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Normal functors in the coarse category

V. Frider

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Abstract. We define the canonical coarse structure on the spaces of the form FX, where F is a finitary normal functor of finite degree and show that every finitary (i.e., preserving the class of finite spaces) normal functor of finite degree in **Comp** has its counterpart in the coarse category.

1. Introduction

The coarse category first introduced by J. Roe [1] turned out to be an appropriate universe for different areas of modern mathematics (see, e.g. [2], [3]). In [4] the authors considered the hyperspace functor in the coarse category and established some properties of the monad generated by this functor. In particular, it was proved in [4] that the G-symmetric power functor can be extended onto the Kleisli category of this monad.

Both the hyperspace functor and the G-symmetric power functor have their counterparts in the category **Comp** of compact Hausdorff spaces. These functors in **Comp** are examples of normal functors in the sense of Shchepin [5]. The theory of normal functors was developed by different authors [6]; in [5] this theory was applied to the topology of nonmetrizable compact Hausdorff spaces and in [7] to the topology of infinite-dimensional manifolds.

The aim of this paper is to show that every finitary (i.e., preserving the class of finite spaces) normal functor of finite degree in **Comp** has its counterpart in the coarse category. These functors are completely determined by their restrictions onto the category of spaces of cardinality $\leq n$, where n denotes the degree of the functor. We show, in particular,

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that our construction coincides with the Vietoris coarse structure on the hypersymmetric power functors (defined in [4]).

2. Preliminaries

2.1. Coarse structures

For the convenience of reader we recall some definitions of the coarse topology; see, e.g. [1], [8] for details.

Let X be a set and $M, N \subset X \times X$. The *composition* of M and N is the set

$$MN = \{(x,y) \in X \times X \mid \text{ there exists } z \in X \text{ such that } (x,z) \in M,\ (z,y) \in N\},$$

the *inverse* of M is the set $M^{-1} = \{(x,y) \in X \times X \mid (y,x) \in M\}$.

A *coarse stucture* on a set X is a family $\mathcal{E}$ of subsets, which are called the *entourages*, in the product $X \times X$ that satisfies the following properties:

1. any finite union of entourages is contained in an entourage;
2. for every entourage M, its inverse M^{-1} is contained in an entourage;
3. for every entourages M, N their composition MN is contained in an entourage;
4. $\cup\mathcal{E} = X \times X$.

A *coarse space* is a pair $(X, \mathcal{E})$, where $\mathcal{E}$ is a coarse structure on a set X.

A coarse structure on X is called *unital,* if the diagonal Δ_X is contained in an entourage. A coarse structure on X is called *anti-discrete,* if $X \times X$ is an entourage.

If $\mathcal{E}_1$, $\mathcal{E}_2$ are coarse structures on X, then $\mathcal{E}_1 \leq \mathcal{E}_2$ means that for every $M \in \mathcal{E}_1$ there is $N \in \mathcal{E}_2$ such that $M \subset N$.

Two coarse structures $\mathcal{E}_1$ and $\mathcal{E}_2$ are said to be *equivalent,* if $\mathcal{E}_1 \leq \mathcal{E}_2$ and $\mathcal{E}_2 \leq \mathcal{E}_1$. We usually identify coarse spaces with equivalent coarse structures. The *weight* of the coarse structure is the cardinal number $w(\mathcal{E}) = min\{|\mathcal{E}'| \mid \mathcal{E} \text{ and } \mathcal{E}' \text{ are equivalent }\}$.

If $\mathcal{E}$ is a coarse structure on X, then, obviously, the coarse structure $\mathcal{E}_1 = \{M \cup M^{-1} \mid M \in \mathcal{E}\}$ is equivalent to $\mathcal{E}$ and is *symmetric* in the sense that $N^{-1} \in \mathcal{E}_1$ for every $N \in \mathcal{E}_1$.

Given $M \in \mathcal{E}$ and $A \subset X$ we define the *M-neighborhood* $M(A)$ of A as follows: $M(A) = \{x \in X \mid (a, x) \in M \text{ for sone } a \in A\}$. We use the notation $M(\{a\})$ instead of $M(a)$. A set $A \subset X$ is *bounded* if there exists $x \in X$ such that $A \subset M(x)$.

