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# SINE-GORDON TRANSFORMATIONS IN NONEQUILIBRIUM SYSTEMS OF BROWNIAN PARTICLES *

# ПЕРЕТВОРЕННЯ СИНУС-ГОРДОН У НЕРІВНОВАЖНИХ СИСТЕМАХ БРАУНІВСЬКИХ ЧАСТИНОК

Finite volume grand canonical correlation functions of nonequilibrium systems of $d$-dimensional Brownian particles, interacting through a regular (long-range) pair potential with integrable first partial derivatives, are expressed in terms of expectation values of Gaussian random field. The initial correlation functions coincide with the Gibbs correlation functions corresponding to a more general pair long range potential. Nonequilibrium Euclidean action is introduced, satisfying a thermodynamic stability property.

Для нерівноважної багатокомпонентної системи браунівських частинок, що взаємодіють завдяки (далекосяжному) парному потенціалу з інтегровними частковими похідними другого порядку, в області скінченного об'єму кореляційні функції великого канонічного ансамблю виражені у термінах математичного сподівання функцій кількох гауссівських випадкових полів. Початкові кореляційні функції збігаються з гіббсівськими кореляційними функціями, що відповідають більш загальному парному потенціалу взаємодії. Введена нерівноважна евклідівська дія, що задовольняє умову термодинамічної стійкості.

**1. Introduction.** Nonequilibrium system of $n$ interacting Brownian particles is described by the Smoluchowski equation for the probability density $\rho_0(X_n, t)$ of finding $n$ particles in the point $X_n = (x_1, \ldots, x_n)$ of $nd$-dimensional space [1]

$$\frac{d}{dt}\rho_0(X_n, t) = \sum_{j=1}^{n} \nabla_j (\beta^{-1} \nabla_j \rho_0(X_n, t) + \rho_0(X_n, t) \nabla_j U_0(X_n)),$$

where $U_0(X_m) = \sum_{1 \le k < j \le n} \phi_0(x_j - x_k)$ is the potential energy of the particles, $\beta$ is the inverse temperature, $\nabla_j = \frac{\partial}{\partial x_j}$. It is the forward Kolmogorov equation for the gradient stochastic differential equations

$$\frac{\partial}{\partial t} x_j = -(\nabla_j U_0)(X_n) + w_j(t), \quad j = 1, \ldots, n,$$

where $w_j(t)$ are independent processes of white noise. Solutions of the infinite particle gradient stochastic differential equation $(j = 1, 2, \ldots, \infty)$ for different subsets of the set of infinite locally finite configurations were constructed in [2 – 6]. The infinite particle system of Brownian particles, described by the thermodynamic limit of grand canonical (nonequilibrium) correlation functions, was investigated in series of papers [7 – 12] for the case of short range (integrable) pair potential $\phi_0$ and Gibbs initial correlation functions.

In this paper, we consider the grand canonical correlation functions of $d$-dimensional systems of interacting Brownian particles with a long-range regular pair potential $\phi_0$, inclosed in a compact domain $\Lambda$ at initial moment, whose boundary does not have any influence upon particles afterwards (it is possible to impose other boundary conditions valid at any moment). We assume that the components of the vector-valued function $(\nabla\phi_0)$ are

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integrable and that at the initial moment the correlation functions coincide with the Gibbs correlation functions of the system with the long range pair potential $\frac{1}{2}\phi_0(x) + \phi_1(x)$, where $\phi_1(x)$ may also be long range.

The considered systems include 1-d systems of charged particles, interacting via the regularized 3-d Coulomb potential. We generalize the Sine–Gordon (SG) transformation for the systems in which a particles has the charge $e \in E_c\{r\}$, where $E_c\{r\}$ is the set on a real line with $r$ elements.

The considered initial Gibbs correlation functions are given by

$$\rho^{\Lambda}(X_m, (e)_m; t) = \\ = \Xi_{\Lambda}^{-1} \sum_{n>0} \frac{1}{n!} \sum_{(e')_n} Z_{(e)_m,(e')_n} \int \rho_0^{\Lambda}(X_m, X'_n, (e)_m, (e')_n; t|\phi_1)\, dX'_n, \tag{1}$$

where summation and integration are performed over the set $E_c\{r\}$ and $d$-dimensional space, respectively.

$$(e)_n = (e_1, \ldots, e_n), \quad Z_{(e)_m} = \prod_{s=1}^{m} z_{e_s},$$

$z_e$ is the activity of the particle with the charge $e$,

$$\rho_0^{\Lambda}(X_n, (e)_n; 0|\phi_1) = \chi_{\Lambda}(X_n) \exp\{-\beta\, U(X_n, (e)_n)\},$$

$$U(X_n, (e)_n) = \frac{1}{2} U_0(X_n, (e)_n) + U_1(X_n, (e)_n),$$

$$U_{0(1)}(X_n, (e)_n) = \sum_{1 \le k < j \le n} e_k e_j \phi_{0(1)}(x_j - x_k).$$

These correlation functions are generalized solutions of the gradient diffusion Bogoliubov hierarchy [7, 13]

$$\frac{\partial}{\partial t}\rho(X_m, (e)_m; t) = \\ = \frac{\partial}{\partial X_m}\left\{\beta^{-1}\frac{\partial \rho(X_m, (e)_m; t)}{\partial X_m} + \rho(X_m, (e)_m; t)\frac{\partial U_0(X_m, (e)_m)}{\partial X_m} + \right. \\ \left. + \sum_{e_{m+1}} \int dx_{m+1}\, \rho(X_{m+1}, (e)_{m+1}; t)\frac{\partial U_0(x_{m+1}, e_{m+1}|X_m, (e)_m)}{\partial X_m}\right\},$$

where

$$U_0(x_{m+1}, e_{m+1}|X_m) = \\ = U_0(x_{m+1}, (e)_{m+1}) - U_0(x_m, (e)_m) = \sum_{j=1}^{m} e_{m+1} e_j \phi_0(x_j - x_{m+1}),$$

summation and integration in the third term are performed over the set $E_c\{r\}$ and $d$-dimensional space, respectively, $\frac{\partial}{\partial X_m}$ is $m\,d$-valued operation of differentiation, suggesting that applying twice it demands the summation over indices of $X_m$. In this notation; the Smoluchovski equation coincide with the above equation if we put the third term in its right side equal to zero.

The proposed SG transformations express the correlation functions in terms of expectation values of functions of random fields

$$\rho^{\Lambda}(X_m, (e)_m; t) = \Xi_{\Lambda}^{-1} \int \mu(d(\varphi)_s)\, \rho^{\Lambda}(X_m, (e)_m; t|(\varphi)_s) \exp\{L^{\Lambda}((\varphi)_s; t)\}, \quad (2)$$

$$\Xi_{\Lambda} = \int \mu(d(\varphi)_s) \exp\{L^{\Lambda}((\varphi)_s; t)\}, \qquad s = 1, 2, 3,$$

where $\mu(d(\varphi)_s) = \prod_{l=1}^{s} \mu_l(\varphi_l)$, $(\varphi)_s = (\varphi_1, \ldots, \varphi_s)$, $\varphi_s$, $s = 1, 2$, depend on $x$ and $\varphi_3$ depends on $(x, t)$. Measures $\mu_l$ are homogeneous Gaussian with the covariances $\phi_l(x)$, $l = 0, 1$; $(-\Delta\phi_0)(x)\delta(t)$, $l = 3$. Functions $L^{\Lambda}$, $\rho^{\Lambda}$ will be called nonequilibrium (Euclidean) action and (random) correlation functions, respectively.

The most important property of the Lagrangian $L^{\Lambda}$ is the property of the thermodynamic stability

$$|L^{\Lambda}((\varphi)_s); t)| \le |\Lambda| l(t),$$

where $|\Lambda|$ is the volume of the compact domain $\Lambda$. The functions $\rho$ appearing on the right-hand side of eq. (2) are uniformly bounded in $\Lambda$.

