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RESEARCH ARTICLE

Characterization of clones of boolean operations by identities

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Abstract. In [4] the authors characterized all clones of Boolean operations (Boolean clones) by functional terms. In this paper we consider a Galois connection between operations and equations and characterize all Boolean clones by using of identities. For each Boolean clone we obtain a set of equations with the property that an operation f belongs to this clone if and only if it satisfies these equations.

1. Preliminaries

Let A be the two-element set $A = \{0, 1\}$. An n-ary Boolean operation is a map $f^A : A^n \to A$. We denote by $O_A^{(n)}$ the set of all n-ary operations defined on A. Let $O_A := \bigcup_{n\geq 1} O_A^{(n)}$ be the set of all operations defined on A. On the set O_A we may define the following composition operations $S_m^{n,A} : O_A^{(n)} \times \left(O_A^{(m)}\right)^n \to O_A^{(m)}$ by setting $S_m^{n,A}(f^A, g_1^A, \ldots, g_n^A) := f^A(g_1^A, \ldots, g_n^A)$, where $f^A(g_1^A, \ldots, g_n^A) \in O_A^{(m)}$ is defined by

$$f^A(g_1^A, \ldots, g_n^A)(a_1, \ldots, a_m) := f^A(g_1^A(a_1, \ldots, a_m), \ldots, g_n^A(a_1, \ldots, a_m))$$

for all m-tuples $(a_1, \ldots, a_m) \in A^m$.

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Further, we consider projections $e_i^{n,A} : A^n \to A, 1 \le i \le n$, defined by $e_i^{n,A}(a_1, \ldots, a_n) := a_i$, as nullary operations.

A clone of Boolean operations, for short a Boolean clone, is a class of Boolean operations that contains all projections and is closed under all composition operations $S_m^{n,A}, m, n \ge 1, m, n \in \mathbb{N}$. All clones of Boolean operations form a lattice, where the lattice operation meet is the intersection. The second lattice operation applied to clones is defined to be the smallest clone that contains the union of both clones. Since any clone can be regarded as a multi-based algebra, all Boolean clones form a complete lattice which is the lattice of all subclones of the clone $O_{\{0,1\}}$, originally described in [5], (see also [6]).

Boolean clones can be characterized by relations in the form $Pol_A\rho$. Here $Pol_A\rho$ is the set of all operations f^A defined on A preserving the h-ary relation ρ in the sense that from

$$(a_{11}, \ldots, a_{1h}) \in \rho, \ldots, (a_{n1}, \ldots, a_{nh}) \in \rho$$

it follows $(f^A((a_{11}, \ldots, a_{n1}), \ldots, f^A(a_{1h}, \ldots, a_{nh})) \in \rho$. It is easy to see that all sets of operations which have the form $Pol_A\rho$ are clones. Conversely, each clone can be presented in this way by relations.

In this paper we characterize all Boolean clones by equations. This was also done in [4], but in our paper we will use the equational theory of Universal Algebra for a description of clones by equations.

If $f^A \in O_A^{(n)}$, then one can consider the algebra $\mathcal{A} = (A; \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus, \neg, \underline{\mathbf{0}}^A, \underline{\mathbf{1}}^A, f^A)$ of type $\tau = (2,2,2,2,2,1,0,0,n)$. To define the language over this algebra, we use the following notation,

K is the operation symbol corresponding to the conjunction $\wedge$,
D is the operation symbol corresponding to the disjuction $\vee$,
I is the operation symbol corresponding to the implication $\Rightarrow$,
E is the operation symbol corresponding to the equivalence $\Leftrightarrow$,
M is the operation symbol corresponding to the addition modulo 2,
N is the operation symbol corresponding to the negation $\neg$,
$\underline{\mathbf{0}}$ is the operation symbol corresponding to the constant $\mathbf{0}$,
$\underline{\mathbf{1}}$ is the operation symbol corresponding to the constant $\mathbf{1}$,
F is the operation symbol corresponding to the operation f^A.

If $f^A \in O_A^{(n)}$, then the dual operation $(f^A)^d$ can be defined by $(f^A)^d(a_1, \ldots, a_n) := \neg f^A(\neg a_1, \ldots, \neg a_n)$ for all $(a_1, \ldots, a_n) \in A^n$.

Let $f^A \in O_A^{(n)}$ and $i \in \{1, \ldots, n\}$. We say that the $i-th$ variable of f^A is essential (or f^A depends essentially on the $i-th$ variable) if there are n-tuples

$$\underline{a} = (a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n), \quad \underline{a'} = (a_1, \ldots, a_{i-1}, c, a_{i+1}, \ldots, a_n)$$

such that $b \neq c$ and $f^A(\underline{a}) \neq f^A(\underline{a}')$. Otherwise the $i-th$ variable is called fictitious (or non-essential).

We denote by $Alg(\tau)$ the class of all algebras of type τ. Terms of type τ over a set X of variables are defined as follows,

(i) $x_i \in X$ is a term of type τ and $\underline{\mathbf{0}}, \underline{\mathbf{1}}$ are terms of type τ,

(ii) if $t_1, \ldots, t_n$ are terms of type τ and if F is the $n-$ary operation symbol, then $t = F(t_1, \ldots, t_n)$ is a term of type τ, if t_1, t_2 are terms, then t_1Kt_2, t_1Dt_2, t_1It_2, t_1Et_2, t_1Mt_2, Nt_1 are terms of type τ.

We denote the set of all terms of type τ by $W_\tau(X)$. If $X_m = \{x_1, \ldots, x_m\}$ is a finite set of variables, then by (i) and (ii) the set $W_\tau(X_m)$ of $m-$ary terms is defined.

For every term $t \in W_\tau(X_m)$ and for every algebra $\mathcal{A} = (A; \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus, \neg, \underline{\mathbf{0}}^A, \underline{\mathbf{1}}^A, f^A)$ we define an operation $t^A \in O_A^{(m)}$, called term operation, inductively by the following steps,

(i) if $t = x_i \in X_m$, then $x_i^A = e_i^{m,A}$ (the $m-$ary projection on the i-th input, $1 \leq i \leq m$),

(ii) if t_1, t_2 are terms of type τ, then $(t_1Kt_2)^A = t_1^A \wedge t_2^A$, $(t_1Dt_2)^A = t_1^A \vee t_2^A$, $(t_1It_2)^A = t_1^A \Rightarrow t_2^A$, $(t_1Et_2)^A = t_1^A \Leftrightarrow t_2^A$, $(t_1Mt_2)^A = t_1^A \oplus t_2^A$, $(Nt_1)^A = \neg t_1^A$ and $\underline{\mathbf{0}}^A = 0$, $\underline{\mathbf{1}}^A = 1$,

(iii) if $t = F(t_1, \ldots, t_n)$ and $t_1^A, \ldots, t_n^A$ are the term operations which are induced by $t_1, \ldots, t_n$, then $t^A = f^A(t_1^A, \ldots, t_n^A)$. Here $f^A(t_1^A, \ldots, t_n^A)$ is the operation defined by

$$f^A(t_1^A, \ldots, t_n^A)(a_1, \ldots, a_n) = f^A(t_1^A(a_1, \ldots, a_n), \ldots, t^A(a_1, \ldots, a_n)).$$

Since later on we will replace only the symbol F by $n-$ary elements of a clone instead of the correct notations t_1Kt_2, t_1Dt_2, t_1It_2, t_1Et_2, t_1Mt_2, Nt_1 we will use $t_1 \wedge t_2$, $t_1 \vee t_2$, $t_1 \Rightarrow t_2$, $t_1 \Leftrightarrow t_2$, $t_1 \oplus t_2$, $\neg t_1$.

A pair $s \approx t$ of terms from $W_\tau(X)$ is called an identity in the algebra $\mathcal{A}$ if $s^A = t^A$, i.e. if the induced term operations are equal. In this case we write $\mathcal{A} \models s \approx t$.

Let $Id_B\mathcal{A}$ be the set of all identities satisfied in $\mathcal{A}$. For the class $K \subseteq Alg(\tau)$ we denote by Id_BK the set of all identities satisfied by each algebra $\mathcal{A}$ from K. If Σ is a set of equations $s \approx t$ consisting of terms from $W_\tau(X)$, then we denote the class of all algebras satisfying each equation from Σ as identity by $Mod_B\Sigma$.

Then we get a Galois connection (Id_B, Mod_B), i.e. the following properties are satisfied:
$\Sigma_1 \subseteq \Sigma_2 \Rightarrow Mod_B\Sigma_2 \subseteq Mod_B\Sigma_1$,
$C_1 \subseteq C_2 \Rightarrow Id_BC_2 \subseteq Id_BC_1$,
$\Sigma \subseteq Id_BMod_B\Sigma$,
$K \subseteq Mod_BId_BC$.