Let $(X_i, \mathcal{E}_i)$, $i = 1, 2$, be coarse spaces. A map $f \colon X_1 \to X_2$ is called *coarse*, if the following two conditions hold:

1. for every $M \in \mathcal{E}_1$ there exists $N \in \mathcal{E}_2$ such that $(f \times f)(M) \subset N$;

2. for any bounded subset A of X_2 the set $f^{-1}(A)$ is bounded.

Let $f, g \colon X_1 \to X_2$ be coarse maps. If there exists $U \in \mathcal{E}_2$ (here $\mathcal{E}_2$ is the coarse structure on X_2) such that $(f(x), g(x)) \in U$ for every $x \in X_1$ then the maps f, g are said to be *U-close*.

Let (X, d) be a metric space. The family

$$\mathcal{E}_d = \{\{(x, y) \in X \times X \mid d(x, y) < n\} \mid n \in \mathbb{N}\}$$

forms a *metric coarse structure* on X.

It is easy to see that the coarse spaces and coarse maps form a category. We denote it by **CS**. Let U : **CS** $\to$ **Set** denote the forgetful functor into the category **Set** of sets.

The *product* of coarse spaces $(X_1, \mathcal{E}_1)$, $(X_2, \mathcal{E}_2)$ is the coarse space $(X_1 \times X_2, \mathcal{E}_1 \times \mathcal{E}_2)$, where $\mathcal{E}_1 \times \mathcal{E}_2 = \{U_1 \times U_2 \mid U_1 \in \mathcal{E}_1,\ U_2 \in \mathcal{E}_2\}$. By induction, we define the product of coarse spaces $(X_i, \mathcal{E}_i)$, $i = 1, \dots, n$. We denote the n-th power of a coarse space $(X, \mathcal{E})$ by $(X^n, \mathcal{E}^n)$. Note that X^n can be naturally identified with $C(n, X)$, the set of all maps from n to X; we say that $f, g \in C(n, X)$ are *U-close* whenever $(f, g) \in U^n$, where $U \in \mathcal{E}$.

2.2. Normal functors

We briefly recall some notions from the theory of normal functors in the category **Comp** of compact Hausdorff spaces; see, e.g., [5] for details. An endofunctor F in **Comp** is called *normal* if it is continuous , monomorphic, epimorphic, preserves weight of infinite compacta, intersections, preimages, singletons and empty set.

Suppose that F is a monomorphic functor that preserves the intersections, and $a \in FX$. The *support* of a is the set

$$\operatorname{supp}_{F,X}(a) = \cap\{A \subset X \mid A \text{ is closed and } a \in FA\}.$$

In the obvious situation the notation $\operatorname{supp}_{F,X}(a)$ is abbreviated to $\operatorname{supp}_F(a)$ or even to $\operatorname{supp}(a)$. We say that a functor is *finitary* if it preserves the finite spaces.

Examples of normal finitary functors are:

1) The hypersymmetric power functor $\exp_n$. Here

$$\exp_n X = \{A \subset X \mid 1 \leq A \leq n\},\ \exp_n f = f(A).$$

2) The G-symmetric power functor SP^n_G, where G is a subgroup of the symmetric group S_n. Define an equivalence relation $\sim_G$ on X^n by the conditions: $(x_1, \dots, x_n) \sim_G (y_1, \dots, y_n)$ if and only if there exists $\sigma \in G$ such that $x_i = y_{\sigma(i)}$ for all $i = 1, \dots, n$. We denote by $[x_1, \dots, x_n]_G$ the equivalence class that contains $(x_1, \dots, x_n)$. By the definition, the G-symmetric power of X is $SP^n_G X = X^n/_{\sim_G}$.

