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Generators and relations for the semigroups of increasing functions on $\mathbb{N}$ and $\mathbb{Z}$

Vadym Doroshenko

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Abstract. The semigroups of all increasing functions over $\mathbb{N}$ and $\mathbb{Z}$ are considered. It is shown that both these semigroups do not admit an irreducible system of generators. In their subsemigroups of cofinite functions all irreducible systems of generators are described. The last semigroups are presented in terms of generators and relations.

Introduction

Semigroups of order-preserving transformations are the object of permanent interest during the latter years. In particular, there are a few interesting algebraic and combinatorial results concerning semigroups of order-preserving partial injections defined on finite chains ([1, 2, 3]). The semigroups of order-preserving transformations are also used for a description of generators in the full transformation semigroups on linearly ordered sets ([4]).

In this paper we consider a countable set endowed with a linear order such that the obtained poset is isomorphic to the set $\mathbb{N}$ of all natural numbers or to the set $\mathbb{Z}$ of all integers with the natural order. We define subsemigroups $O(\mathbb{N})$ and $O(\mathbb{Z})$ of the full transformation semigroup semigroup on the corresponding sets, as semigroups, which consist of transformations preserving linear orders. In other terms $O(\mathbb{N})$ and $O(\mathbb{Z})$ are nothing but the semigroups of all monotone increasing functions of the sets $\mathbb{N}$ and $\mathbb{Z}$ respectively.

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The main goal of this paper is to investigate systems of generators in $O(\mathbb{N})$ and $O(\mathbb{Z})$ and in some their subsemigroups.

The material is organized as follows. In Section 1 we introduce semigroups $O(\mathbb{N})$ and $O(\mathbb{Z})$ and define their subsemigroups $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$ of cofinite functions (i.e. functions with finite complements to their images). In Sections 2, 3 we prove that the semigroups $O(\mathbb{N})$ and $O(\mathbb{Z})$ do not contain a minimal (in other terms irreducible) system of generators. In Section 4 the unique irreducible system of generators of $O_{fin}(\mathbb{N})$ is presented. In a similar manner in the same Section we describe all irreducible systems of generators of $O_{fin}(\mathbb{Z})$. Then in Sections 5 and 6 we find presentations of $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$ respectively in terms of generators and relations. Finally we prove that in finitely generated subsemigroups of $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$ the word problem is solvable.

We refer the reader to [5] for all notions, which will not be defined in this paper.

1. Main definitions and elementary properties

Let $\mathbb{Z}$ and $\mathbb{N}$ be the sets of all integers and all positive integers respectively equipped with the natural linear order. Let M be the set $\mathbb{N}$ or the set $\mathbb{Z}$.

Define $O(M)$ to be the semigroup of all increasing functions from M to M under usual composition. We use the notation fg for the composition $f(g), f, g \in O(M)$.

We call a mapping $f : M \to M$ co-finite if the set $M \setminus f(M)$ is finite. It is easy to see that all co-finite functions from $O(M)$ form a subsemigroup. Denote it by $O_{fin}(M)$. Note that $O(\mathbb{N})$ and $O(\mathbb{Z})$ are uncountable, whereas $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$ are countable. For the convenience we will formally assume that $f(0) = 0$ for each $f \in O(\mathbb{N})$. Note that all increasing functions are injections. Denote by id the identity function. The element id is the unique invertible element of $O(\mathbb{N})$.

For arbitrary integer k define the shift $L_k \in O_{fin}(\mathbb{Z})$ by $L_k(n) = n + k, n \in \mathbb{Z}$.

Note that $L_k L_i(n) = L_k(n + i) = n + k + i$, therefore $L_k L_i = L_{k+i}$. Denote $\mathbf{L} = \{L_k : k \in \mathbb{Z}\}$. Hence $\mathbf{L}$ is a group, isomorphic to the group $(\mathbb{Z}, +)$ and the mapping $L_n \longrightarrow n$ is an isomorphism. It is easy to see that $\mathbf{L}$ is the group of units in $O(\mathbb{Z})$.

Let $f \in O(M)$ and $n \in M$. Define $\Delta f(n) = f(n + 1) - f(n) - 1$. Since f is increasing, we have $\Delta f(n) \geq 0$.

Let us say that f has a jump in point n if and only if $\Delta f(n) > 0$. In this case we say that $\Delta f(n)$ is the height of the jump in n and n is the point of the jump.

Lemma 1. *Let $f \in O(M)$ and $n, m \in M$, $n < m$. Then $f(m) - f(n) \geq m - n$. If $f(m) - f(n) = m - n$ then for each integer k such that $n \leq k \leq m$ the equality $f(k) = f(n) + k - n$ holds.*

Proof. We will prove the statement for $M = \mathbb{N}$. In the case $M = \mathbb{Z}$ the proof is analogous. Let $f \in O(\mathbb{N})$ and $n, m \in \mathbb{N}$, $n < m$. Then

$f(m) - f(n) = (f(m) - f(m-1)) + (f(m-1) - f(m-2)) + \ldots + (f(n+1) - f(n)) = \Delta f(m-1) + \ldots + \Delta f(n) + m - n \geq m - n$.