A starting point in the derivation of the representation is the following relation for the solution of the Smoluchowski equation

$$\rho^{0}(X_n, t) = \int \mu(d(\varphi)_s)\, \rho^{0}_{(\varphi)_s}(X_n, t), \qquad s = 1, 2, 3.$$

A description of the system in terms $\rho^{0}_{(\varphi)_s}$ (eq. (9)) is achieved after a reduction of the Smoluchowski equation to the heat equation for $\hat{\rho}_0^{\Lambda}$ (eq. (6)) and applying the FK (Feynmann – Kac) formula (3.1) [14]. The FK formula reduces our $n$ particle system to the Gibbs system defined on a space of $nd$-dimensional Wiener paths with "potential energy" expressed in terms of three positive-definite two-particle potentials and a three-particle potential. The first two two-particle potentials $(\phi_0, \phi_1)$ are defined on the Wiener paths at initial and final moments, the third one $(-\nabla^2\phi_0)$ is defined on the paths on all time interval. The Gibbs factor with these potentials can be eliminated then by introduction of the above three Gaussian measures by a standard equality used in deriving the usual SG transformation (eqs. (15), (16))

$$\exp\left\{-\frac{\beta}{2}\sum_{i,j=1}^{n} e_j e_k \phi_{0(1)}(x_j - x_k)\right\} =$$

$$= \int \mu_{1(2)}(d\varphi_{1(2)}) \exp\left\{ i\sqrt{\frac{\beta}{2}} \sum_{l=1}^{n} e_l\, \varphi_{1(2)}(x_l)\right\}, \quad (3)$$

$$\exp\left\{-\sum_{j,k=1}^{n} e_k e_j \int_0^{\beta^{-1}t} (-\Delta\phi_0)(w_j(\tau) - w_k(\tau))\, d\tau\right\} =$$

$$= \int \mu_3(d\varphi_3) \exp\left\{\sum_{j=1}^{n} e_j \int_0^{\beta^{-1}t} \varphi_3(w_j(\tau), \tau)\, d\tau\right\}. \quad (4)$$

The nonequilibrium action is given by

$$L^{\Lambda}((\varphi)_s, t) = \ln \Xi_{\Lambda}((\varphi)_s),$$

where $\Xi_\Lambda((\varphi)_s)$ is the grand partition function of the distribution $\rho^0_{(\varphi)_s}$. If $\phi_1$ is short-range, then all the functions $L^\Lambda$ exist on a finite time interval for small values of the activities (depending on time). If this potential is long-range, then thermodynamically stable $L^\Lambda(\varphi_1)$ does not exist and the thermodynamically stable $L^\Lambda((\varphi)_s)$, $s = 2, 3$, are well defined on a finite time interval. For $s = 3$ the function has the most simple form. It corresponds to the Gibbs system defined on Wiener paths with the three particle potential

$$\int_0^{\beta^{-1}t} (\nabla\phi_0)(w_1(\tau))(\nabla\phi_0)(w_2(\tau))\,d\tau .$$

The Gibbs factor containing this potential can be reduced (eq. (17)) by an analog of above formulas $(\cdot)$ to the Gibbs factor containing imaginary two-body potential defined on the extended space of Wiener paths. The Lagrangians are expressed in terms of series (high-temperature expansion) involving all the connected parts of correlation functions of a Gibbs system defined on Wiener paths. In a usual Gibbs system the necessary condition of convergence of the expansion is the condition of integrability of a pair potential. The analog of this condition in our system is the integrability of the function $|\nabla\phi_0|$.

Our approach was inspired by the Ginibre's approach [15] based on a reduction of the Gibbs quantum system with the help of the FK formula to the classical Gibbs system on Wiener paths. The substantial difference between our and Ginibre's approach is that we have to treat the system on Wiener paths with a three-particle potential. Our technique permits to solve the Kirkwood – Saltsbourg (KS) equation for classical and quantum systems with this sort of a three-particle potential. This approach establishes an interesting correspondence between the inverse temperature in quantum systems and time in nonequilibrium systems of interacting Brownian particles, making possible the application of low-temperature expansions of the former in studing a long-time behavior in the latter.

The proposed generalized SG transformation might permit one to construct for the systems an analog of the Glimm – Jaffe – Spencer – Federbush – Brydges cluster expansion [16], applied by us to nonequilibrium systems of Brownian particles with short-range (regularized Yukawa potential) [12], perform the mean-field (Debye – Huckel) limit [17 – 18] and compute quasiequilibrium fluctuations (results concerning equilibrium hydrodynamical fluctuations in nonequilibrium systems of interacting Brownian particles are obtained in [9]; see also [20]). We hope that a more profound generalization of the SG transformation may be devised [21] in dealing with the 3-d system of charged particles. Our results can be generalized to the case of stable short-range pair potential $\phi_1$ and to the case of the potential energy $U_1$, expressed through many-particle short-range potentials.

Our paper is organized as follows: in the second section, we formulate our two main theorems and give formal algebraic formulae for the calculation of the action $L^\Lambda((\varphi)_s)$. The formulae for the case $s = 3$ are given in an abstract form and their understanding demands an application of the FK formula, given in the third section. In latter we also introduce the generalized Gibbs ensemble on Wiener paths with complex pair potential and write down the KS equation [22] in the nonintegrated form (last equalities of the paragraph). In the fourth section we give estimates of $L^p$ norms of functions satisfying this equation and establish the convergence of high-temperature expansion with the help of three propositions, proving the theorems.

**2. Main results. SG transformation.** In this section, we introduce a generating functional of the sequence $\rho^\Lambda_t$ of our nonequilibrium correlation functions $\rho^\Lambda(X_m,$

$(e)_m ; t)$, Gaussian measures $\mu_l(d\varphi_l)$, mentioned in the introduction, new equilibrium correlation functions $\rho^{\Lambda}_{(\varphi)_s}$, $s = 1, 2, 3$, given by eq. (9) whose connection with $\rho^{\Lambda}_t$ is given by eq. (10). The functions correrspond to the grand canonical ensemble, related to the $n$-particle distributions $\hat{\rho}^{\Lambda}_{0(\varphi)_s}(X_n, (e)_n ; t)$, which solve the Cauchy problem for the heat equation, to which the Smoluchowski equation is reduced by the "Euclidean" gauge transformation (5). For $s = 2$ the functions correspond to the initial data with the factorized dependence on the coordinates of $n$ particles. For $s = 3$ the functions are given by eq. (8). The clarification of this formula with the help of the FK formula is postponed to the next section. The section ends with the formal algebraic formulas (12), (13), expressing the grand partition function

$$\Xi_{\Lambda}((\varphi)_s) = \sum_{n \geq 0} \frac{1}{n!} \sum_{(e')_n} Z^{(s)}_{(e')_n} \int \rho^{\Lambda}_{0(\varphi)_s}(X_n, (e)_n ; t) dX_n$$

and the correlation functions $\rho^{\Lambda}_{(\varphi)_s}$ through the connected parts $\hat{v}^{\Lambda}_{0(\varphi)_s}$ of distributions $\hat{\rho}^{\Lambda}_{(\varphi)_s}$.

The considered correlation functions can be represented in the following form

$$\rho^{\Lambda}(X_m, (e)_m ; t) = \frac{\delta^n \hat{F}^{\Lambda}(j ; t)}{\delta j(x_1, e_1) \ldots \delta j(x_m, e_m)} \Bigg| \, j = 1, \ \hat{F}^{\Lambda} = \Xi^{-1}_{\Lambda} F^{\Lambda},$$

where

$$F^{\Lambda}(j ; t) = \sum_{n \geq 0} \frac{1}{n!} \sum_{e_1, \ldots, e_n} Z_{(e)_n} \int dX_n \rho^{\Lambda}_0(X_n, (e)_n ; t) J_n(X, e),$$

$$J_n(X, e) = \prod_{s=1}^{n} j(x_s, e_s).$$

Let $\mu_{1(2)}(d\varphi_{1(2)})$, $\mu_3(d\varphi_3)$ be the Gaussian measures on the probability space $\Omega_j$, $j = 1, 2, 3$, with the covariances

$$\int \mu(d\varphi_{1(2)}) \varphi_{1(2)}(x) \varphi_{1(2)}(y) = \phi_{0(1)}(x-y),$$

$$\int \mu_3(d\varphi_3) \varphi_3(x, t) \varphi_3(y, s) = (-\Delta\phi_0)(x-y) \delta(t-s),$$

where $\Delta = \nabla^2$ is the $d$-dimensional Laplacian.