Given a map $f\colon X \to Y$, we define a map $SP^n_G f\colon SP^n_G X \to SP^n_G Y$ by the formula:

$$SP^n_G f([x_1, \dots, x_n]_G) = [f(x_1), \dots, f(x_n)]_G.$$

3) The subfunctor $P_{n,k}$ of the functor of probability measures, $k \in \mathbb{N}$. Let

$$P_{n,k} = \left\{ \sum_{i=1}^{n} \alpha_i \delta_{x_i} \mid x_1, \dots, x_n \in X,\ \alpha_i = \frac{p_i}{k},\ p_i \in \{0, 1, \dots, \}, \sum_{i=1}^{n} \alpha_i = 1 \right\}.$$

Let F be a normal functor. The *degree of a point* $a \in FX$ is the cardinality of $\operatorname{supp}(a)$, whenever $\operatorname{supp}(a)$ is finite and ∞, otherwise. The degree of a is denoted by $\deg(a)$. The maximal possible value of $\deg(a)$ is called the *degree of the functor* F and denoted by $\deg(F)$. If F is not a functor of degree n for any $n \in \mathbb{N}$ then is said to be a functor of *infinite degree* (this is denoted by $\deg F = \infty$). Let F be a finite normal functor of degree n in the category **Comp**. For every normal functor F : **Comp** $\to$ **Comp** construct the functorF_β : **Tych** $\to$ **Tych** by the following manner (**Tych** denotes the category of Tychonov spaces). For every $X \in |\mathbf{Tych}|$ let

$$F_\beta X = \{a \in F\beta X \mid \operatorname{supp}(a) \subset X\} \subset F\beta X$$

(here βX denotes the Čheh-Stone compactification of the space X). Given a map $f\colon X \to Y$ of Tychonov spaces, we have $F(\beta f)(F_\beta X) \subset F_\beta Y$. Let $F_\beta f = F\beta | F_\beta X : F_\beta X \to F_\beta Y$, which is the restriction $F(\beta f)$ on $F_\beta X$. Obviously, F_β is an endofunctor in **Tych**. By definition, F_β is an extension of F onto **Tych**. We keep the notion F for its canonical extension onto **Tych**.

For every $a \in Fn$, the subfunctor F_a of F is defined as follows:

$$F_a = \cap\{F' \mid F' \text{ is a normal subfunctor of} F \text{ such that } a \in F'n\}.$$

Every functor of the form F_a is a quotient functor of the power functor $(-)^{\deg(a)}$ in the sense that there exists a natural transformation $\pi_a\colon (-)^{\deg(a)} \to F_a$ such that all the maps $\pi_a X\colon X^{\deg(a)} \to F_a$ are onto. Indeed, define $\pi_a X$ as follows: $\pi_a X(x_1,\dots,x_n) = Ff(a)$, where $f\colon \{1,\dots,n\} \to X$ is the map sending i to x_i. Note, that from the normality of F, the preimages of the map $\pi_a X$ are finite.

3. Normal functors in the coarse category

3.1. Construction

Let F be finitary normal functor of degree $n \geq 1$, $(X,\mathcal{E})$ a coarse space. For any $U \in \mathcal{E}$ define

$$\begin{aligned}\hat{U} = &\{(a,b) \in FX \times FX \mid \text{ there exist } W_1,\dots,W_k \in \mathcal{E},\\ &f_1,\dots,f_{2k} \in C(n,X),\ c_1,\dots,c_k \in Fn \text{ such that}\\ &W_1\dots W_k \subset U, \text{ are } f_{2i-1}, f_{2i}\ U\text{-close}, i=1,\dots,k,\\ &\text{i } Ff_1(c_1)=a,\ Ff_{2k}(c_k)=b,\\ &Ff_{2j}(c_j) = Ff_{2j+1}(c_{j+1}),\ j=1,\dots,k-1\}.\end{aligned}$$

Note that here we consider the set X as a discrete topological space, that is why it is possible to consider the discrete space FX, which is identified with the underlying set.

Proposition 3.1. *The family $\{\hat{U} \mid U \in \mathcal{E}\}$ forms a coarse structure on FX.*

Proof. It is easy to see that $\widehat{U^{-1}} = (\hat{U})^{-1}$.