2. A Galois Connection between Operations and Equations

Let $f^A \in O_A^{(n)}$ be an n-ary operation and let

$$\mathcal{A} = (A; \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus, \neg, \underline{\mathbf{0}}^A, \underline{\mathbf{1}}^A, f^A)$$

be an algebra of type $\tau = (2, 2, 2, 2, 2, 1, 0, 0, n)$. Let $s, t \in W_\tau(X)$ be terms of type τ. Then $s \approx t$ is satisfied as identity in $\mathcal{A}$, and we write $\mathcal{A} \models s \approx t$ if $s^A = t^A$. Then we define

Definition 2.1. Let $s \approx t$ be an equation consisting of terms s, t of type τ, i.e. $s \approx t \in W_\tau(X)^2$. Then by

$$f^A \vdash s \approx t \Leftrightarrow \mathcal{A} = (A; \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus, \neg, \underline{\mathbf{0}}^A, \underline{\mathbf{1}}^A, f^A) \models s \approx t$$

we define a binary relation $\vdash$ between $O^{(n)}(A)$ and $W_\tau(X)$. If $f^A \vdash s \approx t$ holds, then we say that the operation f^A satisfies the equation $s \approx t$.

For $C \subseteq O^{(n)}(A)$ we define

$$C \vdash f^A \Leftrightarrow \forall f^A \in C \ (f^A \vdash s \approx t)$$

and for $\Sigma \subseteq W_\tau(X)^2$ we set

$$C \vdash \Sigma \Leftrightarrow \forall s \approx t \in \Sigma \ (C \vdash s \approx t).$$

Let $C \subseteq O_A^{(n)}$, $A = \{0, 1\}$, and let $\Sigma \subseteq W_\tau(X)^2$. Then we define two operations $F_BMod : \mathcal{P}(O_A^{(n)}) \to \mathcal{P}(W_\tau(X)^2)$ (where $\mathcal{P}$ denotes the formation of the power set) and $Id_B : \mathcal{P}(W_\tau(X)^2) \to \mathcal{P}(O_A^{(n)})$ by

$$F_BMod\Sigma = \left\{ f^A \mid f^A \in O_A^{(n)} \text{ and } \forall s \approx t \in \Sigma \ \ (f^A \vdash s \approx t) \right\},$$
$$Id_BC = \left\{ s \approx t \mid s, t \in W_\tau(X) \text{ and } \forall f^A \in C \ \ (f^A \vdash s \approx t) \right\}.$$

Then the pair (F_BMod, Id_B) is a Galois connection, i.e. we have
$\Sigma_1 \subseteq \Sigma_2 \Rightarrow F_BMod\Sigma_2 \subseteq F_BMod\Sigma_1$,
$C_1 \subseteq C_2 \Rightarrow Id_BC_2 \subseteq Id_BC_1$,

$\Sigma \subseteq Id_B F_B Mod \Sigma$,
$C \subseteq F_B Mod Id_B C$.

Further, we get two closure operators $Id_B F_B Mod$ and $F_B Mod Id_B$ on $\mathcal{P}\big(W_n(X)^2\big)$ and on $\mathcal{P}\big(O_A^{(n)}\big)$, respectively.

Our main question is whether each clone of Boolean operations has the form $F_B Mod \Sigma$ for a set Σ of equations. We are especially interested to find one-element sets Σ.

3. The Lattice of all Boolean Clones

The set of all clones of Boolean operations, originally described by E. Post ([5], [6]) forms a lattice. These clones and lattice are often called Post's classes and the lattice is denoted as Post's lattice. Post's lattice is countably infinite, complete, algebraic, atomic and dually atomic. It is also known that every clone in the lattice is finitely generated. Post's classes can be described as follows and the following Hasse diagram illustrates the lattice of all Boolean clones.
$C_1 := O_{\{0,1\}}$.
$C_3 := Pol\{0\}$, and dually $C_2 := Pol\{1\}$.
$C_4 := C_2 \cap C_3$.

$A_1 := Pol \leq$, where $\leq := \{(00), (01), (11)\}$, (monotone Boolean functions).
$A_3 := A_1 \cap C_3$, and dually $A_2 := A_1 \cap C_2$.
$A_4 := A_1 \cap C_4$.

$D_3 := PolN$, where $N := \{(01), (10)\}$, (self-dual Boolean functions).
$D_1 := D_3 \cap C_4$.
$D_2 := D_3 \cap A_1$.

$L_1 := Pol_{\rho_G}$, where $\rho_G := \Big\{(x, y, z, u) \in \{0,1\}^4 \mid x + y = z + u\Big\}$, (linear Boolean functions).
$L_3 := L_1 \cap C_3$, dually $L_2 := L_1 \cap C_2$.
$L_4 := L_1 \cap C_4$.
$L_5 := L_1 \cap D_3$.

$F_8^\mu := PolD_\mu$, where $D_\mu := \{0,1\}^\mu \setminus \{(1, \ldots, 1\}$, for $\mu \geq 2$, and dually $F_4^\mu := PolD'_\mu$ with $D'_\mu := \{0,1\}^\mu \setminus \{(0, \ldots, 0\}$.
$F_7^\mu := F_8^\mu \cap A_1$, and dually $F_3^\mu := F_4^\mu \cap A_1$.
$F_6^\mu := F_8^\mu \cap A_4$, and dually $F_2^\mu := F_4^\mu \cap A_4$.
$F_5^\mu := F_8^\mu \cap C_4$, and dually $F_1^\mu := F_4^\mu \cap C_4$.

$F_8^\infty : \bigcap_{\mu=2}^{\infty} PolD_\mu$, and dually $F_4^\infty : \bigcap_{\mu=2}^{\infty} PolD'_\mu$.
$F_7^\infty := F_8^\infty \cap A_1$, and dually F_3^∞.
$F_6^\infty := F_8^\infty \cap A_4$, and dually F_2^∞.
$F_5^\infty := F_8^\infty \cap C_4$, and dually F_1^∞.

$P_1 := \langle\{et\}\rangle$, and dually $S_1 := \langle\{vel\}\rangle$
$P_3 := \langle\{et, \underline{\mathbf{0}}\}\rangle$, and dually $S_1 := \langle\{vel, \underline{\mathbf{1}}\}\rangle$
$P_5 := \langle\{et, \underline{\mathbf{1}}\}\rangle$, and dually $S_5 := \langle\{vel, \underline{\mathbf{0}}\}\rangle$
$P_6 := \langle\{et, \underline{\mathbf{0}}, \underline{\mathbf{1}}\}\rangle$, and dually $S_6 := \langle\{vel, \underline{\mathbf{0}}, \underline{\mathbf{1}}\}\rangle$.

$O_9 := \left\langle\left\{e_1^1, \underline{\mathbf{0}}, \underline{\mathbf{1}}, non\right\}\right\rangle = \langle\{non, \underline{\mathbf{0}}\}\rangle$.
$O_8 := \left\langle\left\{e_1^1, \underline{\mathbf{0}}, \underline{\mathbf{1}}\right\}\right\rangle$.
$O_6 := \left\langle\left\{e_1^1, \underline{\mathbf{0}}\right\}\right\rangle$, and dually $O_5 := \left\langle\left\{e_1^1, \underline{\mathbf{1}}\right\}\right\rangle$.
$O_4 := \left\langle\left\{e_1^1, non\right\}\right\rangle = \langle\{non\}\rangle$.
$O_1 := \left\langle\left\{e_1^1\right\}\right\rangle$.

Note that $\underline{\mathbf{0}}, \underline{\mathbf{1}}$ are the unary constant functions with values 0 and 1, respectively, e_1^1 is the identity, and *vel*, *et* and *non* are $\vee$, $\wedge$ and $\neg$, respectively.

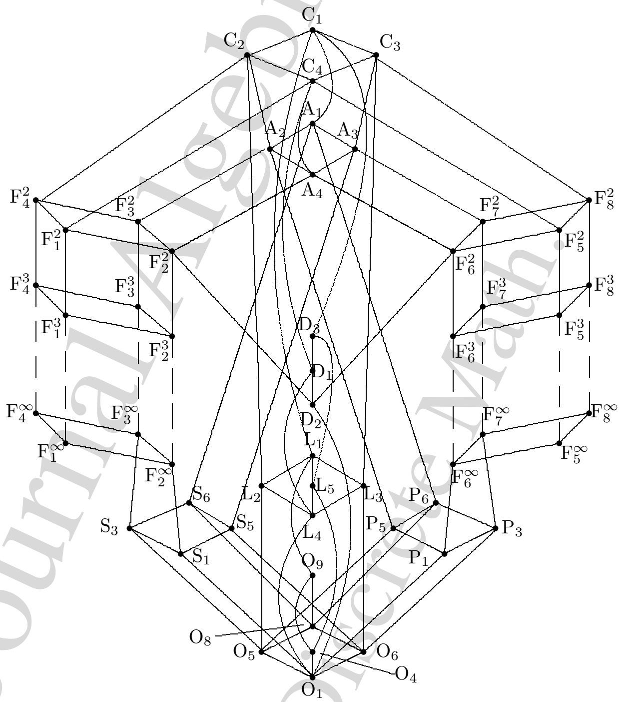

Figure 1: Post's Lattice of all Boolean Clones

4. Identities for Clones of Boolean Operations

There are several methods to characterize clones of Boolean operations. In [4] the authors characterized all clones of Boolean operations by so-called functional terms. Using the langauge of Universal Algebra for each clone of Boolean operations we obtain a set of characterizing identities satisfied in the algebra $\mathcal{A} = (\{0,1\}\,; \wedge, \vee, \Rightarrow, \Leftrightarrow, \oplus, \neg, \mathbf{\underline{0}}^A, \mathbf{\underline{1}}^A, f^A)$. The results are given in the following table. Our notation goes partly back to E. L. Post [6].