If $f(m) - f(n) = m - n$ then $\Delta f(l) = 0$ for all $n \leq l < m$ and the statements follows. □

Note that for any $f \in O(\mathbb{Z})$ the image of f is not bounded from above, and for $f \in O(\mathbb{Z})$ the image of f is not bounded from below.

Lemma 2. *Let $f, g \in O(M)$. If $fg = f$ or $gf = f$ then $g = id$.*

Proof. Since any f is injective, $fg = f$ implies $g = id$.

Let $f, g \in O(M)$ be such that $gf = f$. If $g \neq id$, we have $g(x) \neq x$ for some x. Assume $g(x) > x$ (the case $g(x) < x$ is analogous). Then $g(y) > y$ for any $y > x$ since g is increasing, and hence $gf(t) > f(t)$ for any t such that $f(t) > x$. Such t exists since the image of f is not bounded from above (see remark above). A contradiction. □

Let f be from $O_{fin}(M)$. Define $|f|$ as the cardinality of $M \backslash f(M)$.

Note that if $f \in O_{fin}(\mathbb{N})$ and $|f| = 0$ then $f = id$. If $f \in O_{fin}(\mathbb{Z})$ and $|f| = 0$ then $f \in \mathbf{L}$.

2. There is no minimal system of generators in $O(\mathbb{N})$

Let us define an order on the semigroup $O(\mathbb{N})$.

For $f_1, f_2 \in O(\mathbb{N})$ set $f_1 \prec f_2$ if and only if $f_1 \neq f_2$ and $f_1(n) \leq f_2(n)$ for arbitrary n.

Lemma 3. *Let $f \in O(\mathbb{N}), f \neq id$. Then*

1) for arbitrary $n \in \mathbb{N} : f(n) \geq n$;

2) there exists $N \in \mathbb{N}$ such that $f(n) > n$ for arbitrary $n \geq N$.

Proof. Since f is injective, we have $|f(\{1, ..., n\})| = n$ for every n. Since f is increasing, this implies $f(n) \geq n$. If $f \neq id$, there should exist x such that $f(x) \neq x$ and hence $f(x) > x$, implying $f(y) > y$ for all $y > x$. □

Lemma 4. *Let $f_1 \in O(\mathbb{N})$,$f_2 \in O(\mathbb{N})$, $g = f_1 f_2$, $f_1 \neq id, f_2 \neq id$. Then $f_1 \prec g, f_2 \prec g$.*

Proof. By Lemma 2 $g \neq f_1$ and $g \neq f_2$. By Lemma 3 we have $f_1(f_2(n)) \geq f_2(n)$ for arbitrary n. Hence $f_2 \prec g$.

$f_2(n) \geq n$ and f_1 is increasing imply $f_1(f_2(n)) \geq f_1(n)$. Therefore $f_1 \prec g$. □

Lemma 5. *Let $f \in O(\mathbb{N})$ and one of the following statements hold:*

1) f has at least 2 jumps;

2) f has only 1 jump, moreover, the height of this jump is at least 2.

Then there exist $f_1, f_2 \in O(\mathbb{N})$ such that $f = f_1 f_2, f_1 \neq id, f_2 \neq id$

Proof. 1) Let f have at least two jumps. Let $n_1, n_2, \ldots$ be all points of jumps and $n_1 < n_2 < \ldots$ Define $f_1, f_2 \in O(\mathbb{N})$ by the following equalities

$$f_1(n) = \begin{cases} n, & n \leq n_1 \\ n + \Delta f(n_1), & n > n_1, \end{cases}$$

$$f_2(n) = \begin{cases} f(n), & n \leq n_1 \\ f(n) - \Delta f(n_1), & n > n_1 \end{cases}$$

Then $f = f_1 f_2$ and $f_1, f_2 \neq id$.

2) Let f have exactly 1 jump, moreover, assume that the height of this jump is at least 2. Let n_1 be the point of this jump. Define f_1, f_2 by the following equalities

$$f_1(n) = \begin{cases} n, & n \leq n_1 \\ n + 1, & n > n_1 \end{cases}$$

$$f_2(n) = \begin{cases} n, & n \leq n_1 \\ n + \Delta f(n_1) - 1, & n > n_1 \end{cases}$$

Then $f = f_1 f_2$ and $f_1, f_2 \neq id$. □

Theorem 2.1. *There is no minimal system of generators in $O(\mathbb{N})$.*

Proof. Let $\mathcal{G}$ be a system of generators of $O(\mathbb{N})$. It is easy to see that the elements of the $O(\mathbb{N})$ are in one-to-one correspondence with infinite subsets of $\mathbb{N}$. Hence the semigroup $O(\mathbb{N})$ has uncountable cardinality. Therefore $\mathcal{G}$ is uncountable. The set of functions that have exactly 1 jump is countable. Hence there exists a function $f \in \mathcal{G}$ such that f has more than 1 jump.

By Lemma 5 there exist $f_1, f_2 \in O(\mathbb{N})$ such that $f_1 \neq id, f_2 \neq id$, $f = f_1 f_2$. Since $\mathcal{G}$ is a system of generators, f_1 and f_2 are products of elements from $\mathcal{G}$:

$$f_1 = g_1 \ldots g_n, f_2 = h_1 \ldots h_m,$$

where $g_1, \ldots, g_n, h_1, \ldots h_m \in \mathcal{G}$.