Suppose that $\hat{U},\hat{V} \in \hat{\mathcal{E}}$. Let us show that for any $W \in \mathcal{E}$ we have $\hat{U}\hat{V} \subset \hat{W}$. Let $(a,b) \in \hat{U}\hat{V}$, this means that there exists $c \in FX$ such that $(a,c) \in \hat{U}$, $(c,b) \in \hat{V}$. There exist $U_1,\dots,U_k,V_1,\dots,V_l \in \mathcal{E}$, $a_1,\dots,a_k,b_1,\dots,b_l \in Fn$, and $f_1,f_2,\dots,f_{2k},g_1,g_2,\dots,g_{2l}\colon n \to X$ such that the following holds:

1. $U_1\dots U_k \subset U$, $V_1\dots V_l \subset V$;

2. $Ff_1(a_1)=a$, $Ff_{2i}(a_i) = Ff_{2i+1}(a_{i+1})$ for every $i=1,2,\dots,k-1$, $Ff_k(a_k) = Fg_1(b_1) = c$, $Fg_{2j}(b_j) = Fg_{2j+1}(b_{j+1})$ for every $j = 1,2,\dots,l-1$, $Fg_{2l}(b_l) = b$;

3. the maps f_{2i-1}, f_{2i} are U_i-close, $i=1,\dots,k$ and the maps g_{2j-1}, g_{2j} are V_j-close, $j=1,\dots,l$.

Since $U_1 \dots U_k V_1 \dots V_l \subset UV \subset W$, from the definition of $\mathcal{E}$ it follows that $(a, b) \in \hat{W}$.

We are going to show that $\cup\hat{\mathcal{E}} = FX \times FX$. Choose arbitrary $a, b \in FX$ and $x \in X$ and check that $(a, \eta_X(x)) \in \hat{U}$, $(\eta_X(x), b) \in \hat{U}$ for some $U \in \mathcal{E}$. There exists $U \in \mathcal{E}$ such that $(y, x) \in U$ for every $y \in \operatorname{supp}(a)$. Let $a' \in Fn$ and $f \in X^n$ be such that $Ff(a') = a$. Denote by $g\colon n \to X$ the constant map with value $\{x\}$. Since f and g is U-close, we see that $(a, \eta_X(x)) \in \hat{U}$. Similarly, one can show that $(\eta_X(x), b) \in \hat{U}$ for some $U \in \mathcal{E}$.

□

In the sequel, we will denote the defined above coarse structure on FX by $\mathcal{E}_F$.

Proposition 3.2. *The coarse structure $\mathcal{E}_F$ is unital if and only if so is $\mathcal{E}$.*

Proof. Choose an arbitrary $a \in FX$. Since $\mathcal{E}$ is a unital coarse structure, there exists $U \in \mathcal{E}$ such that $(x_i, x_i) \in U$ for every $i = 1, \dots, n$, where $x_i \in \operatorname{supp}(a)$.

Let us consider the map $\phi : n \to X$ acting in the following way: $\phi(i) = x_i$ and $b \in Fn$ such that $F\phi(b) = a$. Notice that $F\phi(b) = a$ and for every $i = 1, \dots, n$, $(\phi(i), \phi(i)) \in U$. Thus $(a, a) \in \hat{U}$. Therefore, we obtain that $\Delta_{FX} \subset \hat{U}$

□

Lemma 3.3. *Let $(X, \mathcal{E})$ be a coarse space and F a finitary functor of finite degree. Let $a \in FX$, $x \in X$ and $a \in \hat{U}(\eta_X(x))$. Then* $\operatorname{supp}(a) \subset U(x)$.

Proof. Since $(a, \eta_X(x)) \in \hat{U}$, there exist $U_1, \dots, U_k \in \mathcal{E}$, $a_1, \dots, a_k \in Fn$, and maps $f_1, f_2, \dots, f_{2k} : n \to X$ such that the following holds:

(1) $U_1 \dots U_k \subset U$;

(2) $Ff_1(a_1) = a, Ff_{2k}(a_k) = \eta_X(x)$, and $Ff_{2i}(a_2 i) = Ff_{2i+1}(a_{2i+1})$ for every $i = 1, \dots, k-1$;

(3) the maps f_{2i-1}, f_{2i} are U_i-close, $i = 1, \dots, k$.

Let $y \in \operatorname{supp}(a)$. By induction, one can find points $z_i \in \operatorname{supp}(a_i)$, $i = 1, \dots, k$, such that $f_1(z_1) = y, f_{2i}(z_i) = f_{2i+1}(z_{i+1})$, $i = 1, \dots, k-1$. Since $(f_{2i-1}(z_i), f_{2i}(z_i)) \in U_i$, $i = 1, \dots, k$, we conclude that $(y, x) \in U_1 \dots U_k \subset U$. □

Let now $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a coarse map between coarse spaces. It is natural to define the map $Ff\colon FX \to FY$.