Clones	Identities
C_1	$1 \approx 1$
C_3	$\neg F(\underline{0}) \approx 1$
C_2	$F(\underline{1}) \approx 1$
C_4	$\neg F(\underline{0}) \wedge F(\underline{1}) \approx 1$
A_1	$F(\underline{x}) \rightarrow F(\underline{x} \vee \underline{y}) \approx 1$
A_3	$\neg F(\underline{0}) \wedge (\neg F(\underline{x} \vee F(\underline{x} \vee \underline{y})) \approx 1$
A_2	$F(\underline{1}) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1$
A_4	$\neg F(\underline{0}) \wedge F(\underline{1}) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1$
D_3	$F(\underline{x}) \oplus F(\overline{x}) \approx 1$
D_1	$\neg F(\underline{0}) \wedge (F(\underline{x}) \oplus F(\overline{x})) \approx 1$
D_2	$(F(\underline{x}) \oplus F(\bar{x})) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1$
L_1	$1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1$
L_3	$1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1$
L_2	$F(\underline{1}) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1$
L_4	$F(\underline{1}) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1$
L_5	$(F(\underline{1}) \oplus F(\underline{0}) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}))$ ≈ 1
P_6	$F(\underline{x}) \wedge F(\underline{y}) \approx F(\underline{x} \wedge \underline{y})$
P_3	$\neg F(\underline{0}) \wedge (F(\underline{x}) \wedge F(\underline{y}) \Leftrightarrow F(\underline{x} \wedge \underline{y})) \approx 1$
P_5	$F(\underline{1}) \wedge (F(\underline{x}) \wedge F(\underline{y}) \Leftrightarrow F(\underline{x} \wedge \underline{y})) \approx 1$
P_1	$\neg F(\underline{0}) \wedge F(\underline{1}) \wedge (F(\underline{x}) \wedge F(\underline{y}) \Leftrightarrow F(\underline{x} \wedge \underline{y})) \approx 1$
S_6	$(F(\underline{x}) \vee F(\underline{y})) \approx F(\underline{x} \vee \underline{y})$
S_5	$\neg F(\underline{0}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y})) \approx 1$
S_3	$F(\underline{1}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y})) \approx 1$
S_1	$\neg F(\underline{0}) \wedge F(\underline{1}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y})) \approx 1$
F_8^μ	$\bigwedge\limits_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y}) \approx 1$
F_4^μ	$\bigwedge\limits_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{y}) \approx 1$
F_8^∞	$\bigwedge\limits_{\mu=2}^{\infty} (\bigwedge\limits_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1$

Clones	Identities
F_4^∞	$\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}}\vee\underline{y})) \approx 1$
F_5^μ	$F(\underline{1})\wedge(\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}}}\wedge\underline{y})) \approx 1$
F_1^μ	$\neg F(\underline{0})\wedge(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}}\vee\underline{y})) \approx 1$
F_5^∞	$F(\underline{1})\wedge(\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}}}\wedge\underline{y}))) \approx 1$
F_1^∞	$\neg F(\underline{0})\wedge(\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}}\vee\underline{y}))) \approx 1$
F_7^2	$F(\underline{x}) \Rightarrow \neg F(\bar{x})\wedge F(\underline{x}\vee\underline{y}) \approx 1$
F_7^μ	$\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}})})\wedge F(\underline{x_1}\vee\underline{x_2}) \approx 1$
F_3^2	$\neg F(\underline{x}) \Rightarrow F(\bar{x})\wedge\neg F(\underline{x}\wedge\underline{y}) \approx 1$
F_3^μ	$\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}})\wedge\neg F(\underline{x_1}\wedge\underline{x_2})$ ≈ 1
F_7^∞	$\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}})})\wedge F(\underline{x_1}\vee\underline{x_2}))$ ≈ 1
F_3^∞	$\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}})\wedge\neg F(\underline{x_1}\wedge\underline{x_2}))$ ≈ 1
F_6^2	$F(\underline{1})\wedge(F(\underline{x}) \Rightarrow \neg F(\overline{x})\wedge F(\underline{x}\vee\underline{y})) \approx 1$
F_6^μ	$F(\underline{1})\wedge(\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}}})\wedge$ $F(\underline{x_1}\vee\underline{x_2})))) \approx 1$
F_2^2	$\neg F(\underline{0})\wedge\neg F(\underline{x}) \Rightarrow F(\overline{x})\wedge\neg F(\underline{x}\wedge\underline{y})) \approx 1$
F_2^μ	$\neg F(\underline{0})\wedge(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}})\wedge$ $\neg F(\underline{x_1}\wedge\underline{x_2})) \approx 1$
F_6^∞	$F(\underline{1})\wedge(\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1}\wedge\ldots\wedge\underline{x_{\mu-1}}})$ $\wedge F(\underline{x_1}\vee\underline{x_2}))) \approx 1$
F_2^∞	$\neg F(\underline{0})\wedge(\bigwedge_{\mu=2}^{\infty}(\bigwedge_{i=1}^{\mu-1}\neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1}\vee\ldots\vee\underline{x_{\mu-1}}})\wedge$ $\neg F(\underline{x_1}\wedge\underline{x_2}))) \approx 1$
O_9	$((F(\underline{x}\Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x}\Leftrightarrow F(\underline{x}\wedge\underline{y})))\wedge$ $(1\oplus f(\underline{0})\oplus F(\underline{x})\oplus F(\underline{y})\oplus F(\underline{x}\oplus\underline{y})) \approx 1$

Clones	Identities
O_4	$(F(\underline{0}) \oplus F(\underline{1})) \wedge ((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x})$ $\Leftrightarrow F(\underline{x} \wedge \underline{y})))\wedge$ $(1 \oplus f(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1$
O_8	$(F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus f(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y})$ $\oplus F(\underline{x} \oplus \underline{y})) \approx 1$
O_6	$(F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}))$ ≈ 1
O_5	$F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus f(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y})$ $\oplus F(\underline{x} \oplus \underline{y})) \approx 1$
O_1	$F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y})$ $\oplus F(\underline{x} \oplus \underline{y})) \approx 1$

The main result is that every clone of Boolean operations can be characterized by a set of identities. We will give a complete proof of this result. We denote the n-tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$ by $\underline{0}$ and $\underline{1}$, respectively. The proof can be shortened using the following observations.

Lemma 4.1. Let $\Sigma = \{s \approx t\}$, then $F_B Mod\Sigma \cap C_2 = F_B Mod\{F(\underline{1}) \wedge s \approx t\}$.

Proof. Let $f^A \in F_B Mod\{s \approx t\} \cap C_2$, then $f^A \vdash s \approx t$ and $f^A(1, \ldots, 1) = 1$. Now we get $s^A = t^A$ and then $F(\underline{1})^A \wedge s^A = t^A$. Therefore $[F(\underline{1}) \wedge s]^A = t^A$. Thus $f^A \in F_B Mod\{F(\underline{1}) \wedge s \approx t\}$. Let $f^A \in F_B Mod\{F(\underline{1}) \wedge s \approx t\}$, then $[F(\underline{1}) \wedge s]^A = t^A$. Further $F(\underline{1})^A \wedge s^A = t^A$. Therefore $f^A(\underline{1}) = 1$ and $s^A = t^A$. Hence $f^A \in C_2$ and $f^A \in F_B Mod\{s \approx t\}$. □

The following lemma can be proved in a similar way.

Lemma 4.2. Let $\Sigma = \{s \approx t\}$, then $F_B Mod\Sigma \cap C_3 = F_B Mod\{\neg F(\underline{0}) \wedge s \approx t\}$.

Lemma 4.1 and Lemma 4.2 are needed only for the special case when $\Sigma = \{s \approx 1\}$. Then both follow from the the next lemma.

Lemma 4.3. Let $\Sigma_1 = \{s \approx 1\}$ and $\Sigma_2 = \{t \approx 1\}$, then $F_B Mod\Sigma_1 \cap F_B Mod\Sigma_2 = F_B Mod\{s \wedge t \approx 1\}$.

Proof. Let $f^A \in F_B Mod\Sigma_1 \cap F_B Mod\Sigma_2$, then $f^A \vdash s \approx 1$ and $f^A \vdash t \approx 1$. Further we get $s^A \equiv 1$ and $t^A \equiv 1$. Therefore from $s^A \wedge t^A = [s \wedge t]^A \equiv 1$ implies $f^A \vdash s \wedge t \approx 1$. Hence $f^A \in F_B Mod\{s \wedge t \approx 1\}$.