By Lemma 4 $g_i \prec f_1 \prec f, i = 1 \ldots n$, $h_j \prec f_2 \prec f, j = 1 \ldots m$. This implies that f does not coincide with any of the functions from

$$g_1, \ldots g_n, h_1, \ldots h_m.$$

Hence $\mathcal{G} \backslash \{f\}$ is a system of generators of $O(\mathbb{N})$, implying that $\mathcal{G}$ is not a minimal system of generators. $\square$

3. There is no minimal system of generators in $O(\mathbb{Z})$

Denote by S_1 the set of all functions from $O(\mathbb{Z})$ such that the set of jump points of each function from S_1 is bounded from below, i.e. $f \in S_1$ if and only if there exist $N \in \mathbb{Z}$ such that $\Delta f(n) = 0$ for arbitrary $n \leq N, n \in \mathbb{Z}$.

Denote by S_2 the set of all function from $O(\mathbb{Z})$ such that the set of jump points of each function from S_2 is bounded from above.

It is easy to see that S_1, S_2 are subsemigroups of $O(\mathbb{Z})$.

Lemma 6. *The set $O(\mathbb{Z}) \backslash S_1$ is an ideal of $O(\mathbb{Z})$.*

Proof. Let $f \in O(\mathbb{Z}) \setminus S_1$ and $g \in O(\mathbb{Z})$. Then the set of jump points of f is not bounded from below. Note that $n \in \mathbb{Z}$ is a jump point of gf provided that n is a jump point of f. Then $gf \in O(\mathbb{Z}) \setminus S_1$. Since the image of g is not bound from below, for arbitrary jump point n of f there exists $m \in \mathbb{Z}$ such that $g(m) \leq n < g(m+1)$. Then m is a jump point for fg. This implies $fg \in O(\mathbb{Z}) \setminus S_1$. $\square$

Lemma 7. *Let H be a semigroup and K be a subsemigroup of H such that $H \backslash K$ is an ideal. If there are no minimal systems of generators in K then there are no minimal systems of generators in H.*

Proof. Assume that K does not contain any minimal system of generators. Let $\mathcal{G}$ be a system of generators of H. Since $H \backslash K$ is an ideal, each element from K can be expressed only as a product of elements from K. Hence $\mathcal{G}$ contains system of generators of K. Hence $\mathcal{G} \bigcap K$ is a system of generators for K, which can not be minimal by assumptions. Thus $\mathcal{G}$ is not minimal either. $\square$

For $f \in O(\mathbb{Z})$ define $\rho(f) = \sup\limits_{n \in \mathbb{Z}} \Delta f(n)$. Note that $\rho(f) \leq +\infty$.

Lemma 8. *Let $f_1, f_2 \in S_1$.*

1) If $\rho(f_1) < \infty$ and $\rho(f_2) < \infty$ then $\max(\rho(f_1), \rho(f_2)) \leq \rho(f_1 f_2) \leq (\rho(f_1)+1)(\rho(f_2)+1) - 1$.

2) If $\rho(f_1) = \infty$ or $\rho(f_2) = \infty$ then $\rho(f_1 f_2) = \infty$.

Proof. 1) There exist $n_1, n_2 \in \mathbb{Z}$ such that $\rho(f_1) = \Delta f(n_1), \rho(f_2) = \Delta f(n_2)$. Let $f = f_1 f_2$.

Then

$$\Delta(f_1f_2)(n_2) = (f_1f_2)(n_2+1) - (f_1f_2)(n_2) - 1 \geq \\ f_2(n_2+1) - f_2(n_2) - 1 = \Delta f(n_2).$$

Hence $\rho(f) \geq \rho(f_2)$.

There exists $m \in \mathbb{Z}$ such that $f_2(m) \leq n_1 < f_2(m+1)$. Then

$$\Delta(f_1f_2)(m) = f_1(f_2(m+1)) - f_1(f_2(m)) - 1 \geq \\ f_1(n_1+1) - f_1(n_1) - 1 = \Delta f_1(n_1).$$

Hence $\rho(f) \geq \rho(f_1)$, and thus $\max(\rho(f_1), \rho(f_2)) \leq \rho(f_1 f_2)$.

We have

$$\Delta(f_1f_2)(n) = f_1(f_2(n+1)) - f_1(f_2(n)) - 1 = \sum_{k=f_2(n)}^{f_2(n+1)-1} (\Delta f_1(k)+1) - 1 \leq \\ (\rho(f_1)+1)(f_2(n+1) - f_2(n)) - 1 \leq (\rho(f_1)+1)(\rho(f_1)+1) - 1$$

Thus $\rho(f_1f_2) \leq (\rho(f_1)+1)(\rho(f_2)+1) - 1$.

2) Let $\rho(f_2) = \infty$. Then there exist a sequence $\{n_k : k \geq 1\} \subset \mathbb{Z}$ such that $\lim_{k\to\infty} \Delta f_2(n_k) = \infty$. From the proof of part 1) of this Lemma $\Delta f(n_k) \geq \Delta f_2(n_k)$. Therefore $\lim_{k\to\infty} \Delta f(n_k) = \infty$ and $\rho(f) = \infty$.