Proposition 3.4. *The map $Ff\colon (FX, \mathcal{E}_{XF}) \to (FY, \mathcal{E}_{YF})$ is coarse if so is f.*

Proof. Let us verify that the map Ff is coarsely uniform. Given $\hat{U} \in \hat{\mathcal{E}}_{XF}$ and $(a,b) \in \hat{U}$, find $U_1, \dots, U_k \in \mathcal{E}$, $a_1, \dots, a_k \in Fn$, and $f_1, \dots, f_{2k} : n \to X$ such that the following holds:

(1) $U_1 \dots U_k \subset U$;

(2) $Ff_1(a_1) = a, Ff_{2k}(a_k) = b$, and $Ff_{2i}(a_2 i) = Ff_{2i+1}(a_{2i+1})$ for every $i = 1, \dots, k-1$;

(3) the maps f_{2i-1}, f_{2i} are U_i-close, $i = 1, \dots, k$.

Since f is coarsely uniform, for every $i = 1, \dots, k$, there exists $V_i \in \mathcal{E}_Y$ such that $(f \times f)(U_i) \subset V_i$. Find $V \in \mathcal{E}_Y$ such that $V_1 \dots V_k \subset V$. Let $g_i = ff_i : n \to Y$. With these $g_1, \dots, g_{2k}, V_1, \dots, V_k$ and $a_1, \dots, a_k$ we see that $(Ff(a), Ff(b)) \in \hat{V}$. This means that $(Ff \times Ff)(\hat{U}) \subset \hat{V}$ and the coarse uniformity of Ff is proved.

Let us demonstrate that the map Ff is coarsely proper. Suppose that a set $\mathcal{A}$ is bounded in FY, i.e. there exists $V \in \mathcal{E}_Y$ and $y \in Y$ such that $\mathcal{A} \subset \hat{V}(\eta_Y(y))$. Since f is coarsely proper, there exists $x \in X$ and $U \in \mathcal{E}_X$ such that $f^{-1}(Y(y)) \subset U(x)$.

We are going to show that $(Ff)^{-1}(\mathcal{A}) \subset \hat{V}(\eta_X(x))$.

Suppose then there exists $a \in FX$ with $Ff(a) = b \in \mathcal{A}$. Since $(b, \eta_Y(y)) \in \hat{V}$, by Lemma 4.2, $\mathrm{supp}(b) \subset V$ and, therefore $\mathrm{supp}(a) \subset U(x)$. It is obvious that then $(a, \eta_X(x)) \in \hat{U}$. □

3.2. Hypersymmetric power

Let $(X, \mathcal{E})$ be a coarse space. On the hypersymmetric power $\exp_n$ one can consider, in addition to the Vietoris coarse structure, $\widetilde{\mathcal{E}}$, the defined above structure $\hat{\mathcal{E}}$.

Proposition 3.5. *The coarse structures $\widetilde{\mathcal{E}}$ and $\hat{\mathcal{E}}$ on the set* $\exp_n X$ *are coarsely equivalent.*

Proof. Consider an arbitrary entourage $U \in \mathcal{E}$ and show that $\widetilde{U} \subset \hat{U}(A,B) \in \hat{U}$. Given $(a,b) \in \hat{U}$, find $a_1, \dots, a_k \in \exp_n n$ and $f_1, \dots, f_{2k} : n \to X$ such that the following holds:

(1) $U_1 \dots U_k \subset U$;

(2) $\exp_n f_1(a_1) = a, \exp_n f_{2k}(a_k) = b$, and
$\exp_n f_{2i}(a_2 i) = \exp_n f_{2i+1}(a_{2i+1})$ for every $i = 1, \dots, k-1$;

(3) the maps f_{2i-1}, f_{2i} are U_i-close, $i = 1, \dots, k$.

