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Classification of the local isometry groups of rooted tree boundaries

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Dedicated to V.I. Sushchansky on the occasion of his 60th birthday

Abstract. Classification of the groups of local isometries of the rooted tree boundaries is established for transitive groups.

1. The notion of local isometry is defined naturally on a metric space. A bijection is a local isometry if it acts as an isometry in a neighborhood of each point. More precisely, let (X_1, d_1) and (X_2, d_2) be metric spaces. A bijection $\alpha : X_1 \to X_2$ is called a *local isometry* if for every $x \in X_1$ there exists a neighborhood U_x of x such that for every $x_1, x_2 \in U_x$ we have

$$d_2(x_1^\alpha, x_2^\alpha) = d_1(x_1, x_2).$$

It is clear that the set of local isometries of a metric space (X, d) is a group.

The group of the local isometries of the boundary of a rooted tree is investigated in [LS], [Lav], [MI]. In particular, in [Lav] it was proved that if the group of the local isometries of the boundary of a locally finite rooted tree is transitive, then it is complete, i.e., all its automorphisms are inner and its center is trivial.

It is known that the full isometry groups of the boundary of a locally finite spherically homogeneous rooted tree are isomorphic if and only if the trees are isometric [Nek, Proposition 2.10.7]. The main result of this work is the analogous assertion for the local isometries.

2. We start with necessary definitions on rooted trees and groups acting on such trees. Let T be a locally finite rooted tree with the root vertex

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v_0. For every two vertices u, v of the tree T ($u, v \in V(T)$) we define the *distance* between u and v, written $d(u, v)$, to be the length of the shortest path connecting them. For an integer $n \geq 0$ we define the nth level (the sphere of the radius n) to be the set

$$V_n(T) = \{v \in V(T) : d(v_0, v) = n\}.$$

If the degree of a vertex $v \in V_n(T)$ depends only on n, then the tree T is called *spherically homogeneous*. *Spherical index* of a spherically homogeneous tree T is the sequence $\Omega = (a_0, a_1, \ldots)$, where a_0 is degree of the root and $a_n + 1$ is degree of any vertex of n-th level.

Let T be a spherically homogeneous rooted tree with the root v_0 and spherical index Ω. All such trees are isomorphic to the tree T_Ω with the set of vertices equal to the set of all finite sequences $(i_0, i_1, \ldots, i_{n-1})$, where $i_k \in \{1, 2, \ldots, a_k\}$ and $n \geq 0$ is an integer. We also include the empty sequence (corresponding to $n = 0$). Two vertices are adjacent if and only if they are of the form $(i_0, \ldots, i_{n-1}), (i_0, \ldots, i_{n-1}, i_n)$.

An end of a rooted tree is an infinite path starting in the rooted vertex and having no repetitions. We will denote by ∂T (boundary) the set of all ends of T.

Let $\bar{\lambda} = \{\lambda_n\}_{n=0}^{\infty}$ be a strictly decreasing sequence of positive numbers tending to zero. We can introduce a natural ultrametric on ∂T by putting $\rho(x_1, x_2) = \lambda_n$, where n is the length of the maximal common part of the paths x_1 and x_2. The topology induced by the metrics ρ (or for convenience $\bar{\lambda}$) is compact, totally disconnected and has a base of open sets (balls or cylinder sets) of the form $U_v = \{x \in \partial T \mid v \in x\}$, $v \in V(T)$. This compact ultrametric space will be denoted by $(\partial T, \bar{\lambda})$ or simply by ∂T.

3. In this work we study the classification problem of the full local isometry groups of the boundaries of rooted trees. We prove the following theorem.

Theorem 1. Let T_1 and T_2 be locally finite rooted trees such that the groups of the local isometries act on their boundaries transitively. The full local isometries groups of $(\partial T_1, \bar{\mu}_1)$ and $(\partial T_2, \bar{\mu}_2)$ are isomorphic if and only if $(\partial T_1, \bar{\lambda}_1)$ and $(\partial T_2, \bar{\lambda}_2)$ are locally isometric for some metrics $\bar{\lambda}_1$ and $\bar{\lambda}_2$. In other words there are positive integers i and j, such that for every natural s the equality

$$|V_{i+s}(T_1)| = |V_{j+s}(T_2)|$$

holds.