If $\rho(f_1) = \infty$ then the proof is analogous. □

Define $T = \{f \in S_1 | \rho(f) < \infty\}$. By Lemma 8 T is a subsemigroup of S_1.

Lemma 9. *The set $S_1 \backslash T$ is an ideal in S_1.*

Proof. If $f \in S_1 \backslash T$ then $\rho(f) = \infty$. For $g \in S_1$ by Lemma 8 we have $\rho(fg) = \rho(gf) = \infty$. □

Lemma 10. *If $f \in T$ is such that $\rho(f) > 1$ then there exist $f_1, f_2 \in T$ such that $f_1 f_2 = f$ and inequalities $\rho(f_1) < \rho(f), \rho(f_2) < \rho(f)$ hold.*

Proof. Let $n_1 < n_2 < \ldots$ be the all jump points of f. Define f_1, f_2 in the following way

$$f_1(n) = \begin{cases} f(n), & \text{if } n \leq n_1 \\ f(n) - k, & \text{if } n_k < n \leq n_{k+1}, k \geq 1, \end{cases}$$

$$f_2(n) = \begin{cases} n, & n \leq f(n_1) \\ n + k, & f(n_k) - k + 1 < n \leq f(n_{k+1}) - k. \end{cases}$$

The set of jump point of f_1 is bounded from below by n_1 as $f \in S_1$. One verifies that $f = f_2 f_1$, $\Delta f_1(n) = \Delta f(n) - 1$ and $\Delta f_2(n) \leq 1$. Then $\rho(f_1) = \rho(f) - 1 > 0$ and $\rho(f_2) = 1 < \rho(f)$. $\square$

Define $T_1 = \{f \in T | \rho(f) \leq 1\}$. This means that the height of an arbitrary jump of function from T_1 is equal to 1.

From Lemma 10 it follows that T_1 is a system of generators for T .

Lemma 11. *The set $T \backslash T_1$ is an ideal in T.*

Proof. The statement is an immediate corollary of the part 1 of Lemma 8. $\square$

Let $f \in T_1$ and $n_1 < n_2 < \ldots$ be all jump points of f. Define $\tau(f) = \min\limits_{k \geq 1}(n_{k+1} - n_k)$ if f has at least two jump points and $\tau(f) = 0$ otherwise.

Lemma 12. *Let $f_1, f_2 \in T_1$. If the product $f = f_1 f_2$ is contained in T_1 then equalities $\tau(f) \leq \tau(f_1), \tau(f) \leq \tau(f_2)$ hold.*

Proof. Let $n_1 < n_2 < \ldots$ and $m_1 < m_2 < \ldots$ be all jump points of f_1,f_2 respectively. Since f has jumps in m_k for each $k \geq 1$ then $\tau(f) \leq \tau(f_2)$.

For arbitrary $k \geq 1$ there exists l_k such that $f_2(l_k) \leq n_k < f_2(l_k + 1)$. If for some $k \geq 1$ we have $f_2(l_k) < n_k$ then l_k is a point of jump of f_2 and $\Delta f(l_k) = f_1(f_2(l_k + 1)) - f_1(f_2(l_k)) - 1 \geq f_2(l_k + 1) - f_2(l_k) = 2 > 1$. We have $f \notin T_1$. Hence $f_2(l_k) = n_k$. It is easy to see that f has jumps in $l_k, k \geq 1$ and $n_{k+1} - n_k = f_2(l_{k+1}) - f(l_k) \geq l_{k+1} - l_k$. Then $\tau(f) \leq \tau(f_1)$. $\square$

Lemma 13. *If $f \in T_1$ then there exist $f_1, f_2 \in T_1$ such that $f = f_1 f_2$ and $\tau(f) < \tau(f_1), \tau(f) < \tau(f_2)$.*

Proof. Let $n_1 < n_2 < \ldots$ be all jump points of f. Define f_1, f_2 by equalities

$$f_1(n) = n + |\{l \geq 1 | f(n_{2l}) < n\}|$$

and

$$f_2(n) = f(n) - |\{l \geq 1 | n_{2l} < n\}|$$

for $n \in \mathbb{Z}$.

One verifies that $f = f_1 f_2$. The set of all jump points of f_2 is $\{n_{2k-1} : \ k \geq 1\}$. Hence $\tau(f) < \tau(f_2)$. The set of all jump points f_1 is $\{f_2(n_{2k}) : \ k \geq 1\}$. Since between the jump points n_{2k+2} and n_{2k} of f there is exactly one jump point of f_2 we have $f_2(n_{2k+2}) - f_2(n_{2k}) = n_{2k+2} - n_{2k} + 1$. Hence $\tau(f) < \tau(f_1)$. $\square$

Theorem 3.1. *There is no minimal system of generators in $O(\mathbb{Z})$.*

Proof. Let $\mathcal{G}$ be a system of generators of T. By Lemma 11 the subsemigroup generated by $\mathcal{G}_1 = \mathcal{G} \bigcap T_1$ contains T_1.

