other hand, the two pieces $\cdots T_2 T_1 T_0$ and $T_0 T_1 T_2 \cdots$ in (4.31) are congruent via reflection, while the pieces $W_{\rho'}$ and $W_{\rho''}$ in (4.21) are different. We have seen above that for a rational $\rho > 1$, the infinitely many, equally good approximations of L_ρ are made up from infinitely many rational hooks H_ρ. The corresponding tilings of L_ρ are thus of the form

$$\cdots W_\rho T^{\pm} W_\rho T^{\pm} W_\rho T^{\pm} W_\rho \cdots \tag{4.32}$$

where the finite pieces W_ρ of this composition are again congruent and separated by T^+ or T^-.

If ρ is quadratic over $\mathbb{Q}$, the continued fraction expansion (4.24) is periodic. Assume that ρ satisfies an equation

$$\rho^2 = m\rho + 1 \tag{4.33}$$

with an integer $m \geqslant 1$. Then $\rho = m + \frac{1}{\rho}$, and the sequence (a_n) in (4.28) is constant with $a_n = m$. In this case, the sequence (T_n) in (4.31) can be generated by the substitution rule

$$L \mapsto L^m S; \quad S \mapsto L. \tag{4.34}$$

Indeed, for a given tiling w, if we replace every L by $L^m S$ and every S by L, we get a new tiling w'. Then a simple induction shows that the tilings w_n defined in (4.28) and (4.29) satisfy

$$w_n' L^m = w_{n+1}. \tag{4.35}$$

Therefore, an iterated substitution (4.34) gives the infinite tiling $T_0 T_1 T_2 \cdots$ in (4.31).

Since the tilings $w_n = W_{p_n/q_n}$ are symmetric, the reverse tiling $\cdots T_2 T_1 T_0$ ends with w_n, for every $n \geqslant 1$. This implies that the two quasicrystals (4.31) are mirror-images with respect to the substitution rule (4.34). In fact, Eq. (4.35) yields

$$\begin{aligned}(\cdots w_n L S w_n \cdots)' &= \cdots w_n' L^m S L w_n' \cdots = \cdots w_{n+1} S L w_n' \cdots \\ (\cdots w_n S L w_n \cdots)' &= \cdots w_n' L L^m S w_n' \cdots = \cdots w_{n+1} L S w_n' \cdots\end{aligned}$$

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