For any $x \in a$ there exists $c_1 \in a_1 \subset n$ with $f_1(c_1) = x$. Since $\exp_n f_{2i}(a_i) = \exp_n f_{2i+1}(a_{i+1})$, one can choose a sequence $c_i \in a_i$, for $i = 2, \dots, k$ so that $(f_{2i-1}(c_i), f_{2i}(c_i)) \in U_i$, $i = 1, \dots, k$. Let $f_{2k}(c_{2k}) = y$. Then $(x,y) \in U_1 \dots U_k \subset U$. Therefore, $a \subset U(b)$. One can similarly show that $b \subset U(a)$. This means that $(a,b) \in \widetilde{U}$.

Let $(a,b) \in \widetilde{U}$, i.e. $a \subset U(b)$ and $b \subset U(a)$. For every $x \in a$, find $h(x) \in b$ such that $(x, h(x)) \in U$. Similarly, for every $y \in b$, find $g(y) \in a$ such that $(g(y), y) \in U$. Denote by $r : a \to g(b)$ the retraction such that $r(x) = g(h(x))$, for every $x \in a \setminus g(b)$.

Let $f_1 : n \to X$ be a map for which $f_1(n) = a$, $f_2 = rf_1$, $f_4 : n \to X$ be a map for which $f_4(n) = b$, $f_3 = gf_4$, then

$$f_2(n) = rf_1(n) = r(a) = g(b) = gf_4(n) = f_3(n).$$

Since, for every $i \in n$, $(f_1(i), h(f_1(i))) \in U$ and $(h(f_1(i)), g(h(f_1(i)))) \in U$, we see that $(f_1(i), f_2(i)) \in U^2$. Similarly, $(f_3(i), f_4(i)) \in U$, therefore the maps f_1 and f_4 are U^3-close. Finally, $\widetilde{U} \subset \hat{U^3}$.

$\square$

3.3. G-symmetric power

The G-symmetric power functor SP^n_G is endowed with the coarse structure $\widehat{\mathcal{E}}$ defined in the following way:

$\widehat{M} \in \widehat{\mathcal{E}} \Leftrightarrow \widehat{M} = \{([x_1, \ldots, x_n]_G, [y_1, \ldots, y_n]_G) \mid$ there exist $\sigma \in G$, such that for all $i = 1, \ldots, n$, $(x_i, y_{\sigma(i)}) \in M \in \mathcal{E}\}$ (see [4] for details).

Further we will denote $\mathcal{E}_{SP^n_G}$ by $\widetilde{\mathcal{E}}$.

Proposition 3.6. *The coarse structures $\widehat{\mathcal{E}}$ and $\widetilde{\mathcal{E}}$ are equivalent.*

Proof. Consider an arbitrary entourage $M \in \mathcal{E}$ and show that $\widetilde{M}$ and $\widehat{M}$ are coarsely equivalent.

Let $([x_1, \ldots, x_n]_G, [y_1, \ldots, y_n]_G) \in \widehat{M}$, this means that there exists $\sigma \in G$ such that $(x_i, y_{\sigma(i)}) \in M$ for all $i = 1, \ldots, n$. Taking the maps $f_1 : n \to X$ sending i to x_i and $f_2 : n \to X$ sending i to $y_{\sigma(i)}$ we obtain that $SP^n_G f_1([1, \ldots, n]_G) = [x_1, \ldots, x_n]_G$ and $SP^n_G f_2([1, \ldots, n]_G) = [y_1, \ldots, y_n]_G$. Note that f_1 and f_2 are M-close.
Hence, $([x_1, \ldots, x_n]_G, [y_1, \ldots, y_n]_G) \in \widetilde{M}$.

To show that every $([x_1, \ldots, x_n]_G, [y_1, \ldots, y_n]_G) \in \widetilde{M}$ is containing in $\widehat{M}$ we choose entourages $U_1, \ldots, U_k \in \mathcal{E}$, points $a_1, \ldots, a_k \in SP^n_G n$ and maps $f_1, \ldots, f_{2k} \in C(n, X)$ for which

$SP^n_G f_1(a_1) = [x_1, \ldots, x_n]_G$, $SP^n_G f_{2k}(a_k) = [y_1, \ldots, y_n]_G$,

$SP^n_G f_{2i}(a_i) = SP^n_G f_{2i+1}(a_{i+1})$ for $i = 1, \ldots, n$ and

f_{2i-1}, f_{2i} are U_i-close for every $i = 1, \ldots, k$, where $U_1 \cdot \ldots \cdot U_k \subset M$.