Let $f \in \mathcal{G}_1$. By Lemma 13 there exist $f_1, f_2 \in T_1$ such that $f = f_1 f_2$ and $\tau(f) < \tau(f_1), \tau(f) < \tau(f_2)$. There exist elements

$$g_1 \dots, g_n, h_1, \dots, h_m \in \mathcal{G}_1$$

such that $f_1 = g_1, \dots g_n, f_2 = h_1 \dots h_m$. We have

$$f = f_1 f_2 = g_1 \dots g_n h_1 \dots h_m.$$

From Lemma 12 it follows that

$$\tau(f) < \tau(g_1) \dots \tau(f) < \tau(g_n), \tau(f) < \tau(h_1) \dots \tau(f) < \tau(h_m).$$

Hence f does not coincide with any element of $g_1, \dots, g_n, h_1, \dots, h_m$. Therefore $\mathcal{G} \backslash \{f\}$ is a system of generators of T. Hence $\mathcal{G}$ is not a minimal system of generators of T.

By Lemmas 7 and 9 we have that there are no minimal system of generators in S_1.

By Lemmas 7 and 6 we have that there are no minimal system of generators in $O(\mathbb{Z})$. □

4. Minimal systems of generators in $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$

Lemma 14. *Let M be the set $\mathbb{N}$ or the set $\mathbb{Z}$. Then for arbitrary $f_1, f_2 \in O_{fin}(M)$ the equality $|f_1 f_2| = |f_1| + |f_2|$ holds.*

Proof. By the definition of $|\cdot|$ we have $|f_1 f_2| = |M \setminus (f_1 f_2)(M)|$, $|f_1| = |M \setminus f_1(M)|$, $|f_2| = |M \setminus f_2(M)|$.

Let $x \in M \backslash (f_1 f_2)(M)$. This means that $x \in M \backslash f_1(M)$ or $x \in f_1(M)$ and the pre-image $f_1^{-1}(x)$ belongs to $M \backslash f_2(M)$.

Hence $|f_1 f_2| = |f_1| + |f_2|$. □

Lemma 15. *1) For arbitrary $f \in O_{fin}(\mathbb{N})$ there exist $f_1, f_2 \dots f_n \in O_{fin}(\mathbb{N})$ such that $|f_1| = |f_2| = \dots = |f_n| = 1$ and $f = f_1 f_2 \dots f_n$, where $n = |f|$.*

2) Let $f \in O_{fin}(\mathbb{N})$ and $|f| = 1$. Then the equality $f = f_1 f_2$ implies $f_1 = id$ or $f_2 = id$.

Proof. 1) We prove this statement by induction on $|f|$. If $|f| = 1$ the statement holds. Let $|f| > 1$. By Lemma 5 there exist $f_1, f_2 \in O(\mathbb{N})$,

$|f_1| \neq 0, |f_2| \neq 0$ such that $f = f_1 f_2$. By Lemma 14 $|f| = |f_1| + |f_2|$. Hence $|f_1| < |f|, |f_2| < |f|$ and the statement follows.

2) Let $|f| = 1$ and $f = f_1 f_2$. By Lemma 14 $|f| = |f_1| + |f_2|$. Hence $|f_1| = 0$ or $|f_2| = 0$. □

Theorem 4.1. *There exists the unique minimal system of generators in $O_{fin}(\mathbb{N})$, namely $\{id, e_1, e_2, ...\}$, where*

$$e_k(n) = \begin{cases} n, & n \leq k \\ n+1, & n > k \end{cases}.$$

Proof. It is easy to see that the set $e_0, e_1, \ldots$ is the set of all function from $O_{fin}(\mathbb{N})$ which have exactly 1 jump, moreover, this jump has height 1.

Let $\mathcal{G} = \{e_k : k \geq 0\} \bigcup \{id\}$. Let us prove that $\mathcal{G}$ is the unique minimal system of generators of $O_{fin}(\mathbb{N})$.

All elements of $O_{fin}(\mathbb{N})$ are products of elements from $\mathcal{G}$ by the first part of Lemma 15. Hence $\mathcal{G}$ is a system of generators.

By the second part of Lemma 15 each element of $\mathcal{G}$ is indecomposable into a non-trivial product of non-identity elements of $O_{fin}(\mathbb{N})$. Hence $\mathcal{G}$ is minimal.

Let $\mathcal{G}_1$ be a minimal system of generators of $O_{fin}(\mathbb{N})$. By part 2 of Lemma 15 $\mathcal{G}_1$ contains all elements of $\mathcal{G}$. Hence $\mathcal{G}$=$\mathcal{G}_1$.

□

Lemma 16. *1) For arbitrary $f \in O_{fin}(\mathbb{Z})$ there exist $f_1, f_2 \ldots f_n \in O_{fin}(\mathbb{Z})$ such that $|f_1| = |f_2| = \ldots = |f_n| = 1$ and $f = f_1 f_2 \ldots f_n$, where $n = |f|$.*

2) Let $f \in O_{fin}(\mathbb{Z})$ and $|f| = 1$. Then the equality $f = f_1 f_2$ implies $f_1 \in \mathbf{L}$ or $f_2 \in \mathbf{L}$.