In the work [Lav] a sufficient condition for isomorphism for the local isometry groups was established

Proposition 2 ([Lav]). If there exist ultrametrics $\bar{\mu}_1$ and $\bar{\mu}_2$ such that the spaces $(\partial T_1, \bar{\mu}_1)$ and $(\partial T_2, \bar{\mu}_2)$ are locally isometric then their full local isometry groups are isomorphic.

In that work it was also established that the boundary of a rooted tree on which the local isometries act transitively is locally isometric to the boundary of some spherically homogeneous tree.

Proposition 3 ([Lav]). If the group $\mathrm{LIsom}(\partial T, \bar{\lambda})$ acts transitively on ∂T then there is a spherically homogeneous tree T_Ω and a sequence $\bar{\lambda}_\Omega$ such that $(\partial T, \bar{\lambda})$ and $(\partial T_\Omega, \bar{\lambda}_\Omega)$ are locally isometric.

Due to Propositions 2 and 3, in order to prove Theorem 1, we only need to show that if the groups of local isometries of the boundaries ∂T_1 and ∂T_2 of a spherically homogeneous tree are isomorphic then $(\partial T_1, \bar{\lambda}_1)$ and $(\partial T_2, \bar{\lambda}_2)$ are locally isometric for some metrics $\bar{\lambda}_1$ and $\bar{\lambda}_2$.

4. Let us fix some notations. Let T_i $(i = 1, 2)$ be spherically homogeneous trees with spherical indices containing no ones. We will denote ∂T_i by X_i. We assume that the full local isometry groups of X_1 and X_2 are isomorphic and $\phi : \mathrm{LIsom}\, X_1 \to \mathrm{LIsom}\, X_2$ is an isomorphism. By Rubin's theorem [Rub, Corollary 3.13c] there exists a homeomorphism $h : X_1 \to X_2$ such that for every $g \in \mathrm{LIsom}\, X_1$ the equality $\phi(g) = hgh^{-1}$ holds. The action of $\mathrm{LIsom}\, X_i$ on X_i is ergodic with respect to the Bernoulli measure. Actually the only probabilistic measure on X_i with respect to which the action of $\mathrm{LIsom}\, X_i$ is ergodic is the Bernoulli measure (see [GNS]). Thus, we can fix the unique probabilistic $\mathrm{LIsom}\, X_i$-invariant measures m_i on sigma-algebras of Borel subsets of X_i.

Lemma 4. For the homeomorphism h and the ball U_v^1 $(v \in V(T_1))$ there are positive integers l and k such that

$$h(U_v^1) = \bigcup_{i=1}^{k} U_{v_i}^2,$$

where $\{v_1, \ldots, v_k\} \subset V_l(T_2)$.

Proof. Since U_v^1 is compact and balls in X_2 do not intersect or one contains the other, we have what is required. $\square$

Lemma 5. For every ball U_v^1 $(v \in V_k(T_1))$ $(k \in \mathbb{N})$ the equality

$$m_2(h(U_v^1)) = m_1(U_v^1),$$

holds.

Proof. The measures m_1, m_2 are probabilistic, i.e. $m_1(X_1) = m_2(X_2) = 1$. Since m_1 is an LIsom X_1-invariant measure,

$$m_1(X_1) = m_1\left(\bigsqcup_{w\in V_k(T_1)} U_w^1\right) = |V_k(T_1)| m_1(U_v^1).$$

Since every local isometry preserves the measure, and $\phi(g) = hgh^{-1}$ for all $g \in \mathrm{LIsom}\, X_1$ we have $m_2(h(U_u^1)) = m_2(h(U_w^1))$ for any $u, w \in V_k(T_1)$. Therefore

$$m_2(X_2) = m_2(h(X_1)) = m_2\left(h\left(\bigsqcup_{w\in V_k(T_1)} U_w^1\right)\right) =$$
$$= m_2\left(\bigsqcup_{w\in V_k(T_1)} h(U_w^1)\right) = |V_k(T_1)| m_2(h(U_v^1)).$$

Thus, $m_2(h(U_v^1)) = m_1(U_v^1)$. □

Lemma 6. Let $w \in V_m(T_2)$ and $v \in V_l(T_1)$ be such that $h^{-1}(U_w^2) = U_v^1$. Then $|V_m(T_2)| = |V_l(T_1)|$.