**Theorem 1.** *Let $\phi_0$, $\phi_1$ be bounded positive-definiter functions.*

1. *If $\Delta\phi_0$, $\phi_1$ are bounded integrable functions and (components of) $(\nabla\phi_0)$ bounded by a bounded monotone decreasing integrable functions,*

*then there exists a function $A_0(t)$ increasing in $t$ and a function*

$$L^{\Lambda}(j, t \,|\, (\varphi)_s), \qquad s = 1, 2, 3,$$

*on $\Omega^s \times \mathbb{R}^+$ such that for $|j(x)| \leq 1$ and $\sum_e |z_e| \leq \exp\{-A_0(t)\}$ the following equality*

$$F^{\Lambda}(j ; t) = \int \mu(d(\varphi)_s) \exp\{L^{\Lambda}(j ; t | (\varphi)_s)\}$$

*and the inequality*

$$|L^{\Lambda}(j;t|(\varphi)_s| \leq \left(1 - \sum_e |z_e| \exp\{A_0(t)\}\right)^{-1} |\Lambda|,$$

*hold for* $s = 1, 2, 3$. *Here*

$$A_0(t) = A(t) + \frac{1}{4}\beta e^2[\phi_0(0) + \phi_1(0) + 2t(-\phi_0)(0)].$$

2. *If* $(\nabla\phi_0)^2$ *is bounded by a bounded monotone decreasing integrable functions, then the above equality and inequality hold for* $s = 3$.

3. *If condition 2 is satisfied and* $\Delta\,\phi_0$ *is an integrable function, then the previous equality and inequality hold for* $s = 2, 3$.

**Theorem 2.** *If condition 1 of Theorem 1 is satisfied, then there exist a function* $L^{\Lambda}((\varphi)_s, t)$ *on* $\Omega^s \times \mathbb{R}^+$ *and a function* $\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_n; t)$ *on* $\mathbb{R}^{md} \times E_c r^m \times \mathbb{R}^+$ *such that the representation* (2) *is true for* $s = 1, 2, 3$ *and* $\sum_e |z_e| \leq$ $\leq \exp\{-A_0(t)\}$. *Moreover, for the function* $L^{\Lambda}$ *the inequality from Theorem 1 holds and*

$$|\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_m; t)| \leq |Z_{(e)_m}| \exp\{A_0(t) m\} \left(1 - \sum_e |z_e| \exp\{A_0(t)\}\right)^{-1}.$$

*If the condition* 2 (3) *of Theorem* 1 *is satisfied, then the above conclusion hold for* $s = 3$ $(s = 2, 3)$.

The theorems are proved with the help of the obvious equalities

$$L^{\Lambda}((\varphi)_s, t) = L^{\Lambda}(1; t|(\varphi)_s),$$

$$\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_m; t) = \left\{\exp\{-L^{\Lambda}(j; t|(\varphi)_s)\} \frac{\delta^n \exp\{L^{\Lambda}(j; t|(\varphi)_s)\}}{\delta j(x_1, e_1) \ldots \delta j(x_1, e_1)}\right\} (j = 1).$$

The first step in proving the theorems is to transform the Smoluchowski equation into the heat equation with an interaction term with the help of a kind of a gauge transformation

$$\rho^{\Lambda}_0(X_n, (e)_n; t|\phi) = \exp\left\{-\frac{\beta}{2} U_0(X_n, (e)_n\right\} \hat{\rho}^{\Lambda}(X_n, (e)_n; t|\phi_1), \tag{5}$$

$$\frac{\partial}{\partial t}\hat{\rho}^{\Lambda}_0(X_n, (e)_n; t|\phi_1) = \beta^{-1}\left(\sum_{j=1}^{n} \frac{\partial^2}{(\partial x_j)^2}\hat{\rho}^{\Lambda}_0(X_n, (e)_n; t|\phi_1) + \right.$$

$$\left. + V(X_n)\hat{\rho}^{\Lambda}_0(X_n, (e)_n; t|\phi_1)\right),$$

$$V(X_n) = \frac{\beta}{2}\sum_{j=1}^{n}\left\{\frac{\partial^2 U_0(X_n)}{\partial x_j^2} - \frac{\beta}{2}\left(\frac{\partial U_0(X_n)}{\partial x_j}\right)^2\right\},$$

$$\hat{\rho}^{\Lambda}(X_n, (e)_n; 0|\phi_1) = \chi_{\Lambda}(X_n) \exp\{-\beta\, U_1(X_n, (e)_n\}.$$

From this and eq. (3) it follows that

$$\rho_0^\Lambda(X_n,(e)_n;t|\phi) = \exp\left\{\frac{\beta}{4}\varphi_0(0)\sum_{j=1}^n e_j^2\right\}\times$$

$$\times\int\mu_1(d\varphi_1)\prod_{j=1}^n \exp\left\{i\sqrt{\frac{\beta}{2}}e_j\varphi_1(x_j)\right\}\hat\rho_0^\Lambda(X_n,(e)_n;t|\phi_1). \qquad (6)$$

Since the solution of the Cauchy problem for the heat equation depends linearly and continuously in different functional spaces on the initial data the similar representation can be obtained with the help of eq. (3) for the function $\hat\rho_0^\Lambda$ appearing on the right-hand side of equality (7)

$$\hat\rho_0^\Lambda(X_n,(e)_n;t|\phi_1) = \exp\left\{\frac{\beta}{2}\phi_1(0)\sum_{j=1}^n e_j^2\right\}\times$$

$$\times\int\mu_2(d\varphi_2)\prod_{j=1}^n \hat\rho^\Lambda_{0(\cdot,\varphi_2)}(X_n,(e)_n;t), \qquad (7)$$

where $\hat\rho^\Lambda_{0(\cdot,\varphi_2)}(X_n,(e)_n;t)$ is the solution of the Cauchy problem of the above heat equation with the following initial data

$$\hat\rho^\Lambda_{0(\cdot,\varphi_2)}(X_n,(e)_n;0) = \prod_{j=1}^n \exp\left\{i(\beta/2)^{1/2}\sum_{j=1}^n e_j\,\varphi_2(x_j)\right\}\chi_\Lambda(X_n).$$

It can be shown (the FK formula and (4) are to be used) that the following equality holds

$$\hat\rho^\Lambda_{0(\cdot,\varphi_2)}(X_n,(e)_n;t) = \prod_{j=1}^n \exp\left\{\frac{t}{2}(-\Delta)(0)\sum_{j=1}^n e_j^2\right\}\times$$

$$\times\int\mu_3(d\varphi_3)\,\hat\rho^\Lambda_{0(\cdot,\varphi_2,\varphi_3)}(X_n,(e)_n;t), \qquad (8)$$

where $\hat\rho^\Lambda_{0(\cdot,\varphi_2,\varphi_3)}$ coincide with the functions $\hat\rho^\Lambda_{0(\cdot,\varphi_2)}$ at the initial moment.

Now let us define the correlation functions $(s=1,2,3)$

$$\rho^\Lambda_{(\varphi)_s}(X_m,(\sigma)_n;t) =$$

$$= \Xi_\Lambda((\varphi)_s)^{-1}\sum_{n>0}\frac{1}{n!}\sum_{(e)_n} Z^{(s)}_{(e')_m,(e')_n}\int dX_n'\,\rho^\Lambda_{0((\varphi)_s)}(X_m,X_n';t), \qquad (9)$$

where

$$\hat\rho^\Lambda_{0(\varphi)_s}(X_n,(e)_n;t) =$$

$$= \exp\left\{i\sqrt{\frac{\beta}{2}}\sum_{j=1}^n e_j\,\varphi_1(x_j)\right\}\hat\rho^\Lambda_{0(\cdot,(\varphi)_{(s\setminus 1)})}(X_n,(e)_n;t), \qquad s=2,3,$$

$$\hat\rho^\Lambda_{0\varphi_1}(X_n,(e)_n;t) = \exp\left\{i\sqrt{\frac{\beta}{2}}\sum_{j=1}^n e_j\,\varphi_1(x_j)\right\}\hat\rho_0^\Lambda(X_n,(e)_n;t|\phi_1),$$

$$z_e^{(1)} = \exp\left\{\frac{\beta}{4}\phi_0(0)e_s^2\right\}z_e, \qquad z_e^{(2)} = \exp\left\{\frac{\beta}{2}\phi_1(0)e_s^2\right\}z_e^{(1)},$$

$$z_e^{(3)} = \exp\left\{\frac{t}{2}(-\Delta\phi_0)(0)e_s^2\right\} z_e^{(2)}, \qquad (s\backslash 1) = (2, \dots, s),$$

and $\Xi_\Lambda((\varphi)_s)$ is the grand partition function of the sequence $\rho^\Lambda_{0(\varphi)_s}$, coinciding with the numerator in the expression for $\rho^\Lambda_{(\varphi)_s}$ for $m = 0$. As a result, we obtain that the following representation holds for correlation functions

$$\rho^\Lambda(X_m, (e)_m; t) = \Xi_\Lambda^{-1} \int \mu((d\varphi)_s) \Xi_\Lambda((\varphi)_s) \rho^\Lambda_{(\varphi)_s}(X_m, (e)_m; t). \tag{10}$$