Proof. The proof is analogous to that of Lemma 15. □

Theorem 4.2. *Let $\mathcal{G}$ be a minimal system of generators of $O_{fin}(\mathbb{Z})$. Then $\mathcal{G}$ has the form $\mathcal{G} = \{f, L_{i_1}, L_{i_2}, \ldots L_{i_n}\}$, where the function f has exactly one jump which, additionally, has height 1, and the set of indices $\{i_1, i_2, \ldots i_n\}$ is a minimal system of generators of the semigroup $(\mathbb{Z}, +)$.*

Proof. Let $\mathcal{G}$ be a minimal system of generators of $O_{fin}(\mathbb{Z})$. The complement of $\mathbf{L}$ in $O_{fin}(\mathbb{Z})$ is an ideal. Hence the intersection $\mathcal{G} \bigcap \mathbf{L}$ contains a minimal system of generators of $\mathbf{L}$. By Lemma 16 the elements with exactly one jump, which, additionally, has height 1, can not be decomposed into a nontrivial product of noninvertible element of $O_{fin}(\mathbb{Z})$. Therefore $\mathcal{G}$ must contain some element f such that $|f| = 1$.

For this element we have

$$f(n) = \begin{cases} n - k_1, & n \le l_1 \\ n - k_1 + 1, & n > l_1 \end{cases},$$

for some $k_1, l_1 \in \mathbb{Z}$.

Then for arbitrary k_2, l_2 we have

$$L_{l_2-l_1+k_2-k_1} f L_{l_1-l_2} = \begin{cases} n - k_2, & n \le l_2 \\ n - k_2 + 1, & n > l_2 \end{cases}.$$

Therefore all elements that have exactly one jump belong to the subsemigroup, generated by f and $\mathbf{L}$, by Lemma 16. So the set, described in the statement of the theorem is a generating system.розписать

This implies that $\mathcal{G} = \{f, L_{i_1}, L_{i_2}, \dots L_{i_n}\}$, where the function f has exactly one jump which, additionally, has height 1 and the set $\{i_1, i_2, \dots i_n\}$ is a minimal system of generators of the semigroup $(\mathbb{Z}, +)$. □

5. Defining relations for $O_{fin}(\mathbb{N})$

Consider the semigroup $O_{fin}(\mathbb{N})$.

Let $e_k(n) = \begin{cases} n, & n \le k \\ n+1, & n > k \end{cases}$

By Section 4 $\{e_k : k \ge 0\} \bigcup \{id\} \subset O_{fin}(\mathbb{N})$ is the unique minimal system of generators of $O_{fin}(\mathbb{N})$.

Lemma 17. *The hollowing equalities hold in* $O_{fin}(\mathbb{N})$

$$e_k e_l = e_{l+1} e_k, \quad k \le l \tag{1}$$

Proof. Let $k \le l$.

$$e_k e_l(n) = \begin{cases} e_k(n), & n \le l \\ e_k(n+1), & n > l \end{cases} = \begin{cases} n, & n \le k \\ n+1, & k < n \le l \\ n+2, & n > l \end{cases}$$

$$e_{l+1} e_k(n) = \begin{cases} e_{l+1}(n), & n \le k \\ e_{l+1}(n+1), & n > k \end{cases} = \begin{cases} n, & n \le k \\ n+1, & k < n \le l \\ n+2, & n > l \end{cases}$$

Hence $e_k e_l = e_{l+1} e_k$. □

Let F be the free semigroup generated by the set $\{x_k : k \geq 0\}$.

We call an element w of F canonical if it has the form

$$x_{n_1}^{t_1} x_{n_2}^{t_2} \dots x_{n_k}^{t_k}, \text{ де } n_1 < n_2 < \dots < n_k, t_i > 0, 1 \leq i \leq n. \tag{2}$$

We say that an element, $g = e_{i_1} e_{i_2} \dots e_{i_m}$, from $O_{fin}(\mathbb{N})$ is written in a canonical form if the word $x_{i_1} x_{i_2} \dots x_{i_m}$ is canonical.

Lemma 18. *Each element of $O_{fin}(\mathbb{N})$ can be uniquely written in a canonical form.*

Proof. Let $g = e_{n_1}^{t_1} e_{n_2}^{t_2} \dots e_{n_k}^{t_k}$ be an element in a canonical form. Then the function g has jumps in points $n_1, \dots n_k$ of heights $t_1, t_2 \dots t_k$ respectively. Since any function from $O_{fin}(\mathbb{N})$ is uniquely determined by the jump points and their heights, each function from $O_{fin}(\mathbb{N})$ can not have more than one canonical form.

Let us prove that each function $f \in O_{fin}(\mathbb{N})$ has a canonical form. Let $n_1 < \dots < n_k$ be all points of jumps of f and $t_1, \dots, t_k$ be their heights. One easily checks that $f = e_{n_1}^{t_1} e_{n_2}^{t_2} \dots e_{n_k}^{t_k}$ is a canonical form. □

We say that the words $w = w_1 x_k x_l w_2$ and $w' = w_1 x_{l+1} x_k w_2$ $k \leq l$, $w_1, w_2 \in F$ are obtained from each other by an elementary transformation.

Define a binary relation $\sim$ on the semigroup F. For $w_1, w_2 \in F$ let $w_1 \sim w_2$ if and only if w_1 can be obtained from w_2 by a finite number of elementary transformations (possibly none). It is easy to see that $\sim$ is an equivalence.