Let $f^A \in F_B Mod\{s \wedge t \approx 1\}$, then $f^A \vdash s \wedge t \approx 1$. Therefore $[s \wedge t]^A \equiv 1$. Then from $s^A \wedge t^A \equiv 1$ we get $s^A \equiv 1$ and $t^A \equiv 1$. Now we have $f^A \vdash s \approx 1$ and $f^A \vdash t \approx 1$. Thus $f^A \in F_B Mod\Sigma_1 \cap F_B Mod\Sigma_2$. □

Let F^d be a new n-ary operation symbol. Let

$$L = \{K, D, E, M, N, \underline{0}, \underline{1}, F, F^d\}.$$

Instead of these symbols we use $L = \{\wedge, \vee, \Leftrightarrow, \oplus, \neg, \underline{0}, \underline{1}, F, F^d\}$. Let $\tau^{'}$ be the type which uses only operation symbols from L.

Definition 4.4. Let $W_{\tau'}(X_m)$ be the set of all m-ary terms of type $\tau^{'}$. Now for each $t \in W_{\tau'}(X_m)$ we define the dual term t^d inductively by the following steps,

(i) if $t = x_i \in X_m$, then $x_i^d = x_i, 1 \leq i \leq m$,

(ii) if $t_1, \ldots, t_n \in W_{\tau'}(X_m)$ and $t_1^d, \ldots, t_n^d$ are dual terms of $t_1, \ldots, t_n$, respectively, then $(t_1 \wedge t_2)^d = t_1^d \vee t_2^d$, $(t_1 \vee t_2)^d = t_1^d \wedge t_2^d$, $(t_1 \Leftrightarrow t_2)^d = t_1^d \oplus t_2^d$, $(\neg t_1)^d = t_1^d$ and $F(t_1, \ldots, t_n) = F^d(t_1^d, \ldots, t_n^d)$.

This gives the set $W_{\tau'}(X_m)^d \subseteq W_{\tau'}(X_m)$. For a subset $M \subseteq W_{\tau'}(X_m)$ we define $M^d := \{t^d \mid t \in M\}$ and for the algebra $\mathcal{A} = (A; M^A, f^A)$ we define $\mathcal{A}^d = (A; (M^d)^A, (f^d)^A)$.

Lemma 4.5. For each $t \in W_{\tau'}(X_m)$ we get $(t^d)^{\mathcal{A}} = (t^{\mathcal{A}})^d$.

Proof. If $t = x_i \in X_m$, then $(t^{\mathcal{A}})^d = (x_i^{\mathcal{A}})^d = (e_i^{m,A})^d = e_i^{m,A} = x_i^{\mathcal{A}} = (t^d)^{\mathcal{A}}$. If $t_1^d, \ldots, t_n^d$ are dual terms of $t_1, \ldots, t_n$, respectivly and $(t_1^d)^{\mathcal{A}} = (t_1^{\mathcal{A}})^d, \ldots, (t_n^d)^{\mathcal{A}} = (t_n^{\mathcal{A}})^d$, then $((t_1 \wedge t_2)^d)^{\mathcal{A}} = (t_1^d \vee t_2^d)^{\mathcal{A}} = (t_1^d)^{\mathcal{A}} \vee (t_2^d)^{\mathcal{A}} = (t_1^{\mathcal{A}})^d \vee (t_2^{\mathcal{A}})^d = (t_1^{\mathcal{A}} \wedge t_2^{\mathcal{A}})^d = ((t_1 \wedge t_2)^{\mathcal{A}})^d$. For $t_1 \vee t_2, t_1 \Leftrightarrow t_2, t_1 \oplus t_2, \neg t_1$ the corresponding equations can be proved similarly. If $t = F(t_1, \ldots, t_n)$, we get $((F(t_1, \ldots, t_n))^d)^{\mathcal{A}} = (F^d(t_1^d, \ldots, t_n^d))^{\mathcal{A}} = (F^d)^{\mathcal{A}}((t_1^d)^{\mathcal{A}}, \ldots, (t_n^d))^{\mathcal{A}} = (F^{\mathcal{A}})^d((t_1^{\mathcal{A}})^d, \ldots, (t_n^{\mathcal{A}})^d) = (F^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}))^d = (F(t_1, \ldots, t_n)^{\mathcal{A}})^d$. □

Lemma 4.6. Let $L^A = \{\wedge, \vee, \Leftrightarrow, \oplus, \neg, \mathbf{0}, \mathbf{1}, f^A, (f^A)^d\}$ and let s, t be terms using only operation symbols from L^A. Let $L^{'A} \subseteq L^A$ be a subset, then

$$\mathcal{A} = (\{0,1\}; L^{'A}, f^A) \models s \approx t \Longleftrightarrow \mathcal{A}^d = (\{0,1\}; (L^{'d})^{\mathcal{A}}, (f^A)^d) \models s^d \approx t^d.$$

Proof. Let $\varphi : \{0,1\} \to \{0,1\}$ be given by $\varphi(0) = 1$ and $\varphi(1) = 0$, i.e. $\varphi = \neg$ is the negation. Let $f_i^A \in L^{'}$. Since

$$\begin{aligned}\varphi((f_i^A)^d(x_1, \ldots, x_n)) = \neg(f_i^A)^d(x_1, \ldots, x_n) = \\ \neg(\neg f_i^A(\neg x_1, \ldots, \neg x_n)) = f_i^A(\neg x_1, \ldots, \neg x_n) = \\ f_i^A(\varphi(x_1), \ldots, \varphi(x_n))\end{aligned}$$

and

$$\varphi((f^A)^d(x_1,\ldots,x_n)) = \neg(f^A)^d(a_1,\ldots,a_n) = \\ \neg(\neg f^A(\neg x_1,\ldots,\neg x_n)) = f^A(\neg x_1,\ldots,\neg x_n) = \\ f^A(\varphi(x_1),\ldots,\varphi(x_n)),$$

φ has the properties of an isomorphism exchanging each operation by the dual one. Since $\mathcal{A} \vdash s \approx t$, then $\mathcal{A}^d \vdash s^d \approx t^d$ and vice versa. □

Corollary 4.7. $(F_B Mod\{s \approx t\})^d = F_B Mod\{s^d \approx t^d\}$.

Proof. Let $(f^A)^d \in (F_B Mod\{s \approx t\})^d$, then $f^A \in F_B Mod\{s \approx t\}$. From Lemma 4.4 we get $(f^A)^d \vdash s^d \approx t^d$. Therefore $(f^A)^d \in F_B Mod\{s^d \approx t^d\}$, i.e. $(F_B Mod\{s \approx t\})^d \subseteq F_B Mod\{s^d \approx t^d\}$.

Let $f^A \in F_B Mod\{s^d \approx t^d\}$, then $\mathcal{A} = (\{0,1\}; K^{\mathcal{A}}, f^A) \models s^d \approx t^d$. From Lemma 4.4 we get $\mathcal{A}^d = (\{0,1\}; K'^{\mathcal{A}}, (f^A)^d) \models (s^d)^d \approx (t^d)^d = s \approx t$. Further $(f^A)^d \in F_B Mod\{s \approx t\}$. Therefore $((f^A)^d)^d \in (F_B Mod\{s \approx t\})^d$. Hence $f^A \in (F_B Mod\{s \approx t\})^d$, i.e. $F_B Mod\{s^d \approx t^d\} \subseteq (F_B Mod\{s \approx t\})^d$. □

Proposition 4.8.

$$\begin{aligned} C_1 &= F_B Mod\{1 \approx 1\}, \\ C_3 &= F_B Mod\{\neg F(\underline{0}) \approx 1\}, \\ C_2 &= F_B Mod\{F(\underline{1}) \approx 1\}, \\ C_4 &= F_B Mod\{\neg F(\underline{0}) \wedge F(\underline{1}) \approx 1\}. \end{aligned}$$

Proof. Obviously, for C_1, the equation $1 \approx 1$ is satisfied by all Boolean operations f^A since F does not occur in our equation. Since $C_3 = C_1 \cap C_3$, $C_2 = C_1 \cap C_2$ and $C_4 = C_3 \cap C_2$, then one can apply Lemma 4.1, Lemma 4.2 and Lemma 4.3, respectively. □

We set $\underline{x} = (x_1,\ldots,x_n)$. If $x_i \leq y_i$ for all $1 \leq i \leq n$, where $\leq$ denotes the usual order on the set $\{0,1\}$, then we write $\underline{x} \preceq \underline{y}$.