Now we are able to prove the equalities in the theorems formally. In order to do this we have to use the sequence of the connected parts $\nu$ of the sequence $\rho$ of correlation functions [16]. They are defined by the equalities $(\rho(z_n), n \geq 0)$

$$\rho = \exp_* \nu, \quad \nu = \ln_*(e + \rho), \quad e_0 = 1, \quad e(Z_n) = 0, \quad n > 0; \quad \rho_0 = 0,$$

where $z_j$ may be elements of arbitrary set, $\exp_*$, $\ln_*$ are operations given with the help of multiplication, defined by the operation $*$ on the space of sequences, as usual functions exp and ln do and

$$(W * V)(Z_n) = \sum_{Y \in Z_n} W(Y) V(Z_n \setminus Y), \qquad W(\varnothing) = W_0.$$

The space of sequences with this multiplication becomes on algebra. Let us define the linear (multiplicative with respect to $*$) functional on it

$$\langle \rho \rangle_j = \sum_{n \geq 0} \frac{1}{n!} \int \prod_{k=1}^{n} j(z_k) d\omega(z_k) \rho(Z_n).$$

Since $\langle W * V \rangle_j = \langle W \rangle_j \langle V \rangle_j$, the following formal equality is true

$$\langle \exp_* \nu \rangle_j = \exp\{\langle \nu \rangle_j\}.$$

Putting $z = (x, e)$ and using the last equality we obtain

$$F^\Lambda(j; t) = \int \mu(d(\varphi)_s) \exp\{\langle \hat{\nu}^\Lambda_{0(\varphi)_s} \rangle_j\}, \qquad s = 1, 2, 3, \tag{11}$$

where $\hat{\nu}^\Lambda_{0(\varphi)_s}$ is the sequence of connected parts of $\hat{\rho}^\Lambda_{0(\varphi)_s}$. Hence, we gave the formal proof of the equalities of the theorems:

$$\Xi^\Lambda((\varphi)_s) = \exp\{L^\Lambda((\varphi)_s, t)\}, \qquad s = 1, 2, 3, \tag{12}$$

where

$$L^\Lambda(j; t | (\varphi)_s) = \langle \hat{\nu}^\Lambda_{0(\varphi)_s} \rangle_j,$$

$$L^\Lambda((\varphi)_s; t) = \langle \hat{\nu}^\Lambda_{0(\varphi)_s} \rangle, \qquad \langle \cdot \rangle = \langle \cdot \rangle_1.$$

Let $\psi^{(-1)}$ denote the sequence such that $\psi * \psi^{(-1)} = e$ and $D_Z$ denote the operation

$$(D_Z \psi)(Y) = \psi(Z, Y).$$

Using the equality $\langle \psi^{-1} \rangle = \langle \psi \rangle^{-1}$ (provided $\langle \psi \rangle$ exists) the following relation is easily proved

$$\rho^\Lambda_{(\varphi)_s}(X_n, (e)_n; t) = \langle \hat{\rho}^{\Lambda(-1)}_{0(\varphi)_s} * D_{X_m, (e)_m} \hat{\rho}^\Lambda_{0(\varphi)_s} \rangle =$$

$$= \sum_{n \geq 0} \frac{1}{n!} \sum_{(e')_n} Z_{(e)_m, (e')_n} \int dX'_n \rho^\Lambda_{(\varphi)_s}(X_m, (e)_m | X'_n, (e')_m). \tag{13}$$

In order to prove the theorems we have to prove the convergence of the series which defines operation $\langle\cdot\rangle(j)$. The most straight forward method to do it is to apply the FK formula, reduce the above connected parts to the connected parts of a generalized Gibbs system and use the reccurent KS equation.

**3. FK formula and KS equation.** In this section we fulfill the reduction of the considered nonequilibrium system to the Gibbs system with an imaginary pair potential defined on the space of Wiener paths. At first we introduce the Gibbs grand canonical correlation functions, depending on the Wiener paths $W_m$, $\rho^{\Lambda}_{(\varphi)_s}(X_m, W_m, (e)_m)$ (eq. (17)) with the help of the FK formula (14). Equality (18) connects the functions with the functions $\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_m; t)$. The FK formula clarifies the Gibbs structure of the functions $\hat{\rho}^{\Lambda}_{0(\varphi)_s}$ (eqs. (15), (16)). Then with the help of eq. (19), which is an analog of eq. (3), we introduce a new additional set of Wiener paths $W_{2,m}$, starting from the origin (the Wiener paths, appearing in the FK formula, start from $X_m$ and are denoted by $W1, m$), and the correlation functions $\rho^{*\Lambda}(W_{(2)m}, (e)_m)$ which correspond to the Gibbs ensemble on Wiener paths $W_{(2)m}$ and complex (imaginary) pair potential $(s=3)$. The imaginary pair potential is defined with the help of a stochastic integral. Equation (20) establishes a relation between the functions and $\rho^{\Lambda}_{(\varphi)_s}$. Further we define the functions $\rho^{*}_{s}(W_{(2)m} | W'_{(2)n})$ which satisfy the recurrent KS equation (23). The connected parts $\hat{v}^{\Lambda}_{0(\varphi)_s}$, correlation functions $\rho^{*\Lambda}_{(\varphi)_s}$, and the action $L^{\Lambda}$ are expressed through them by eqs. (21), (22). Let us write down the expression for the functions $\hat{\rho}^{\Lambda}_0$ using the FK formula [14].

$$\hat{\rho}^{\Lambda}_0(X_n, (e)_n; t) = \\ = \int P_{X_n}(W_n)\chi_{\Lambda}(W_n)(t\beta^{-1}) \exp\{-\beta[V_t(W_n) + U_1(W_n(\beta^{-1}t))]\}, \tag{14}$$

where

$$V_t(W_n) = -\beta^{-1}\int_0^{\beta^{-1}} V(W_n(\beta^{-1}\tau))\,d\tau,$$

$P_{X_n}(dW_n)$ is the Wiener measure on a probability space $(\Omega^d_0)^n$ of $nd$-dimensional paths, concentrated on the paths, starting from $X_n$, $\Omega^d_0$ is the probability space of $d$-dimensional paths. To simplify notation we omit the dependence of $V_t$, $U_{(s)}$ on $(e)_n$.

The function $V_t$ is the sum of two terms $V_t = U^t_2 + U^t_3$,

$$U^t_2(W_n) = -\frac{1}{2}\int_0^{\beta^{-1}t} d\tau\,\frac{\partial^2 U_0(W_n(\tau))}{(\partial W_m)^2},$$

$$U^t_3(W_n) = \beta\frac{1}{4}\int_0^{\beta^{-1}t} d\tau\left\|\frac{\partial U_0(W_n(\tau))}{\partial W_n}\right\|^2,$$

where $\frac{\partial^2}{(\partial W_n)^2}$ is the $nd$-dimensional Laplasian, $\|\cdot\|$ is the Euclidean norm in $nd$-dimensional space.