Proof. By Lemma 5 we have $m_2(U_w^2) = m_1(U_v^1)$. Therefore

$$|V_m(T_2)| m_2(U_w^2) = m_1(U_v^1)|V_l(T_1)| = 1.$$

Thus we have $|V_m(T_2)| = |V_l(T_1)|$. □

Let (X, d) be a metric space. A bijection $\alpha : X_1 \to X_2$ is called *a uniform local isometry* if there exists $\delta > 0$ such that

$$d_2(x_1^\alpha, x_2^\alpha) = d_1(x_1, x_2)$$

for all $x_1, x_2 \in X_1$ such that $d_1(x_1, x_2) < \delta$. It is easy to see that for a compact spaces every local isometry is a uniform local isometry.

Lemma 7. There exist ultrametrics $\bar{\mu}_1$ and $\bar{\mu}_2$ such that the spaces $(\partial T_1, \bar{\mu}_1)$ and $(\partial T_2, \bar{\mu}_2)$ are locally isometric if and only if there exist i and j, such that for every positive integer s the equality

$$|V_{i+s}(T_1)| = |V_{j+s}(T_2)|$$

holds.

Proof. Obviously, if the equality holds then the spaces $(\partial T_1, \bar{\mu}_1)$ and $(\partial T_2, \bar{\mu}_2)$ are locally isometric for some ultrametrics $\bar{\mu}_1$ and $\bar{\mu}_2$. The remaining implication follows from uniformity of local isometry. □

Lemma 8. The spaces X_1 and X_2 are locally isometric for some ultrametrics.

Proof. Let $\{v_i,\ i \in \mathbb{N}\} \subset V(T_2)$ be infinite sequence of vertices such that $v_i \not\preceq v_j$ for all $i \neq j$. Let $h^{-1}(U^2_{v_i}) = \bigsqcup_{j=1}^{k_i} U^1_{u_{ij}}$, where $\{u_{i1}, \ldots, u_{ik_i}\} \subset V_{l_i}(T_1)$.

We also consider infinite sequence of vertices $\{w_i,\ i \in \mathbb{N}\} \subset V(T_2)$ such that $w_i \prec v_i$ and $m_2(U^2_{w_i}) < m_1(U^1_{u_{ij}})$ for all i and j. We can assume that $m_2(U^2_{w_i}) > m_2(U^2_{w_{i+1}})$, that is w_i closer to the root than w_{i+1}. We suppose that $h^{-1}(U^2_{w_i})$ is not a ball for all i. We will show that there exists a sequence $\{g_i,\ i \in \mathbb{N}\} \subset \operatorname{Isom} X_1$ such that

1. g_i acts trivially on $X_1 \setminus h^{-1}(U^2_{v_i})$.
2. $\phi(g_i)(U^2_{w_i}) = hg_ih^{-1}(U^2_{w_i})$ is not a ball.

There exist only two possibilities for $h^{-1}(U^2_{w_i})$:

1. There exist j_1 and j_2 $(1 \leq j_1, j_2 \leq k_i)$ such that $h^{-1}(U^2_{w_i}) \cap U^1_{u_{ij_s}} \neq \emptyset$ $(s = 1, 2)$.
2. There exists j $(1 \leq j \leq k_i)$ such that $h^{-1}(U^2_{w_i}) \subsetneqq U^1_{u_{ij}}$.