Proposition 4.9.

$$\begin{aligned} A_1 &= F_B Mod\{F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y}) \approx 1\}, \\ A_3 &= F_B Mod\{\neg F(\underline{0}) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1\}, \\ A_2 &= F_B Mod\{F(\underline{1}) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1\}, \\ A_4 &= F_B Mod\{\neg F(\underline{0}) \wedge F(\underline{1}) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1\}. \end{aligned}$$

Proof. Let $f^A \in A_1$ and let $\Sigma = \{F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y}) \approx 1\}$, then $f^A(\underline{x}) \leq f^A(\underline{y})$ for every $\underline{x} \preceq \underline{y}$. Assume $f^A(\underline{x}) = 0$, then $(f^A(\underline{x}) \Rightarrow f^A(\underline{x}) \vee \underline{y})) = (0 \Rightarrow f^A(\underline{x}) \vee \underline{y})) = 1$. Assume $f^A(\underline{x}) = 1$, then $f^A(\underline{x} \vee \underline{y}) = 1$ since f^A is monotone and $\underline{x} \preceq \underline{x} \vee \underline{y}$. Moreover $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) = 1$. Let $\underline{x} \preceq \underline{y}$ and assume that $f^A(\underline{x}) > f^A(\underline{y})$, then $f^A(\underline{x}) > f^A(\underline{x} \vee \underline{y})$ since $f^A(\underline{x} \vee \underline{y}) = f^A(\underline{y})$. Therefore $f^A(\underline{x}) = 1$ and $f^A(\underline{x} \vee \underline{y}) = 0$. Hence $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) = 0 \neq 1$, a contradiction. Thus $f^A(\underline{x}) \leq f^A(\underline{y})$. Therefore $f^A \in A_1$.

Since $A_3 = A_1 \cap C_3, A_2 = A_1 \cap C_2$ and $A_4 = A_2 \cap C_3$, then we can apply Lemma 4.1, Lemma 4.2 and Lemma 4.3, respectively. □

For $\underline{x} = (x_1, \ldots, x_n)$ we write $\overline{x} = (\neg x_1, \ldots, \neg x_n)$.

Proposition 4.10.

$$
\begin{aligned}
D_3 = &\quad F_B Mod\{F(\underline{x}) \oplus F(\overline{x}) \approx 1\},\\
D_1 = &\quad F_B Mod\{\neg F(\underline{0}) \wedge (F(\underline{x}) \oplus F(\overline{x})) \approx 1\},\\
D_2 = &\quad F_B Mod\{(F(\underline{x}) \oplus F(\overline{x})) \wedge (\neg F(\underline{x}) \vee F(\underline{x} \vee \underline{y})) \approx 1\}.
\end{aligned}
$$

Proof. Let $f^A \in D_3$ and let $\Sigma = \{F(\underline{x}) \oplus F(\overline{x}) \approx 1\}$, then $f^A(\underline{x}) = f^A(x_1, \ldots, x_n) = \neg f^A(\neg x_1, \ldots, \neg x_n) = \neg f^A(\overline{x})$. Assume $f^A(\underline{x}) = 0$, then $\neg f^A(\overline{x}) = 0$, i.e. $f^A(\overline{x}) = 1$. Therefore $f^A(\underline{x}) \oplus f^A(\overline{x}) = 1$. Assume $f^A(\underline{x}) = 1$, then $\neg f^A(\overline{x}) = 1$, i.e. $f^A(\overline{x}) = 0$. Then $f^A(\underline{x}) \oplus f^A(\overline{x}) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $f^A(\underline{x}) \oplus f^A(\overline{x}) = 1$ for every $\underline{x} \in \{0,1\}^n$. Assume $f^A(\underline{x}) = 0$, then $1 = f^A(\underline{x}) \oplus f^A(\overline{x}) = 0 \oplus f^A(\overline{x})$. Therefore $f^A(\overline{x}) = 1$, i.e. $\neg f^A(\overline{x}) = 0$. Hence $f^A(\underline{x}) = \neg f^A(\overline{x})$. Assume $f^A(\underline{x}) = 1$, then $1 = f^A(\underline{x}) \oplus f^A(\overline{x}) = 1 \oplus f^A(\overline{x})$. Now we get $f^A(\overline{x}) = 0$, i..e. $\neg f^A(\overline{x}) = 1$. Hence $f^A(\underline{x}) = \neg f^A(\overline{x})$. Therefore $f^A \in D_3$.

Since $D_1 = D_3 \cap C_4 = D_3 \cap C_3$ and $D_2 = D_3 \cap A_1$, then the proof can be given using Lemma 4.1 and Lemma 4.3, respectively. □

Instead of $x \wedge y$ we will also write xy.

Proposition 4.11.

$$
\begin{aligned}
L_1 = &\ F_B Mod\{1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1\},\\
L_3 = &\ F_B Mod\{1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1\},\\
L_2 = &\ F_B Mod\{F(\underline{1}) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1\},\\
L_4 = &\ F_B Mod\{F(\underline{1}) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1\},\\
L_5 = &\ F_B Mod\{(F(\underline{1}) \oplus F(\underline{0})) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus\\
&\ F(\underline{x} \oplus \underline{y})) \approx 1\}.
\end{aligned}
$$

Proof. Let $f^A \in L_1$ and let $\Sigma = \{1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y}) \approx 1\}$, then there are $a_0, a_1, \ldots, a_n \in \{0,1\}$ such that $f^A(\underline{x}) = f^A(x_1, \ldots, x_n) = a_0 \oplus a_1x_1 \oplus, \ldots \oplus a_nx_n$. If $a_i = 0$, then $a_ix_i \oplus a_iy_i = 0 \oplus 0 = 0 = 0 \wedge (x_i \oplus y_i) = a_i \wedge (x_i \oplus y_i)$. If $a_i = 1$, then $a_ix_i \oplus a_iy_i = (1x_i) \oplus (1y_i) = x_i \oplus y_i = 1 \wedge (x_i \oplus y_i) = a_i \wedge (x_i \oplus y_i)$. Therefore $(a_ix_i) \oplus (a_iy_i) = a_i(x_i \oplus y_i)$. Then $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1 \oplus (a_0 \oplus a_1x_1 \oplus \ldots \oplus a_nx_n) \oplus (a_0 \oplus a_1y_1 \oplus \ldots \oplus a_ny_n) \oplus (a_0 \oplus a_1(x_1 \oplus y_n) \oplus \ldots \oplus a_n(x_n \oplus y_n)) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Assume that f^A is essentially depending on n variables. If $n = 0$, then f^A is constant. Now we have that f^A is linear. Suppose $n \geq 1$. Let $\alpha = (a_1, \ldots, a_n), \beta = (\neg a_1, a_2, \ldots, a_n)$ and $e_1 = (1, 0, \ldots, 0)$. Since $f^A \vdash \Sigma$, then $1 = 1 \oplus f^A(\underline{0}) \oplus f^A(\alpha) \oplus f^A(\beta) \oplus f^A(\alpha \oplus \beta) = 1 \oplus f^A(\underline{0}) \oplus f^A(\alpha) \oplus f^A(\beta) \oplus f^A(e_1)$. Therefore $f^A(\alpha) = f^A(\beta)$ iff $f^A(\underline{0}) = f^A(e_1)$. Since a_1 is an essential variable, then there exist $x_2, \ldots, x_n \in \{0,1\}^n$ such that $f^A(a_1, x_2, \ldots, x_n) \neq f^A(\neg a_1, x_2, \ldots, x_n)$. Therefore $f^A(\underline{0}) \neq f^A(e_1)$. Hence $f^A(\alpha) \neq f^A(\beta)$ for any $a_2, \ldots, a_n \in \{0,1\}^n$. Thus $f^A(a_1, \ldots, a_n) = a_1 \oplus f^A(0, a_2, \ldots, a_n)$. Applying the same argument to the remaining variable we get $f^A(a_1, \ldots, a_n) = a_1 \oplus a_2 \oplus \ldots \oplus a_n \oplus f^A(0, \ldots, 0)$. Therefore $f^A \in L_1$.

For the proof of the equations for $L_3 = L_1 \cap C_3$, $L_2 = L_1 \cap C_2$, $L_4 = L_1 \cap C_4$, and $L_5 = L_1 \cap D_3$, we can apply Lemma 4.1, Lemma 4.2 and Lemma 4.3. □

Proposition 4.12.

$$
\begin{aligned}
P_6 = &\ F_B Mod\{F(\underline{x}) \wedge F(\underline{y}) \approx F(\underline{x} \wedge \underline{y})\},\\
P_3 = &\ F_B Mod\{\neg F(\underline{0}) \wedge (F(\underline{x}) \wedge F(\underline{y}) \Leftrightarrow F(\underline{x}) \wedge F(\underline{y})) \approx 1\},\\
P_5 = &\ F_B Mod\{F(\underline{1}) \wedge (F(\underline{x}) \wedge F(\underline{y}) \Leftrightarrow F(\underline{x}) \wedge F(\underline{y})) \approx 1\},\\
P_1 = &\ F_B Mod\{\neg F(\underline{0}) \wedge F(\underline{1}) \wedge (F(\underline{x}) \wedge F(\underline{y})) \Leftrightarrow F(\underline{x}) \wedge F(\underline{y})) \approx 1\}.
\end{aligned}
$$

Proof. Let $f^A \in P_6$ and let $\Sigma = \{F(\underline{x}) \wedge F(\underline{y}) \approx F(\underline{x} \wedge \underline{y})\}$. If f^A is constant, then $f^A \vdash \Sigma$. If $f^A \in P_6/\,\{\mathbf{0,1}\}$, then $f^A(\underline{x}) = f^A(x_1, \ldots, x_n) =$