We have $(x_i, f_2(i)) \in U_1$, $(f_3(i), f_4(i)) \in U_2, \ldots, (f_{2k}(i), y_i) \in U_k$ and

$$[f_{2i}(1), \ldots, f_{2i}(n)]_G = SP^n_G f_{2i}(a_i) = SP^n_G f_{2i+1}(a_{i+1}) = \\ = [f_{2i+1}(1), \ldots, f_{2i+1}(n)]_G.$$

Therefore, we conclude that $f_{2i}(j) = f_{2i+1}(j)$ for every $i = 1, \ldots, k$ and $j = 1, \ldots, n$ and that implies $(x_i, y_i) \in U_1 \cdot \ldots \cdot U_k \subset M$.

Now taking $\sigma = e$, where e is the identity permutation we obtain that $[x_1, \ldots, x_n]_G, [y_1, \ldots, y_n]_G) \in \widehat{M}$. □

4. Normality properties

In this section we define counterparts of the properties from the definition of the normal functor for the functors in the asymptotic category.

Definition 4.1. Recall a map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between coarse spaces to be *monomorphic*, if there exists a map $g\colon f(X) \to Y$ such that the following holds: there exists $U_X \in \mathcal{E}_X$, $U_Y \in \mathcal{E}_Y$, such that for every $x \in X$, $y \in f(X)$ we have $(x, gf(x)) \in U_X$, $(y, fg(y)) \in U_Y$.

Proposition 4.2. *The functor $F\colon \mathbf{CS} \to \mathbf{CS}$ defined above preserves the class of monomorphic maps.*

Proof. Let f, g and U_i, $i = 1, 2$ be as above. The map $Ff : FX \to FY$ is monomorphic, this follows from the commutative diagram:

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "Ff"'] \arrow[rr, bend left, dashed, "id_{FX}"] & Ff(FX) \arrow[r, "Fg"] & FX \arrow[d, "Ff"] \\
Ff(FX) \arrow[r, "Fg"'] \arrow[rr, bend right, dashed, "id_{Ff(FX)}"'] & FX \arrow[r, "Ff"'] \arrow[u, "Ff"'] & Ff(FX)
\end{tikzcd}$$

□

Definition 4.3. Recall a map $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ between coarse spaces to be *epimorphic*, if the image $f(X)$ is coarsely dense in Y. This means that exists $U_Y \in \mathcal{E}_Y$ such that $U_Y(f(X)) \supset Y$.

Proposition 4.4. *The functor $F\colon \mathbf{CS} \to \mathbf{CS}$ preserves the class of epimorphic maps.*

Proof. Let $f : X \to Y$ be an epimorphic map. Show that $Ff : FX \to FY$ is also epimorphic.

Fix an arbitrary $b \in FY$. Since $\mathrm{supp}(b) = y_1, \ldots, y_n$, there exists $b^* \in Fn$ such that $F\phi(b^*) = b$, where $\phi : n \to Y$ acts in the following way: $\phi(i) = y_i$.

The map f is epimorphic, this means that there exists $U_Y \in \mathcal{E}_Y$ for which $Y \subset U_Y(f(X))$. Hence, for every i there exists y_I^* with $y_i \in U_Y(y_i^*)$.

Consider the map $\varphi : n \to X,\ \varphi(i) = x_i,\ x_i \in X$ and $f(x_i) = y_i^*$. Let $a = F\varphi(b^*)$. Finally, we have to show that $(Ff(a), b) \in \hat{U}_Y$. With this aim we notice that:

1) $F\psi(b^*) = a,\ f\varphi(b^*) = b$;

2) φ and ψ are U_Y-close, because $\varphi(i) = y_i,\ \psi(i) = x_i$ and $f(x_i) = y_i^*$. We have $(y_i, y_i^*) \in U_Y$, so we are done.