In the first case we choose g_i as follows: g_i acts trivially on $X_1 \setminus h^{-1}(U^2_{v_i})$ and on $U^1_{u_{ij_1}}$ and move some points of $U^1_{u_{ij_2}}$ in such a way that $g_i(U^1_{u_{ij_2}}) = U^1_{u_{ij_2}}$ and $g_i(h^{-1}(U^2_{w_i}) \cap U^1_{u_{ij_2}}) \neq h^{-1}(U^2_{w_i}) \cap U^1_{u_{ij_2}}$. Since $h^{-1}(U^2_{w_i}) \cap U^1_{u_{ij_2}}$ is strictly contained in $U^1_{u_{ij_2}}$, we have that there exists g_i with the required properties.

In the second case we choose g_i as follows: g_i acts trivially on $X_1 \setminus h^{-1}(U^2_{v_i})$ and stabilizes $U^1_{u_{ij}}$ and $g_i(h^{-1}(U^2_{w_i})) \neq h^{-1}(U^2_{w_i})$. Since $h^{-1}(U^2_{w_i})$ is strictly contained in $U^1_{u_{ij}}$, we have that there exists g_i with required properties.

Since $h^{-1}(U^2_{v_i}) \cap h^{-1}(U^2_{v_j}) = \emptyset$ for $i \neq j$ we have that the infinite product $g_1g_2g_3\ldots$ defines an isometry g of the metric space X_1. Thus $\phi(g)$ is a local isometry of X_2. Therefore images of the balls with "small" radii are balls for the mapping $\phi(g)$. That is, there is a positive integer r, such that for every $t > r$ and every vertex $v \in V_t(T_2)$ there exists $w \in V(T_2)$ such that

$$\phi(g)(U^2_v) = U^2_w. \tag{0.1}$$

By the construction, $\phi(g_i)(U^2_{w_i})$ is not a ball for any positive integer i. Hence $\phi(g)(U^2_{w_{r+1}})$ is not a ball either. Since w_{r+1} is in the level with number greater than r, we get a contradiction with (0.1). Thus our assumption is not true. So for any $n > r$ the image $h^{-1}(U^2_{w_n})$ is a ball. Let m be such that w_n is in $V_m(T_2)$. By Lemma 6 we have that $|V_m(T_2)| = |V_l(T_1)|$ for some positive integer l.

Suppose that there exists an infinite increasing sequence of positive integers $\{l_i,\ i \in \mathbb{N}\}$ such that for every i and m the inequality $|V_{l_i}(T_2)| \neq |V_m(T_1)|$ holds. We can consider a sequence $\{u_j,\ j \in \mathbb{N}\} \subset V(T_2)$ such that for every positive integer j the conditions $u_j \prec w_j$ and $u_j \in V_k(T_2)$ hold for $k \in \{l_i,\ i \in \mathbb{N}\}$. Since $m_2(U^1_{u_i}) < m_2(U^2_{w_i}) < m_1(U^1_{u_{i1}})$ for all i, by the supposition and by Lemma 5, we have that $h(U^1_{u_i})$ is not a ball. Therefore we have a contradiction with what we proved above. Thus for every m greater than some n the equality $|V_m(T_2)| = |V_l(T_1)|$ holds for some positive integer l. Since h is a homeomorphism and ϕ is an isomorphism the converse statement is true: for every l greater than some k the equality $|V_l(T_1)| = |V_m(T_2)|$ holds for some positive integer m. Therefore there exist i_0 and j_0 such that for every natural s the following equality holds:

$$|V_{i_0+s}(T_1)| = |V_{j_0+s}(T_2)|.$$

So the spaces X_1 and X_2 are locally isometric for some metrics. □

Now the proof of Theorem 1 follows from Lemmas 7, 8 and Propositions 2 and 3.

Corollary 9. If the ultrametrics on X_1 and X_2 are such that these spaces are locally isometric, then the homeomorphism h is a local isometry.

Proof. Let $f : X_1 \to X_2$ be a local isometry. Then hf^{-1} induces an automorphism of G_1. Since G_1 is a complete group we have that hf^{-1} is a local isometry. So h is also a local isometry. □

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