$x_1 \wedge \ldots \wedge x_n$. Assume $f^A(\underline{x} \wedge \underline{y}) = 0$, then $(x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n) = 0$. Then there exists $x_i \wedge y_i = 0$ for some $i \in \{1, \ldots, n\}$. Therefore from $x_i = 0 \vee y_i = 0$ it follows $f^A(\underline{x}) \wedge f^A(\underline{y}) = (x_1 \wedge \ldots \wedge x_n) \wedge (y_1 \wedge \ldots \wedge y_n) = 0$. Thus $f^A(\underline{x}) \wedge f^A(\underline{y}) = f^A(\underline{x} \wedge \underline{y})$. Assume $f^A(\underline{x} \wedge \underline{y}) = 1$, now we get $(x_1 \wedge y_1) \wedge \ldots \wedge (x_n \wedge y_n) = 1$. Then $x_i = y_i = 1$ for all $i \in \{1, \ldots, n\}$. Therefore $f^A(\underline{x}) = x_1 \wedge \ldots \wedge x_n = 1$ and $f^A(\underline{y}) = y_1 \wedge \ldots \wedge y_n = 1$. Thus $(f^A(\underline{x}) \wedge f^A(\underline{y})) = f^A(\underline{x} \wedge \underline{y})$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $f^A(\underline{x}) \wedge f^A(\underline{y}) = f^A(\underline{x} \wedge \underline{y})$. Assume f^A is not constant. Let $\underline{x} \preceq \underline{y}$, then $x_i \wedge y_i = x_i$ for all $i \in \{1, \ldots, n\}$. Now we have $f^A(\underline{x} \wedge \underline{y}) = f^A(\underline{x})$. Therefore $f^A(\underline{x}) \wedge f^A(\underline{y}) = f^A(\underline{x})$. If $f^A(\underline{x}) = 1$, then $f^A(\underline{y}) = 1$. Therefore $f^A(\underline{x}) \leq f^A(\underline{y})$. Hence f^A is monotone.

Let $B = \{\underline{x} \in \{0,1\}^n \mid f^A(\underline{x}) = 1 \wedge \forall\, \underline{y}\ (\underline{y} \prec \underline{x} \Rightarrow f^A(\underline{y}) = 0)\ \}$. We will show that $|B| = 1$. Assume that $|B| > 1$. We have $\underline{a}, \underline{b} \in B$, $\underline{a} \neq \underline{b}$, and then $f^A(\underline{a}) = 1$ and $f^A(\underline{b}) = 1$. Since $\underline{a} \neq \underline{b}$, $\underline{a} \not\prec \underline{b}$ and $\underline{b} \not\prec \underline{a}$, then there exist $i, j \in \{1, \ldots, n\}$ and $i \neq j$ such that $a_i > b_i$ and $b_j > a_j$. Then $a_i \wedge b_i = b_i < a_i$ and $a_j \wedge b_j = a_j$. Consider $k \in \{1, \ldots, n\}$ such that $k \neq i, j$. If $a_k = b_k$, then $a_k \wedge b_k = a_k$. If $a_k \neq b_k$, then $a_k \wedge b_k = 0 \leq a_k$. Therefore $\underline{a} \wedge \underline{b} \prec \underline{a}$. Further $f^A(\underline{a} \wedge \underline{b}) = 0$. Therefore $f^A(\underline{a}) \wedge f^A(\underline{b}) \neq f^A(\underline{a} \wedge \underline{b})$, a contradiction. Therefore $|B| \leq 1$.

Assume $|B| = \phi$, then there is no $\underline{x} \in \{0,1\}^n$ such that $f^A(\underline{x}) = 1$ or if there is an $\underline{x}$ such that $f^A(\underline{x}) = 1$, then there is a $\underline{y}$ with $\underline{y} \prec \underline{x}$ and $f^A(\underline{y}) = 1$.

If there is no $\underline{x} \in \{0,1\}^n$ such that $f^A(\underline{x}) = 1$, then $f^A(\underline{x}) = 0$ for all $\underline{x} \in \{0,1\}^n$. Therefore f^A is constant $\mathbf{0}$, a contradiction.

If there is an $\underline{x}$ such that $f^A(\underline{x}) = 1$, then there is a $\underline{y}$ with $\underline{y} \prec \underline{x}$ and $f^A(\underline{y}) = 1$. Assume $\underline{y} = \underline{0}$, then $f^A(\underline{0}) = 1$. Therefore f^A is constant $\mathbf{1}$, a contradiction.

Assume $\underline{y} \neq \underline{0}$, now we have $f^A(\underline{y}) = 1$ and since $|B| = \phi$, then there exists $\underline{z_1}$ with $\underline{z_1} \prec \underline{y}$ and $f^A(\underline{z_1}) = 1$. If $\underline{z_1} = \underline{0}$, then $f^A(\underline{0}) = 1$. Therefore f^A is constant $\mathbf{1}$, a contradiction. If $\underline{z_1} \neq \underline{0}$, then there exists $\underline{z_2}$ with $\underline{z_2} \prec \underline{z_1}$ and $f^A(\underline{z_2}) = 1$ otherwise $\underline{z_2} \in B$. Applying the same argument then we get the chain $\underline{z_k} \prec \underline{z_{k-1}} \prec, \ldots, \prec \underline{z_1} \prec \underline{y}$ for some k. Since this chain is finite, then $\underline{z_k} = \underline{0}$. Therefore f^A is constant $\mathbf{1}$, a contradiction. Therefore $|B| \neq \phi$. Hence $|B| = 1$.

Hence f^A is monotone and has the property $|B| = 1$. Since f^A is not constant $\mathbf{0}$, then $f^A(\underline{1}) = 1$. Therefore $B = \{\underline{1}\}$. Hence $f^A(\underline{x}) = 0$ for all $\underline{x} \neq \underline{1}$. Therefore $f^A(\underline{x}) = f^A(x_1, \ldots, x_n) = x_1 \wedge \ldots \wedge x_n$. Since $P_3 = P_6 \cap C_3$, $P_5 = P_6 \cap C_2$ and $P_1 = P_6 \cap C_4$, then one can apply Lemma 4.1, Lemma 4.2 and Lemma 4.3. □

Proposition 4.13.

$$\begin{aligned} S_6 = &\ F_B Mod\{F(\underline{x}) \vee F(\underline{y}) \approx F(\underline{x} \vee \underline{y})\}, \\ S_5 = &\ F_B Mod\{\neg F(\underline{0}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y}))) \approx 1\}, \\ S_3 = &\ F_B Mod\{F(\underline{1}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y}))) \approx 1\}, \\ S_1 = &\ F_B Mod\{\neg F(\underline{0}) \wedge F(\underline{1}) \wedge ((F(\underline{x}) \vee F(\underline{y})) \Leftrightarrow F(\underline{x} \vee \underline{y}))) \approx 1\}. \end{aligned}$$

Proof. Since $S_6 = (P_6)^d$, then by Corollary 4.5 $S_6 = F_B Mod\{F(\underline{x}) \vee F(\underline{y}) \approx F(\underline{x} \vee \underline{y})\}$.

Since $S_5 = S_6 \cap C_3$, $S_3 = S_6 \cap C_2$ and $S_1 = S_6 \cap C_4$, one can apply lemmas 4.1, 4.2 and 4.3. □

Let $\underline{x_i}$ be the n-tuple $\underline{x_i} = (x_{i_1}, \ldots, x_{i_n})$.

Proposition 4.14. For each $\mu \geq 2$,

$$\begin{aligned} F_8^\mu = &\ F_B Mod\{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y}) \approx 1\}, \\ F_5^\mu = &\ F_B Mod\{F(\underline{1}) \wedge (\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1\}, \\ F_7^\mu = &\ F_B Mod\{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}}) \wedge F(\underline{x_1} \vee \underline{x_2}) \approx 1\}, \\ F_6^\mu = &\ F_B Mod\{F(\underline{1}) \wedge (\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}}) \wedge \\ &\ F(\underline{x_1} \vee \underline{x_2})) \approx 1\}. \end{aligned}$$

Proof. Let $f^A \in F_8^\mu$ and let $\Sigma = \{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y}) \approx 1\}$, then for any $\alpha_1, \ldots, \alpha_\mu \in \{0,1\}^n$: if $f^A(\alpha_1) = \cdots = f^A(\alpha_\mu) = 1$, then $\alpha_1 \wedge \ldots \wedge \alpha_\mu \neq (0, \ldots, 0)$. Assume $f^A(\alpha_1) = \cdots = f^A(\alpha_{\mu-1}) = 1$, and let $\beta = \overline{\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}} \wedge \alpha_\mu$, then $\beta \wedge (\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}) = \overline{\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}} \wedge \alpha_\mu \wedge (\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}) = (0, \ldots, 0)$. Therefore by the contrapositive of the implication which defines the elements from F_8^μ we get $f^A(\beta) = 0$. Then $f^A(\alpha_1) \wedge \ldots \wedge f^A(\alpha_{\mu-1}) \Rightarrow \neg f^A(\overline{\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}} \wedge \alpha_\mu) = 1$. If there exists α_i such that $f^A(\alpha_i) = 0$ for some $i \in \{1, \ldots, \mu-1\}$, then $f^A(\alpha_1) \wedge \ldots \wedge f^A(\alpha_{\mu-1}) \Rightarrow \neg f^A(\overline{\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}} \wedge \alpha_\mu) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $\bigwedge_{i=1}^{\mu-1} f^A(\underline{x_i}) \Rightarrow \neg f^A(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{x_\mu}) =$

1. Assume $f^A(\underline{0}) \neq 0$, i.e. $f^A(\underline{0}) = 1$. Since $\bigwedge_{i=1}^{\mu-1} f^A(\underline{0}) = 1$ and $\neg f^A(\overline{0 \wedge \ldots \wedge 0} \wedge \underline{0}) = \neg f^A(\underline{0}) = \neg 1 = 0$, then $\bigwedge_{i=1}^{\mu-1} f^A(\underline{0}) \Rightarrow$

$\neg f^A(\overline{\underline{0} \wedge \ldots \wedge \underline{0}} \wedge \underline{0}) = 0$, a contradiction. Therefore $f^A(\underline{0}) = 0$.