Now the functions $\hat{\rho}^{\Lambda}_{0(\varphi)_s}$ defined in the first section can be written down as follows

$$\hat{\rho}^{\Lambda}_{0(\cdot,\varphi_2)}(X_n,(e)_n;t) = \int P_{X_n}(dW_n)\exp\left\{i(\beta/2)^{1/2}\sum_{j=1}^{n} e_j\,\varphi_2(w_j(\beta^{-1}t))\right\}\times$$
$$\times\exp\{-\beta V_t(W_n)\}\chi_\Lambda(W_n(\beta^{-1}t)), \tag{15}$$

$$\hat{\rho}^{\Lambda}_{0(\cdot,(\varphi)_{(3\setminus1)})}(X_n,(e)_n;t) = \int P_{X_n}(dW_n)\exp\{-\beta U_3^t(W_n)\}\chi_\Lambda(W_n(\beta^{-1}t))\times$$
$$\times\exp\left\{i(\beta/2)^{1/2}\sum_{j=1}^{n} e_j\left[\varphi_2(w_j(\beta^{-1}t)) + \int_0^{\beta^{-1}t}\varphi_3(w_j(\tau))\,d\tau\right]\right\}. \tag{16}$$

Thus, we performed the reduction of the nonequilibrium system of interacting Brownian particles to the generalized Gibbs system with the phase space $\mathbb{R}\times E_c\{r\}\times$ $\times\,\Omega^d$, caracterized by the correlation functions

$$\rho^{\Lambda}_{(\varphi)_s}(X_m,W_m,(e)_m) =$$
$$= \Xi_\Lambda^{-1}\sum_{n\le 0}\frac{1}{n!}\sum_{(e')_n} Z^{(s)}_{(e')_m,(e')_n}\int dX'_n\int P_{X'_n}(dW'_n)\exp\{-\beta U^t_{(s)}(W_m,W'_n)\}\times$$
$$\times\chi_\Lambda(W_m(\beta^{-1}t),W'_n(\beta^{-1}t))\exp\{i(\beta/2)^{1/2}[\varphi^{(m)}_{(s)}(X;W)+\varphi^{(n)}_{(s)}(X';W')]\}, \tag{17}$$

$$U^t_{(s)} = U^t_s+\ldots+U^t_3,\quad s=1,2,3,\qquad U^t_1(W_n) = U_1(W_n(\beta^{-1}t)),$$

$$\varphi^{(m)}_{(3)}(X;W) = \sum_{j=1}^{m} e_j\left[2^{-1/2}\varphi_1(x_j)+\varphi_2(w_j(\beta^{-1}t))+\beta^{-1}\int_0^{\beta^{-1}t}\varphi_3(w_j(\tau))\,d\tau\right],$$

$$\varphi^{(m)}_{(2)}(X;W) = \sum_{j=1}^{m} e_j\,[2^{-1/2}\varphi_1(x_j)+\varphi_2(w_j(\beta^{-1}t))],$$

$$\varphi^{(m)}_{(1)}(X;W) = \sum_{j=1}^{m} e_j\,\varphi_1(x_j) = \varphi^{(m)}_1(X).$$

The following relation is valid

$$\rho^{\Lambda}_{(\varphi)_s}(X_m,(e)_m;t) = \int P_{X_m}(dW_m)\,\rho^{\Lambda}_{(\varphi)_s}(X_m,W_m,(e)_m). \tag{18}$$

It is necessary to perform the thermodynamic limit for these functions in order to prove our theorems. The functions have the algebraic structure of the reduced density matrices of quantum system with two and three particle potentials. If the three-particles potential $\phi_0(x,y) = (\nabla\phi_0(x),\nabla\phi_0(y))$ were not present, then we would immediately use the Ginibre technique to prove the convergence of high temperature expansions for the correlation functions in the thermodynamic limit. To circumvent this difficulty we introduce the sequence of new functional variables $W_{2,n}\in(\times\Omega_d)^n$ which help us to reduce the Gibbs system with the three-body potential to the Gibbs system with a pair imaginary potential with the one-particle phase space $\mathbb{R}^d\times E_c\{r\}\times$ $(\times\Omega_d)^2$:

$$\exp\{-\beta U_3^t(W_{1,n})\} = \int P_0(dW_{2,n})\exp\left\{i\sqrt{\frac{\beta}{2}}\sum_{j=1}^{n}\int_0^{\beta^{-1}t}(dw_{2j}(\tau),\nabla_j U(W_{(1)n}(\tau)))\right\} =$$
$$= \int P_0(dW_{2,n})\exp\{-\beta U_3^{t*}(W_{(2)n})\}, \tag{19}$$

$$U_3^{t*}(W_{(2)n}) = \sum_{1 \le k < j \le n} \phi_3^{t*}(W_{(2),j} | W_{(2),k}),$$

$$\phi_3^{t*}(W_{(2)} | V_{(2)}) = \frac{i}{2}[\phi_t^*(w_1 - v_1; w_2) - \phi_t^*(w_1 - v_1; v_2)],$$

$$\phi_t^*(w_1; w_2) = \int_0^{\beta^{-1}t} (dw_2(\tau), \nabla\phi_0(w_1(\tau))), \qquad W_{(2)n} = (W_{1,n}, W_{2,n}),$$

where $(\cdot,\cdot)$ is the scalar product of the 3-d Euclidean space, and $\phi_t^*(w_1; w_2)$ is the stochastic integral. The function $\phi_t^*(w; v)$ is a measurable function on $L^2((\times\Omega_d)^2, P_0)$. Indeed, this function is defined almost everywhere in $w$ ($w$ is a continuous function) as a limit in the topology of $L^2(\Omega_d, P_0)$ of integral Riemannian sums, i.e., is a measurable function in $v$. This function is also measurable in $w$, since it is defined as a limit of the almost everywhere convergent sequence of measurable functions (a sequence of functions converging in the topology of $L^2(\Omega_d, P_0)$ has a subsequence converging almost everywhere). The integral sums are cylindrical functions in $w$, $v$. Hence, the limit is a measurable function.

Now we express the correlation function $\rho^{\Lambda}_{(\varphi)_s}$ introduced in the first section as follows

$$\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_m; t) = \int P_{(X_m,0)}(dW_{(2)m})_s\, \rho^{*\Lambda}_{(\varphi)_s}(W_{(2)m}, (e)_m), \tag{20}$$

$$\rho^{\Lambda *}_{(\varphi)_s}(W_{(2)m}, (e)_m) = \Xi_\Lambda^{-1}((\varphi)_s) \times$$

$$\times \sum_{n \ge 0} \frac{1}{n!} \sum_{(e')_n} Z^{(s)}_{(e)_m,(e')_n} \int dX'_n \int P_{(X'_n,0)}(dW'_{(2)n}) \exp\{-\beta U^{t*}_{(s)}(W_{(2)m}, W'_{(2)n})\} \times$$

$$\times \chi_\Lambda(W_{1,m}(\beta^{-1}t), W'_{1,n}(\beta^{-1}t)) \exp\{i(\beta/2)^{1/2}[\varphi^{(m)}_{(s)}(X; W) + \varphi^{(n)}(X'; W')]\},$$

$$U^{t*}_{(s)} = U^t_s + \ldots + U^{t*}_3, \quad s = 1, 2, 3.$$

Let $\nu^*_{0(s)}$ be the connected parts of the Gibbs factors $\rho^*_{(s)}$ produced by the potential $U^{t*}_{(s)}$. Let us also define the following functions

$$\rho^*_{(s)}(W_{(2)m}, (e)_m | W'_{(2)n}, (e')_n) = (\hat{\rho}^{*(-1)}_{0(s)} * D_{W_{(2)m},(e)_m}\, \rho^*_{0(s)})(W'_{(2)n}; (e')_n).$$

From the definition of the multiplication $*$, connected parts and the inverse sequence $\rho^{-1}$ given above, we deduce the proposition and the corollary.