□

Definition 4.5. Let $f\colon (X, \mathcal{E}_X) \to (Y, \mathcal{E}_Y)$ be a coarse map and $B \subset Y$. A *coarse preimage* of the set B is the set A together with two maps $g : A \to B, j : A \to X$ for which the following holds:

if there are two coarse maps $\alpha : Z \to X,\ \beta : Z \to B$ such that $f\alpha \sim i\beta$, then exists a map $h : Z \to A$ such that $gh \sim \beta,\ jh \sim \alpha$.

For the convenience of the reader let us introduce this definition by the diagram:

$$\begin{tikzcd}
Z \arrow[dr, dashed, "h"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "\beta"'] & & \\
& A \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"'] \\
& B \arrow[r, "i"'] & Y
\end{tikzcd}$$

Proposition 4.6. *Any two coarse preimages of B are coarsely equivalent.*

Proof. Let $(A, g, j), (A', g', j')$ be different coarse preimages of the set B. For the coarse preimage A put $Z = A',\ \alpha = j', \beta = g'$. We obtain that $f\alpha = fj',\ i\beta = ig'$.

Since A' is also a coarse preimage, $fj' \sim ig'$ implies $f\alpha \sim i\beta$. That is why we can find the map $h : A' \to A$ such that $gh \sim g'$ and $jh \sim j'$.

Similarly, one can show that for A' there exists a map $h' : A \to A'$ for which $g'h' \sim g$ and $j'h' \sim j$. Therefore,

$$g' \sim gh \sim g'h'h \Rightarrow h'h \sim id_{A'}$$
$$j \sim j'h' \sim jhh' \Rightarrow hh' \sim id_A$$

This completes the proof. □

Proposition 4.7. *The functor F preserves coarse preimages.*

Proof. Let $f : X \to Y$ be a coarse map and A be a coarse preimage for B.

$$\begin{tikzcd}
\Gamma_\alpha \arrow[dr, dashed, "\beta"'] \arrow[drr, "\pi_2"] & & \\
& A \arrow[r, "j"'] \arrow[d, "g"'] & X \arrow[d, "f"] \\
& B \arrow[r, phantom, "\subset"] & Y
\end{tikzcd}$$

We build the set $\Gamma_\alpha = \{(z, x) | x \in \operatorname{supp}\alpha(z)\} \subset Z \times X$. Note that $f(\pi_2(\Gamma_\alpha)) \subset B$, where π_2 is a projection map on the second factor of the product. Since A is a coarse preimage of B, there exists $\beta : Z \to A$ such that $j\beta \sim \pi_2$. Moreover, $Ff(\alpha(Z)) \subset FB$, whence $\operatorname{supp}(Ff \circ \alpha(Z)) \subset B$ and $\Rightarrow f(\operatorname{supp}\alpha(Z)) \subset B$. Put $\xi(z) = F\beta \circ \alpha(z)$. This map is sending any point $x \in Z$ to $\xi(z) \in FA$ i.e. $\xi : Z \to FA$. It is easy to see that the diagram

$$\begin{tikzcd}
Z \arrow[dr, "\xi"'] \arrow[drr, "\alpha"] & & \\
& FA \arrow[r, "Fj"'] \arrow[d, "Fg"'] & FX \arrow[d, "Ff"] \\
& FB \arrow[r, phantom, "\subset"] & FY
\end{tikzcd}$$

is commutative, thus $FA \cong (Ff)^{-1}(FB)$. This ends the proof. □

Definition 4.8. A functor $F : \mathbf{CS} \to \mathbf{CS}$ is *normal* in $\mathbf{CS}$ if:

1) F preserves weight;
2) F is monomorphic;
3) F is epimorphic;
4) F preserves preimages;
5) F preserves $\emptyset$ (i.e. bounded coarse spaces).

Remark 4.9. There is now sense to consider properties such as preserving intersections and continuity in coarse category. For example, simple intersection between sets of odd natural numbers and even natural numbers is empty set, but in coarse category their intersection is equivalent to all natural row.

Proposition 4.10. *If F is a finitary normal functor of finite degree $n \geq 1$, then F is normal in* $\mathbf{CS}$.

Proof. The preservation of weights is in a follows from the definition of the coarse structure on FX (see definition). All the other properties follow from propositions 4.7, 4.9 and 4.12. □

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Contact information

V. Frider Ivan Franko National University of Lviv, 1 Universitetska Str. 79000 Lviv, Ukraine

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