Let $\alpha_1, \ldots, \alpha_\mu \in \{0,1\}^n$ such that $f^A(\alpha_1) = \ldots = f^A(\alpha_\mu) = 1$. Because of $(f^A(\alpha_1) \wedge \ldots \wedge f^A(\alpha_{\mu-1})) \Rightarrow \neg f^A(\overline{\alpha_1 \wedge \ldots \alpha_{\mu-1}} \wedge \underline{x}) = 1$ for all $\underline{x} \in \{0,1\}^n$ we have $\neg f^A(\overline{\alpha_1 \wedge \ldots \alpha_{\mu-1}} \wedge \alpha_\mu) = 1$, i.e $f^A(\overline{\alpha_1 \wedge \ldots \alpha_{\mu-1}} \wedge \alpha_\mu) = 0$. Assume $(\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}) \wedge \alpha_\mu = \underline{0}$, then $\overline{\alpha_1 \wedge \ldots \wedge \alpha_{\mu-1}} \wedge \alpha_\mu = \alpha_\mu$. Therefore $f^A(\alpha_\mu) = f^A(\overline{\alpha_1 \wedge \ldots \alpha_{\mu-1}} \wedge \alpha_\mu) = 0$, a contradiction. Hence $(\alpha_1 \wedge \ldots \alpha_{\mu-1}) \wedge \alpha_\mu \neq \underline{0}$. Therefore $f^A \in F_8^\mu$.

Since $F_5^\mu = F_8^\mu \cap C_4$, $F_7^\mu = F_8^\mu \cap A_1$, $F_6^\mu = F_8^\mu \cap A_4$, one can apply Lemma 4.3. $\square$

Proposition 4.15. For each $\mu \geq 2$,

$$\begin{aligned}
F_4^\mu = \quad & F_B Mod\{\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{y}) \approx 1\},\\
F_1^\mu = \quad & F_B Mod\{\neg F(\underline{0}) \wedge (\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{y})) \approx 1\},\\
F_3^\mu = \quad & F_B Mod\{\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}}) \wedge \neg F(\underline{x_1} \wedge \underline{x_2}) \approx 1\},\\
F_2^\mu = \quad & F_B Mod\{\neg F(\underline{0}) \wedge (\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}}) \wedge\\
& \neg F(\underline{x_1} \wedge \underline{x_2})) \approx 1\}.
\end{aligned}$$

Proof. Let $f^A \in F_4^\mu$ and let $\sigma = \{\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{y}) \approx 1\}$, then for any $\alpha_1, \ldots, \alpha_\mu \in \{0,1\}^n$: if $f^A(\alpha_1) = \cdots = f^A(\alpha_\mu) = 0$, then $\alpha_1 \vee \ldots \vee \alpha_\mu \neq (1, \ldots, 1)$. If $f^A(\alpha_1) = \ldots = f^A(\alpha_{\mu-1}) = 0$, we get $\neg f^A(\alpha_i) = 1$ for all $i \in \{0,1\}^n$. Let $\beta = \overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1}} \vee \alpha_\mu$, then $\beta \vee (\alpha_1 \vee \ldots \vee \alpha_{\mu-1}) = (\overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1}} \vee \alpha_\mu) \vee (\alpha_1 \vee \ldots \vee \alpha_{\mu-1}) = (1, \ldots, 1) \vee \alpha_\mu = (1, \ldots, 1)$. Then by the contrapositive of the implication which defines the elements from F_4^μ we get $f^A(\beta) = 1$, i.e. we have $f^A(\beta) = f^A(\overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1}} \vee \alpha_\mu) = 1$. Then $\neg f^A(\alpha_1) \vee \ldots \vee \neg f^A(\alpha_{\mu-1}) \Rightarrow f^A(\overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1}} \vee \alpha_\mu) = 1$. If there exists α_i such that $f^A(\alpha_i) = 1$ for some $i \in \{1, \ldots, \mu-1\}$, then $\neg f^A(\alpha_i) = 0$. Thus $\neg f^A(\alpha_1) \wedge \ldots \wedge \neg f^A(\alpha_{\mu-1}) \Rightarrow f^A(\overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1}} \vee \alpha_\mu) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $\bigwedge_{i=1}^{\mu-1} \neg f^A(\underline{x_i}) \Rightarrow f^A(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{x_\mu}) = 1$. Assume $f^A(\underline{1}) \neq 1$, i.e. $f^A(\underline{1}) = 0$, then $\bigwedge_{i=1}^{\mu-1} \neg f^A(\underline{1}) \Rightarrow f^A(\overline{1 \vee \ldots \vee 1} \vee \underline{1}) = 0$, a contradiction. Therefore $f^A(\underline{1}) = 1$.

Let $\alpha_1, \ldots, \alpha_\mu \in \{0,1\}^n$ such that $f^A(\alpha_1) = \ldots = f^A(\alpha_\mu) = 0$. Then $\neg f^A(\alpha_i) = 1$ for all $i \in \{1, \ldots, \mu\}$. Therefore $\neg f^A(\alpha_1) \wedge \ldots \wedge \neg f^A(\alpha_{\mu-1}) \Rightarrow f^A(\overline{\alpha_1 \vee \ldots \alpha_{\mu-1}} \vee \underline{x}) = 1$ for all $\underline{x} \in \{0,1\}^n$. This gives

$$f^A(\overline{\alpha_1 \vee \ldots \alpha_{\mu-1}} \vee \alpha_\mu) = 1.$$

Assume $(\alpha_1 \vee \ldots \vee \alpha_{\mu-1}) \vee \alpha_\mu = \underline{1}$, then $\overline{\alpha_1 \vee \ldots \vee \alpha_{\mu-1})} \vee \alpha_\mu = \alpha_\mu$. Therefore $f^A(\alpha_\mu) = f^A(\overline{\alpha_1 \vee \ldots \alpha_{\mu-1})} \vee \alpha_\mu) = 1$, a contradiction. Hence $(\alpha_1 \vee \ldots \vee \alpha_{\mu-1}) \vee \alpha_\mu \neq \underline{1}$. Therefore $f^A \in F_4^\mu$.

Since $F_1^\mu = F_4^\mu \cap C_4$, $F_3^\mu = F_4^\mu \cap A_1$, and $F_2^\mu = F_4^\mu \cap A_4$, one can apply Lemma 4.3. □

Proposition 4.16. $O_9 = F_B Mod\{((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x} \Leftrightarrow F(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_9$ and let $\Sigma = \{((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x} \Leftrightarrow F(\underline{x}\wedge\underline{y}))) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$. If f^A is the constant, then $f^A \vdash \Sigma$. If f^A is the identity mapping, then $((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y})) = ((x \Leftrightarrow y) \Rightarrow (x \Leftrightarrow (x \wedge y))) = ((x \Leftrightarrow y) \Rightarrow (x \Leftrightarrow (x \wedge y)))$ and this is a tautology. Since $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1 \oplus 0 \oplus x \oplus y \oplus (x \oplus y) = 1$, then $((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. If f^A is the negation, then $((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = ((y \Leftrightarrow x) \Rightarrow (y \Leftrightarrow \neg(x \wedge y))) \wedge (1 \oplus 1 \oplus y \oplus x \oplus \neg(x \oplus y)) = ((y \Leftrightarrow x) \Rightarrow (y \Leftrightarrow (y \vee x))) \wedge (y \oplus x \oplus \neg(x \oplus y)) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Therefore f^A is linear.