**Proposition 1.** *The folloving equalities are true*

$$\hat{\nu}^{\Lambda}_{0(\varphi)_s}(X_n, (e)_n; t) = \int P_{(X_m,0)}(dW_{(2)m})\, \nu^{*\Lambda}_{0(s)}(W_{(2)n}, (e)_n) \times$$

$$\times \chi_\Lambda(W_{1,n}(\beta^{-1}t)) \exp\{i(\beta/2)^{1/2}\varphi^{(m)}_{(s)}(X; W)\}, \tag{21}$$

$$\nu^{*\Lambda}_{0(s)}(W_{(2)n}, (e)_n) = \rho^*_{(s)}(W_2, e_1 | W_{(2)(n\setminus 1)}, (e)_{(n\setminus 1)}),$$

$$\rho^{\Lambda}_{(\varphi)_s}(X_m, (e)_m | X'_n, (e')_n) =$$

$$= \int P_{(X_m,0)}(dW_{(2)m})_s \int P_{(\mathbf{X}'_n,0)}(dW'_{(2)n})_s\, \rho^*_{(s)}(W_{(2)m}, (e)_m | W'_{(2)n}, (e')_n) \times$$

$$\times \chi_\Lambda(W_{1,m}(\beta^{-1}t), W'_{1,n}(\beta^{-1}t)) \exp\{i(\beta/2)^{1/2}[\varphi^{(m)}_{(s)}(X, W) + \varphi^{(m)}_{(s)}(X', W')]\}.$$

**Corollary.** *The following relations are valid*

$$\rho^*_{(\varphi)_s}(W_{(2)m}, (e)_m) =$$

$$= \sum_{n\geq 0}\frac{1}{n!}\sum_{(e')_n} Z^{(s)}_{(e)_m,(e')_n}\int dX'_n \int P_{(X'_n,0)}(dW_{(2)n})\,\rho^*_{(s)}(W_{(2)m}, (e)_m \,|\, W'_{(2)n}, (e')_n)\times$$

$$\times\chi_\Lambda(W_{1,m}(\beta^{-1}t), W'_{1,n}(\beta^{-1}t))\exp\{i(\beta/2)^{1/2}[\varphi^{(m)}_{(s)}(X,W)+\varphi^{(n)}_{(s)}(X',W')]\}, \quad (22)$$

$$L^\Lambda(j;t|(\varphi)_s) =$$

$$= \sum_{n\geq 0}\frac{1}{n!}\sum_{(e)_n} Z^{(s)}_{(e)_n}\int dX_n \int P_{(X_n,0)}(dW_{(2)n})\,\nu^*_{0(s)}(W_{(2)n},(e)_n)\,J_n(X,e)\times$$

$$\times\chi_\Lambda(W_{1,m}(\beta^{-1}t)\exp\{i(\beta/2)^{1/2}\varphi^{(n)}_{(s)}(X,W)\}.$$

In order to prove our theorems with the help of this corollary we have to use the standard recurrent KS equations

$$\rho^*_{(s)}(W_{(2)m}, (e)_m \,|\, W'_{(2)n}; (e')_n) =$$

$$= \exp\{-\beta U^{t*}_{(s)}(W_{(2)m}\,|\,W_{(2),j})\}\sum_{\{l\}\in(1,\ldots,n)} K^*_{(s)}(W'_{(2)\{l\}}\,|\,W_{(2),j})\times$$

$$\times\rho^*_{(s)}(W_{(2)(m\setminus 1)}, W'_{(2)\{l\}}, (e)_{(m\setminus 1)}, (e')_{\{l\}}\,|\,W'_{(2)(n\setminus\{l\})}, (e')_{(n\setminus\{l\})}), \quad (23)$$

where $W_{(2),j}=(w_{1,j}, w_{2,j})$, $(n\setminus\{l\})=(1,\ldots,n)\setminus\{l\}$, $j\leq m$ and

$$K^*_{(s)}(W'_{(2)n}\,|\,W_{(2),j}) =$$

$$= \prod_{r=1}^{n}[K^*_{(s)}(W'_{(2),r}\,|\,W_{(2),j}) = \exp\{-\beta\phi^{t*}_{(s)}(W'_{(2),r}\,|\,W_{(2),j})\}-1],$$

$$U^{t*}_{(s)}(W_{(2)m}\,|\,W'_{(2),j}) = e_j\sum_{r=1,\,r\neq j}^{m} e_r\,\phi^{t*}_{(s)}(W_{(2),j}\,|\,W_{(2),r}),$$

$$\rho^*_{(s)}(W_{(2)m}\,|\,\varnothing) = \exp\{-\beta U^*_{(s)}(W_{(2)m})\}.$$

For simplicity, we omit a dependence of functions $K^*_{(s)}$ on $e_j$, $e'_s$.

**4. Main bounds.** In this section, the crucial estimate (27) is proved by induction with the help of the KS equation, which yields the convergence of the series in eqs. (22). As a result, our theorems are true, provided $A(t) = A^*_2(t) + \beta B^*(t)$.

Indeed, from eqs. (18), (22), eq. (28) and the Schwartz inequality we derive

$$|\rho^\Lambda(X_m,(e)_m;t)| \leq |Z^{(s)}_{(e)_m}|\sum_{n\geq 0}\left[\sum_e |z^{(s)}_e|\right]^n \|\rho^*\|_{(m,n|q)} \leq$$

$$\leq |Z^{(s)}_{(e)_m}|\exp\{mA(t)\}\left(1-\sum_e\exp\{A(t)\}|z^{(s)}_e|\right)^{-1},$$

$$L^\Lambda(j;t|(\varphi)_s) \leq |\Lambda|\left(1-\sum_e|z^{(s)}_e|\exp\{A(t)\}\right)^{-1}$$

if $\sum_e |z^{(s)}_e| < \exp\{-A(t)\}$.

The most simple case corresponds to $s = 3$. For it we do not need to use the RG (Ruell – Ginibre) symmetrization. For the cases $s = 1, 2$ we need to use it. In order to simplify notation we denote by $\rho^*$, $K^*$, $U^{t*}$ the function $\rho^*_{(1)}$, $K^*_{(1)}$, $U^{t*}_{(1)}$, respectively, and derive the bounds for the case $s = 1$ crucial for our theorems. All bounds for $s = 2, 3$ are majorized by similar bounds for $s = 1$.

RG symmetrization exploits the stability (positive-definiteness) of the potentials $\phi_0$, $\phi_1$. Let $\chi_{(j,m)}(X_m, W_{1,m})$ be the characteristic function on $\Omega_d$ of the set where the following inequality holds

$$\operatorname{Re} U^{t*}(W_{(2)m} | W_{(2),j}) \geq \frac{e_j^2}{2}[\phi_1(0) + \beta^{-1} t(-\Delta\phi_0)(0)].$$

From the stability of the potentials

$$2\sum_{j=1}^{m}\left[U^{t*}(W_{(2)m} | W_{(2),j}) + \frac{e_j^2}{2}(\phi_1(0) + (-\Delta\phi_0)(0))\right] =$$

$$= \int_0^{\beta^{-1}t} d\tau \sum_{k,j=1}^{m} e_k e_j [t^{-1}\beta\phi_1(w_{1,j}(\beta^{-1}t) - w_{1,k}(\beta^{-1}t)) -$$

$$- \Delta\phi_0(w_{1,j}(\tau) - w_{1,k}(\tau))] \geq 0,$$

it follows that

$$\sum_{j=1}^{m} \chi^*_{(j,m)} = 1, \quad \chi^*_{(j,m)} = \left(\sum_{j=1}^{m} \chi_{(j,m)}\right)^{-1} \chi_{(j,m)}. \tag{24}$$

After multiplying both sides of the recurrent KS equation defined at the end of the second section by $\chi^*_{(j,m)}$ and summing over $j$ from 1 to $m$, we obtain the symmetrized recurrent KS equation

$$\rho^*(W_{(2)m}, (e)_m | W'_{(2)n}, (e')_n) = \sum_{j=1}^{m} \chi^*_{(j,m)} \exp\{-\beta U^{t*}(W_{(2)m} | W_{(2),j}) \times$$

$$\times \sum_{\{l\} \in (1,\ldots,n)} K^*(W_{(2)\{l\}} | W_{(2),j}) \times$$

$$\times \rho^*(W_{(2)m}, W'_{(2)\{l\}}, (e)_{(m\setminus 1)}, (e')_{\{l\}} | W'_{(2)(n\setminus\{l\})}, (e')_{(n\setminus\{l\})}). \tag{25}$$