Lemma 19. *The set of all canonical words forms a cross section of the equivalence classes of the relation $\sim$ introduced above.*

Proof. Let $w = x_{n_1} x_{n_2} \dots x_{n_k}$ be a word from the free semigroup F. Call $x_i x_j$ an inversion if $i > j$. Given w with at least one inversion we can change this inversion to $x_j x_{i-1}$ by an elementary transformation, reducing the total number of inversions in w. The proof is completed by induction. □

Theorem 5.1. *The semigroup $O_{fin}(\mathbb{N})$ has the following presentation by generators and relations:*

$$\langle x_0, x_1, \dots | x_k x_l = x_{l+1} x_k, k \leq l \rangle$$

Proof. Define a homomorphism $\varphi : F \to O_{fin}(\mathbb{N})$ by the rule $\varphi(x_k) = e_k, k \geq 0$.

Since the set $\{e_k : k \geq 0\}$ is a system of generators of $O_{fin}(\mathbb{N})$ then $Im\varphi = O_{fin}(\mathbb{N})$.

Let us prove that $Ker\varphi = \sim$. Let $w_1, w_2 \in F$.

If $w_1 \sim w_2$ then $\varphi(w_1) = \varphi(w_2)$ by Lemma 17.

From Lemma 19 canonical words form a cross-section for the equivalence classes with respect to $\sim$. They are mapped to different elements in $O(\mathbb{N})$ by Lemma 18. The statement follows. □

Consider Richard Thompson's group F. It has the presentation

$$F = \langle x_0, x_1, \dots | x_l x_k = x_k x_{l+1}, k < l \rangle$$

(see for example [6]). Then we directly obtain that the semigroup, anti-isomorphic to $O_{fin}(\mathbb{N})$, is a homomorphic image of the semigroup having the same presentation as the group F.

6. Defining relations for $O_{fin}(\mathbb{Z})$ with respect to 3 generators

Consider the semigroup $O_{fin}(\mathbb{Z})$.

Define $f \in O(\mathbb{Z})$ as follows $f(n) = \begin{cases} n, & n \leq 0 \\ n+1, & n > 0 \end{cases}$. By Section 4 $\{f, L_1, L_{-1}\}$ is a minimal system of generators of $O_{fin}(\mathbb{Z})$. Set $f_k = L_1^k f L_{-1}^k, k \in \mathbb{Z}$, where $L_t^k = L_{-t}^{-k}$ for $k < 0$, $t = \pm 1$.

We have

$$f_k(n) = (L_1^k f L_{-1}^k)(n) = L_1^k(f(n-k)) = \begin{cases} n, & n \leq k \\ n+1, & n > k \end{cases}.$$

Lemma 20. *The following equalities hold in $O_{fin}(\mathbb{Z})$*

$$f_k f_l = f_{l+1} f_k, \quad k < l \tag{3}$$

Proof. Direct calculation. □

It is easy to see that the following equalities hold in $O_{fin}(\mathbb{Z})$:

$$L_1 L_{-1} = L_{-1} L_1 = id. \tag{4}$$

Let $g \in O_{fin}(\mathbb{Z})$, $n_1 < \ldots < n_k$, be all points of jumps of g and $t_1, \ldots, t_k$ be their heights. For arbitrary $n \leq n_1$ we have $g(n) - n = g(n_1) - n_1$. We say that $h = g(n_1) - n_1$ is an initial height of g. We have

$$f(n) = \begin{cases} n+h, & n \leq n_1 \\ n+h+\sum_{i=1}^{j} \Delta f(n_i), & n_j < n \leq n_{j+1}, 1 \leq j < k \\ n+h+\sum_{i=1}^{k} \Delta f(n_i), & n_k \leq n \end{cases}.$$

Hence the elements from $O_{fin}(\mathbb{Z})$ are in a one-to-one correspondence with the triples $(h, \{n_1, \dots, n_k\}, \{t_1, \dots, t_k\})$, where h is the initial weight, $\{n_1, \dots, n_k\}$ are jump points, and $\{n_1, \dots, n_k\}$ are heights of the jumps.

Let F be the free semigroup generated by the set $\{x, y_1, y_{-1}\}$. Define $x_k = y_1^k x y_{-1}^k, k \in \mathbb{Z}$, where $y_t^k = y_{-t}^{-k}$, for $k < 0$, $t = \pm 1$. We call an element, w, of F canonical if it has the form

$$y_q^m x_{n_1}^{t_1} x_{n_2}^{t_2} \dots x_{n_k}^{t_k}, \text{ де } n_1 < n_2 < \dots < n_k, t_i > 0, 1 \le i \le n. \qquad (5)$$

We say that an element $g = L_t^k f_{i_1} f_{i_2} \dots f_{i_m}$ from $O_{fin}(\mathbb{N})$ is written in a canonical form if the word $y_t^k x_{i_1} x_{i_2} \dots x_{i_m}$ is canonical.