Next we will show that f^A depends essentially on at most one variable. If f^A does not depend on any varaiable, then f^A is the constant **0** or **1**. Then $f^A \in O_9$. If f^A is essentially depending on one variable, then $f^A \in \{x, \neg x\}$. Then $f^A \in O_9$. Assume f^A depends essentially on more than one variable, i.e. f^A has at least two essential variables. Let x_1, x_2 be essential variables of f^A, then $f^A(\underline{x}) = f^A(x_1, \ldots, x_n) = x_1 \oplus x_2 \oplus f^A(0, 0, x_3, \ldots, x_n)$. Let $\alpha = (0, 1, x_3, \ldots, x_n)$ and $\beta = (1, 0, x_3, \ldots, x_n)$, then $f^A(\alpha) = 0 \oplus 1 \oplus f^A(0, 0, x_3, \ldots, x_n) =$

$1 \oplus 0 \oplus f^A(0,0,x_3,\ldots,x_n) = f^A(\beta)$. Therefore $f^A(\alpha) = f^A(\beta)$. Since $f^A \vdash \Sigma$, then $(f^A(\alpha) \Leftrightarrow f^A(\beta)) \Rightarrow (f^A(\alpha) \Leftrightarrow f^A(\alpha \wedge \beta)) = 1$. Therefore $f^A(\alpha) = f^A(\alpha\wedge\beta)$. Since $f^A(\alpha\wedge\beta) = f^A(0\wedge 1, 1\wedge 0, x_3\wedge x_3, \ldots, x_n\wedge x_n) = f^A(0,0,x_3,\ldots,x_n) = \neg f^A(\alpha)$, then $f^A(\alpha) \neq f^A(\alpha\wedge\beta) = \neg f^A(\alpha)$, a contradiction. Therefore f^A depends essentially on one variable. □

Proposition 4.17. $O_4 = F_B Mod\{(F(\underline{0}) \oplus F(\underline{1})) \wedge ((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x}) \Leftrightarrow F(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_4$ and let $\Sigma = \{(F(\underline{0}) \oplus F(\underline{1})) \wedge ((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x}) \Leftrightarrow F(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$, then $f^A \vdash ((F(\underline{x}) \Leftrightarrow F(\underline{y})) \Rightarrow (F(\underline{x} \Leftrightarrow F(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$. Since f^A is the identity or the negation, then $f^A(\underline{0}) \oplus f^A(\underline{1}) = 1$. Further $(f^A(\underline{0}) \oplus f^A(\underline{1})) \wedge ((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $(f^A(\underline{0}) \oplus f^A(\underline{1})) \wedge ((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x} \Leftrightarrow f^A(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus f^A(\underline{0}) \oplus F(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Now we get $f^A(\underline{0}) \oplus f^A(\underline{1}) = 1$ and $((f^A(\underline{x}) \Leftrightarrow f^A(\underline{y})) \Rightarrow (f^A(\underline{x}) \Leftrightarrow f^A(\underline{x} \wedge \underline{y}))) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then $f^A \in O_9$. Since $f^A(\underline{0}) \oplus f^A(\underline{1}) = 1$, then $f^A \in \{x, \neg x\}$. □

Proposition 4.18. $O_8 = F_B Mod\{(F(\underline{x}) \Rightarrow F(\underline{x}) \vee \underline{y})) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_8$ and let $\Sigma = \{(F(\underline{x}) \Rightarrow F(\underline{x}) \vee \underline{y})) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$. Since $O_8 \subseteq A_1$ and $O_8 \subseteq L_1$, then $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y}) = 1$ and $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Therefore $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Hence $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Therefore $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y}) = 1$ and $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Hence $f^A \in A_1$ and $f^A \in L_1$. Thus $f^A \in \{\mathbf{0}, \mathbf{1}, x\}$. □

Proposition 4.19. $O_6 = F_B Mod\{(F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_6$ and let $\Sigma = \{(F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$. If f^A is the constant, then $f^A \vdash \Sigma$. If f^A is the identity, then $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y}) = x \Rightarrow (x \vee y) = 1$ and $1 \oplus$

$f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1 \oplus x \oplus y \oplus (x \oplus y) = 1$. Then $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Thus $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y}) = 1$ and $1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Therefore $f^A \in A_1$ and $f^A \in L_3$. Hence $f^A \in O_6$. □

Proposition 4.20. $O_5 = F_B Mod\{F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_5$ and let $\Sigma = \{F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{0}) \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$. If f^A is the constant **1**, then $f^A \vdash \Sigma$. If f^A is the identity, then $f^A(\underline{1}) = 1$ and $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then $f^A(\underline{1}) \wedge (f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1 \wedge 1 \wedge 1 = 1$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $f^A(\underline{1}) \wedge (f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then$f^A(\underline{1}) = 1$, $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y}) = 1$ and $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y}) = 1$. Therefore $f^A \in C_2$, $f^A \in A_1$ and $f^A \in L_3$. Hence $f^A \in O_6$ □

Proposition 4.21. $O_1 = F_B Mod\{F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$.

Proof. Let $f^A \in O_1$ and let $\{F(\underline{1}) \wedge (F(\underline{x}) \Rightarrow F(\underline{x} \vee \underline{y})) \wedge (1 \oplus F(\underline{x}) \oplus F(\underline{y}) \oplus F(\underline{x} \oplus \underline{y})) \approx 1\}$, then $f^A(\underline{1}) = 1$ and $(f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then $f^A(\underline{1}) \wedge (f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Therefore $f^A \vdash \Sigma$. Let $f^A \in F_B Mod\Sigma$, then $f^A(\underline{1}) \wedge (f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Then $f^A(\underline{1}) = 1$ and $f^A(\underline{x}) \Rightarrow f^A(\underline{x} \vee \underline{y})) \wedge (1 \oplus f^A(\underline{x}) \oplus f^A(\underline{y}) \oplus f^A(\underline{x} \oplus \underline{y})) = 1$. Therefore $f^A \in C_2$ and $f^A \in O_6$. Assume $f^A(\underline{0}) \neq 0$, i.e. $f^A(\underline{0}) = 1$, then $1 \oplus f^A(\underline{0}) \oplus f^A(\underline{1}) \oplus f^A(\underline{0} \oplus \underline{1}) = 0$, a contradiction. Then $f^A(\underline{0}) = 0$. Therefore $f^A \in O_1$. □

Proposition 4.22.

$$F_8^{\infty} = F_B Mod\{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1 \mid \mu \geq 2\},$$

$$F_5^{\infty} = F_B Mod\{F(\underline{1}) \wedge (\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1$$

$$\mid \mu \geq 2\},$$

$$
\begin{aligned}
F_7^\infty =& F_B Mod\{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}}) \wedge F(\underline{x_1} \vee \underline{x_2}) \approx 1 \\
& \mid \mu \geq 2\},\\
F_6^\infty =& F_B Mod\{F(\underline{1}) \wedge (\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}}) \\
& \wedge F(\underline{x_1} \vee \underline{x_2})) \approx 1 \mid \mu \geq 2\}.
\end{aligned}
$$

Proof. Let $f^A \in F_8^\infty$ and let $\Sigma = \{\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1 \mid \mu \geq 2\}$. Since $\bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) \approx 1$ is satisfied by any operation from F_8^μ for all $\mu \geq 2$ and $f^A \in F_8^\infty = \bigcap_{\mu \geq 2} F_8^\mu$, then $f^A \vdash \bigwedge_{i=1}^{\mu-1} F(\underline{x_i}) \Rightarrow \neg F(\overline{x_1 \wedge \ldots \wedge x_{\mu-1}} \wedge \underline{y})) \approx 1$ for all $\mu \geq 2$. Therefore $f^A \vdash \Sigma$.

Let $f^A \in F_B Mod\Sigma$, then $\bigwedge_{i=1}^{\mu-1} f^A(\underline{x_i}) \Rightarrow \neg f^A(\overline{\underline{x_1} \wedge \ldots \wedge \underline{x_{\mu-1}}} \wedge \underline{y})) = 1$ for all $\mu \geq 2$. Therefore $f^A \in F_8^\mu$ for all $\mu \geq 2$. Hence $f^A \in \bigcap_{\mu \geq 2} F_8^\mu = F_8^\infty$. Since $F_5^\infty = F_8^\infty \cap A_4$, $F_7^\infty = F_8^\infty \cap A_1$, and $F_6^\infty = F_8^\infty \cap A_4$, then one can apply Lemma 4.3. □

Proposition 4.23.

$$
\begin{aligned}
F_4^\infty = & \quad F_B Mod\{\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \underline{y}) \\
& \quad \approx 1 \mid \mu \geq 2\},\\
F_1^\infty = & \quad F_B Mod\{\neg F(\underline{0}) \wedge (\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}} \vee \\
& \quad \underline{y})) \approx 1 \mid \mu \geq 2\},\\
F_3^\infty = & \quad F_B Mod\{\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow \neg F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}}) \wedge \neg F(\underline{x_1} \\
& \quad \wedge \underline{x_2}) \approx 1 \mid \mu \geq 2\},\\
F_2^\infty = & \quad F_B Mod\{\neg F(\underline{0}) \wedge (\bigwedge_{i=1}^{\mu-1} \neg F(\underline{x_i}) \Rightarrow F(\overline{\underline{x_1} \vee \ldots \vee \underline{x_{\mu-1}}}) \wedge \\
& \quad \neg F(\underline{x_1} \wedge \underline{x_2})) \approx 1 \mid \mu \geq 2\}.
\end{aligned}
$$

The proof is similar to the proof of Proposition 4.22.

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