Let us set

$$|\rho^*|_n(W_{(2)m}) = \int dX'_n \int P_{X'_n,0}(dW_{(2)n}) |\rho^*(W_{(2)m}, (e)_m | W'_{(2)n}, (e')_n)|, \tag{26}$$

$$|\rho^*|_{n,q}(W_{(2)m} | W'_{1,n}) = \left(\int P_0(dW_{2,n})(|\rho^*|_n(W_{(2)m}, W'_{(2)n})\right)^{1/q},$$

$$(K^*_q(W_2 | W'_{1,n}))^q = \int P_0(dW'_{2,n})(K^*(W_2 | W'_{(2)n}))^q.$$

Once more we shall not write down a dependence of $|\rho^*|_n$, $|\rho^*|_{n,q}$ on $(e)_m$. From the Hölder inequality and the symmetrized recurrent KS equation it follows that

$$|\rho^*|_n(W_{(2)m}) \leq \sum_{j=1}^{m} \chi^*_{(j,m)} \exp\{\beta B^*(t)\} \sum_{l=1}^{n} \frac{n!}{l!(n-l)!} \int dX'_l \int P_{X'_l}(dW'_{1,l}) \times$$

$$\times K^{*}_{q/(q-1)}(W_2 \,|\, W'_{1,l})\,|\rho^{*}|_{n,q}(W_{(2)m(j)} \,|\, W'_{1,(n\setminus l)}), \quad (n\setminus l) = (1, \ldots, n)\setminus(1, \ldots, l),$$

where

$$B^{*}(t) = \frac{\bar{e}^2}{2}[\phi_1(0) + \beta^{-1} t(-\Delta\phi_0)(0)],$$

$$\bar{e} = \max_{E_c\{r\}} e, \quad m(j) = (1, \ldots, j-1, j+1, \ldots, m).$$

Let us take the product over $j$ of the KS equation and use the generalized Hölder inequality

$$\int \prod_{j=1}^{q} |F_j(x)|^q \mu(dx) \le \prod_{j=1}^{q} \left(\int |F_j(x)|^q \mu(dx)\right)^{1/q}.$$

As a result

$$\int P_0(dW_{2,m})|\rho^{*}|^q_n(W_{(2)m}) \le$$

$$\le \exp\{q\beta B^{*}(t)\} \sum_{j_1,\ldots,j_q} \sum_{l_1,\ldots,l_q} \left(\prod_{r=1}^{q} \chi^{*}_{(j_r,m)} \frac{n_{l_r}!}{l_r!(n-l_r)!}\right) \times$$

$$\times \int \int \prod_{r=1}^{q} dX'^{(r)}_{l_r} P'^{(r)}_{X_{l_r}}(dW'^{(r)}_{1,l_r}) \left[\int P_0(dW_{2,j_r}) K^{*}_{q/(q-1)}(W_{2,j_r} \,|\, W'^{(r)}_{1,l_r})\right]^{1/q} \times$$

$$\times \left[\int P_0(dW_{2,m(j_r)}) (|\rho^{*}|_{n,q}(W_{(2)m(j_r)}, (e)_{m(j)} \,|\, W'^{(r)}_{1,(n\setminus l_r)}, (e')_{(n\setminus l_r)}))^q\right]^{1/q}.$$

By applying the Hölder inequality twice, using (24) and extending a summation over $l_r$ from 0 to $\infty$, we obtain the estimate (27) for $m+n$, assuming it holds for $m+n-$ $-1$, denoting

$$K^{*}_{l,q} = \left[\max_{(e)_m,(e')_n} \operatorname*{ess\,sup}_{w_2 \in \Omega_d} \int dX_l \int P_{X_l}(dW_{1,l}) P_0(dw_2) |K^{*}_q(W_2 \,|\, W'_{1,l})|^q\right]^{1/q}.$$

Thus, we proved the following proposition.

**Proposition 2.** *If* $A^{*}_q(t) = \ln \sum_{n\ge 0} \frac{1}{n!} K^{*}_{n,q}$ *is bounded, then*

$$\|\rho^{*}\|_{(m,n|q)} = \max_{(e)_m} \operatorname*{ess\,sup}_{W_{1,m}} \left[\int P_0(dW_{2,m})(|\rho|_n(W_{2(m)})^q\right]^{1/q} \le$$

$$\le n! \exp\{(m+n)[A^{*}_q(t) + \beta B^{*}(t)]\}. \tag{27}$$

With the help of the next proposition, we shall prove that $A^{*}_q(t)$ is finite.

**Proposition 3.** *If the conditions of the Theorem* 1 *are true then there exists the constant* $K^{*}_q$ *such that the following inequality holds*

$$K^{*}_{n,q} \le (n!)^{1/2}(K^{*}_q)^n. \tag{28}$$

***Proof.*** Applying once more the Hölder inequality, we obtain

$$K^{*}_{n,q} \le \max_{(e')_n} \operatorname*{ess\,sup}_{w_1 \in \Omega_d} \int dX'_n \int P_{X'_n}(dW'_{1,n}) K^{*}_q(w_1 \,|\, W'_{1,n}),$$

$$(K_q^*(w_1|W'_{1,n}))^q = \int P_0(dw_2)\,P_0(dW'_{2,n})\,|K^*(W_2|W'_{(2)n})|^q.$$

Form the definition of $K^*$ we immediately derive

$$|K^*(W_2|W'_{(2)n})| \le (\beta\bar{e}^2\exp\{\beta B^*(t)\})^n \prod_{j=1}^{n} |\phi^{t*}(W_2|W'_{2,j})|.$$

Now let us take into consideration that $\phi^{t*} = \phi_1^t + \phi_2^t + \phi_3^{t*}$ represent the above product to the power $q$ as a sum over $\bigcup\{n_s\} = (1, \dots, n)$, of products $|\phi_1^t|^{n_1}|\phi_2^t|^{n_2}|\phi_3^{t*}|^{n_3}$ multiplied by $2^{qn}$, and using the bounds (in the first the generalized Hölder inequality is used)

$$\int P_0(dw_2)\,P_0(dW'_{2,n}) \prod_{j=1}^{n} |\varphi_3^{t*}(W_2|W'_{2,n})|^q \le$$

$$\le 2^{qn}\prod_{j=1}^{n}\Big[\int P_0(dw'_2)\,|\varphi_t^*(w_1 - w_{1,j}|w'_2)|^{qn}\Big]^{1/n} =$$

$$= 2^{qn}\frac{(qn)!}{\left(\frac{q}{2}n\right)!}\prod_{j=1}^{n}(\phi_3^t(w_1 - w_{1,j}))^q, \qquad \phi_3^t(w) = \left[\int\limits_0^{\beta^{-1}t}(\nabla\phi_0(w(\tau))^2\,d\tau\right]^{1/2},$$

$$3^{-n}n^n \le n! \le n^n, \qquad (qn)!\left(\left(\frac{q}{2}n\right)!\right)^{-1} \le 18^{qn/2}q^{qn/2}(n!)^{q/2},$$

we obtain

$$K_q^*(w_1|W'_{1,n}) \le (n!)^{1/2}\,K_{0q}^{*n}\prod_{j=1}^{n}|\phi|_{(3)}^t(w_1 - w'_{1,j}),$$

where

$$K_{0q}^{*n} = \frac{1}{2}(12(2q)^{1/2}\bar{e}^2\beta\exp\{\beta B^*(t)\} + 1), \qquad |\phi|_{(3)}^t = |\phi_1^t| + |\phi_2^t| + |\phi_3^t|.$$

From this we immediatly derive

$$K_q^* = K_{0q}^*K_{(3)}^*, \qquad K_{(3)}^* = \operatorname*{ess\,sup}_{w\in\Omega_d}\int dx'\int P_{x'}(dw')\,|\phi|_{(3)}^t(w_1 - w'),$$

$$K_{(3)}^* = K_1^* + K_2^* + K_3^*, \qquad K_s^* = \operatorname*{ess\,sup}_{w\in\Omega_d}\int dx'\int P_{x'}(dw')\,|\phi_s^t(w_1 - w')|.$$

It is obvious that after changing the order of intagration in the expressions for $K_s^*$, $s = 1, 2$, we get

$$K_1^* \le \int dx|\phi_1(x)|, \qquad K_2^* \le \beta^{-1}t\int dx|\Delta\phi_0(x)|.$$

To prove proposition (25) we have to prove that $K_3^*$ is finite. In all our following estimates we use the fact that for monotone integrable bounded functions $h$, $g$ the following bound is valid

$$\int dy\,f(|x-y)|)h(|y|) \le f\left(\frac{|x|}{2}\right)\|h\|_{L^1} + h\left(\frac{|x|}{2}\right)\|f\|_{L^1}. \qquad (29)$$

This bound is obtained by splitting the integral into integrals over two domains:

$$|y| \geq \frac{|x|}{2}, \quad |y| \leq \frac{|x|}{2},$$

using the monotonicity of the functions, the fact that in these domains either $|y| \geq \frac{|x|}{2}$ or $|x-y| \geq \frac{|x|}{2}$, and after that enlarging the domains to the whole space.