Lemma 21. *Each element of $O_{fin}(\mathbb{Z})$ can be uniquely written in a canonical form.*

Proof. Let $g = y_q^m f_{n_1}^{t_1} f_{n_2}^{t_2} \dots f_{n_k}^{t_k}$ be an element in a canonical form. Then the function g has the jumps of heights $t_1, t_2 \dots t_k$ in points $n_1, \dots n_k$ respectively and the initial height is mq. Since each function from $O_{fin}(\mathbb{Z})$ is uniquely determined by the jump points, their height and the initial height, each function from $O_{fin}(\mathbb{Z})$ can not have more then one canonical form.

Let us prove that each function $f \in O_{fin}(\mathbb{Z})$ has a canonical form. Let $n_1 < \dots < n_k$ be all jump points of f and $t_1 \dots t_k$ be their heights and m be initial height of f. We have $f = h_{sgn(m)}^{|m|} f_{n_1}^{t_1} f_{n_2}^{t_2} \dots f_{n_k}^{t_k}$ is a canonical form (where $sgn(m)$ is a sign of m). $\square$

We say that the words $w = w_1 x_k x_l w_2$ and $w' = w_1 x_{l+1} x_k w_2 \;\; k \le l$, $w_1, w_2 \in F$ are obtained from each other by an elementary transformation.

Define a binary relation $\sim$ on the semigroup F. For $w_1, w_2 \in F$ let $w_1 \sim w_2$ if and only if w_1 can be obtained from w_2 by a finite number (possibly none) of elementary transformations. It is easy to see that $\sim$ is an equivalence.

Lemma 22. *The set of all canonical words forms a cross section of the equivalence classes of the relation $\sim$ introduced above.*

Proof. Let $w = y_{p_0}^{t_0} x y_{p_1}^{t_1} x \dots x y_{p_m}^{t_m}$, where $p_i = \pm 1, t_i \ge 0, 0 \le i \le n$. Using elementary transformations we obtain

$$\begin{aligned} & y_{p_0}^{t_0} x \dots x y_{p_m}^{t_m} \sim \\ & \quad y_{p_0}^{t_0} x \dots x y_{p_{m-1}}^{t_{m-1}} y_{p_m}^{t_m} (y_{-p_m}^{t_m} x y_{p_m}^{t_m}) \sim y_{p_0}^{t_0} x \dots x y_{p'_{m_1}}^{t'_{m_1}} (y_{-p_m}^{t_m} x y_{p_m}^{t_m}) \sim \\ & \qquad y_{p_0}^{t_0} x \dots y_{p'_{m_1}}^{t'_{m_1}} (y_{-p'_{m_1}}^{t'_{m_1}} x y_{p'_{m_1}}^{t'_{m_1}}) x_{-p_m} \sim \dots \sim y_l^k x_{-n'_1} x_{-n'_2} \dots x_{-n'_m}. \end{aligned}$$

We can transform obtaining word to canonical how in proof of Lemma 19.

□

Theorem 6.1. *The semigroup $O_{fin}(\mathbb{Z})$ has the following presentation by generators and relations:*

$$\langle x, y_1, y_{-1} | x_k x_l = x_{l+1} x_k, k \leq l, y_1 y_{-1} = 1, y_{-1} y_1 = 1 \rangle.$$

Proof. Define a homomorphism $\varphi : F \to O_{fin}(\mathbb{Z})$ by the rule $\varphi(x) = f, \varphi(y_1) = L_1, \varphi(y_{-1}) = L_{-1}$.

Analogously to the proof theorem 5.1 we have

1)$Im\varphi = O_{fin}(\mathbb{Z})$

2)$Ker\varphi = \sim$.

The statement of the theorem follows. □

Corollary 1. *The word problem is solvable in each finitely generated subsemigroup of $O_{fin}(\mathbb{N})$ and $O_{fin}(\mathbb{Z})$.*

Proof. Let us proof the statement for $O_{fin}(\mathbb{N})$, the proof for $O_{fin}(\mathbb{Z})$ is analogous.

Let H be a finitely generated subsemigroup of $O_{fin}(\mathbb{N})$ and $h_1, \ldots, h_n$ be generators of H. Let $\{e_k : k \geq 0\}$ be the minimal system of generators of $O_{fin}(\mathbb{N})$ from Section 4. Let $h_i = e_{l_{i1}} e_{l_{i2}} \ldots e_{l_{im_i}}$ for some $1 \leq i \leq n, n_i \geq 0$. Denote by N the largest number among $l_{ij}, 1 \leq i \leq n, 1 \leq j \leq m_i$. Then the semigroup H is contained in the subsemigroup of $O_{fin}(\mathbb{N})$, generated by $e_0, e_1 \ldots e_N$.

Let $h_{i_1} h_{i_2} \ldots h_{i_q}$ and $h_{j_1} h_{j_2} \ldots h_{j_r}$ be words over the generating system of H, which represent some elements w and v from H. We can represent w and v over $e_0, \ldots, e_N$. Using elementary transformations from Section 5 we can write w and v in canonical forms. Since any element of $O_{fin}(\mathbb{N})$ has a unique canonical form then $w = v$ if and only if their canonical forms are the same. □

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Contact information

V. Doroshenko Kyiv Taras Shevchenko University,
Department of Mechanics and Mathematics,
Volodymyrska, 60
Kyiv 01033, Ukraine
E-Mail: dvadim@univ.kiev.ua

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