Now let us put $f = P^{\tau}$, $\tau \leq \beta^{-1} t$, where

$$P^{\tau}(|x|) = (4\tau\pi)^{-d/2} \exp\left\{-\frac{|x|^2}{4\tau}\right\}.$$

Then using the bound

$$P^{\tau}(|x|) = \exp\left\{-\beta \frac{|x|^2}{8t}\right\} P^{[2]\tau}, \quad P^{[2]\tau}(|x|) = P^{\tau}\left(\frac{|x|}{\sqrt{2}}\right),$$

we derive

$$\int dy\, P^{\tau}(|x-y)|) h^2(|y|) \leq h^2\left(\frac{|x|}{2}\right) \|P^{\tau}\|_{L^1} + \exp\left\{-\beta \frac{|x|^2}{8t}\right\} \|h^2 P^{[2]\tau}\|_{L^1} \leq$$

$$\leq h^2\left(\frac{|x|}{2}\right) + \exp\left\{-\beta \frac{|x|^2}{8t}\right\} 2^{d/2} \|h^2\|_{L^\infty} \leq h_t(|x|). \tag{30}$$

Here, we computed the norm of $P^{[2]\tau}$: $\|P^{[2]\tau}\|_{L^1} = 2^{d/2}$. From the Hölder inequality, assuming that $|\nabla\phi_0| \leq h$ and $n$ is a monotone bounded integrable function, we obtain with the help of the above bound

$$K_3^* = \operatorname*{ess\,sup}_{\Omega_d,\, w(0)=0} \int dx \left[\int_0^{\beta^{-1}t} d\tau \int P^{\tau}(|y|)\, h^2(|y-x-w(\tau)|)\, dy\right]^{1/2} \leq$$

$$\leq \operatorname{ess\,sup} h_t^*(w),$$

where

$$h_t^*(w) = \int dx \left[\int_0^{\beta^{-1}t} d\tau\, h_t(|w(\tau)-x|)\, dy\right]^{1/2}: \tag{31}$$

Now, to prove that $K_3^*$ is bounded, it is sufficient to prove that there exists the positive number $\overline{h}$ such that

$$\int (h_t^*(w))^n P_0(dw) \leq a_0 \overline{h}^n,$$

for some $a_0$ and all positive integers. Indeed, assuming that $h_t^*$ is not bounded, it is easily shown that the inverse inequality holds, since on a set of nonzero measure $a_0$ the function is greater than arbitrary number $\overline{h}$.

Let us put

$$(I(X_n))^2 = n! \int_{\tau[\beta^{-1}t]_{(n)}} d\tau_{(n)} \int dY_n\, P^{\tau_1}(|y_1|)\, h_t(|y_1 - x_1|) \times$$

$$\times \prod_{j=2}^{n} P^{\tau_j-\tau_{j-1}}(|y_j-y_{j-1}|)\, h_t(|y_j-x_j|),$$

where $\tau[t]_{(n)} = 0 \le \tau_1 \le \tau_2 \le \ldots \le t$. Then with the help of the hölder inequality and the definition of the Wiener integral we derive

$$\int P_0(dw)\,(h_t^*(w))^n \le I = \int dX_n\, I(X_n).$$

Let us apply the inequality derived from inequalities (29), (30) and trivial bound $\exp\left\{-\beta\frac{|x|^2}{8\tau}\right\} = \exp\left\{-\beta\frac{|x|^2}{16\tau}\right\}$

$$\int P^{\tau}(|x-y)|)\,h_t(|y|)\,dy \le 2\left(1+\sqrt{2^d}\right) h_t\left(\frac{|x|}{2}\right). \tag{32}$$

As a result,

$$(I(X_n))^2 \le n!\,2\left(1+\sqrt{2^d}\right) \int_{\tau[\beta^{-1}t]_{(n)}} d\tau_{(n)} \int dY_n\, P^{\tau_1}(|y_1|)\, h_t(|y_1-x_1|) \times$$

$$\times \prod_{j=1}^{n} P^{\tau_j-\tau_{j-1}}\, h_t(|y_j-x_j|)\, h_t(|y_{n-1}-x_n|).$$

Repeating the above argument as in the case of inequality (30) and with its help we obtain $(\tau \le \beta^{-1}t)$

$$\int P^{\tau}(|y_{n-1}-y_{n-2}|)\, h_t(|y_{n-1}-x_{n-1}|)\, h_t\left(\frac{|y_{n-1}-x_n|}{2^l}\right) dy_{n-1} \le$$

$$\le h_t\left(\frac{|x_n-x_{n-1}|}{2^{l+1}}\right) \int P^{\tau}(|y_{n-1}-y_{n-2}|)\, h_t(|y_{n-1}-x_{n-1}|)\, dy_{n-1} +$$

$$+ h_t\left(\frac{|x_n-x_{n-1}|}{2}\right) \int P^{\tau}(|y_{n-1}-y_{n-2}|)\, h_t\left(\frac{|y_{n-1}-x_n|}{2^l}\right) dy_{n-1} \le$$

$$\le 2\left(1+\sqrt{2^{\nu}}\right)\left[h_t\left(\frac{|x_n-x_{n-1}|}{2^{l+1}}\right) h_t\left(\frac{|y_{n-2}-x_n|}{2}\right) + \right.$$

$$\left. + h_t\left(\frac{|x_n-x_{n-1}|}{2}\right) h_t\left(\frac{|y_{n-2}-x_n|}{2^{l+1}}\right)\right].$$

Since $\tau_j - \tau_{j-1} \le \beta^{-1}t$, for $l=1$, we have

$$I_n \le \left(2\left(1+\sqrt{2^d}\right)\right)^{1/2}\left[I_{n,1}^1 \int dx \left(h_t\left(\frac{|x|}{4}\right)\right)^{1/2} + I_{n,1}^2 \int dx\left(h_t\left(\frac{|x|}{2}\right)\right)^{1/2}\right],$$

$$I_{n,1}^l = n! \int dX_{n-1}\left[\int_{\tau[\beta^{-1}t]_{(n)}} d\tau_{(n)} \int dY_{n-1}\, P^{\tau_1}(|y_1|)\, h_t(|y_1-x_1|) \times\right.$$

$$\left. \times \prod_{j=1}^{n-2} P^{\tau_j-\tau_{j-1}}(|y_j-y_{j-1}|)\, h_t(|y_j-x_j|)\, h_t\left(\frac{|y_{n-2}-x_{n-1}|}{2^l}\right)\right]^{1/2}, \quad l=1,2.$$

Here we used the fact that the square root of a finite sum is less than the sum of square roots of its elements. Iterating this bound $n$ times we obtain $2^n$ terms of the following form

$$\int_{\tau[\beta^{-1}t]_n} d\tau_{(n)} \left(2\left(1+\sqrt{2^{\nu}}\right)\right)^{n/2} \prod_{j=1}^{n} \int dx \left(h_t\left(\frac{|x|}{2^{l_i}}\right)\right)^{1/2},$$

$$0 \le l_i \le n, \quad \sum_i l_i = n.$$

Hence,

$$I_n \le \left(\sqrt{t\beta^{-1}}\, a\|h_t^{1/2}\|_{L^1}\right)^n = \bar{h}^n, \quad a = 2^{1+d/2}\left(1+\sqrt{2^d}\right), \quad a_0 = 1.$$

Thus we proved the following proposition.

**Proposition 4.** *The function* $h_t^*(w)$ *defined by eq.* (31) *is bounded almost everywhere if* $h$ *is an integrable function.*

By proposition 4 we end the proofs of inequality (28) and propositions $1-3$.


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