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A UNIFIED APPROACH FOR UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS USING HADAMARD PRODUCT

For given analytic functions $\varphi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ in $U = \{z:\ |z| < 1\}$ with $\lambda_n \geq 0$, $\mu_n \geq 0$, and $\lambda_n \geq \mu_n$ and for α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$, let $E(\varphi, \psi; \alpha, \beta)$ be the class of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in U such that $f(z) * \psi(z) \neq 0$ and $|(f(z) * \varphi(z))/((f(z) * \psi(z)) - 1| < \beta |(f(z) * \varphi(z))/((f(z) * \psi(z)) + (1 - 2\alpha)|$ for $z \in U$, where $*$ denotes the Hadamard product. Let T be the class of functions $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ that are analytic and univalent in U, and let $E_T(\varphi, \psi; \alpha, \beta) = E(\varphi, \psi; \alpha, \beta) \cap T$. Coefficient estimates, extreme points, distortion properties, etc. are determined for the class $E_T(\varphi, \psi; \alpha, \beta)$ when the second coefficient is kept fixed. The results thus obtained, for particular choices of $\varphi(z)$ and $\psi(z)$, not only generalize various known results but also give rise to several new results.

Нехай в $U = \{z:\ |z| < 1\}$ задані аналітичні функції $\varphi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$ та $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$, де $\lambda_n \geq 0$, $\mu_n \geq 0$ і $\lambda_n \geq \mu_n$, і $E(\varphi, \psi; \alpha, \beta)$, $0 \leq \alpha < 1$, $0 < \beta \leq 1$, — клас аналітичних функцій $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ в U таких, що $f(z) * \psi(z) \neq 0$ і $|(f(z) * \varphi(z))/((f(z) * \psi(z)) - 1| < \beta |(f(z) * \psi(z))/((f(z) * \varphi(z)) + (1 - 2\alpha)|$ при $z \in U$, де через $*$ позначено добуток Адамара. Нехай T — клас функцій $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ аналітичних і унівалентних в U, і нехай $E_T(\varphi, \psi; \alpha, \beta) = E(\varphi, \psi; \alpha, \beta) \cap T$. Коефіцієнтні оцінки, екстремальні точки, властивості дисторсії тощо встановлені для класу $E_T(\varphi, \psi; \alpha, \beta)$ при фіксованому другому коефіцієнті. При спеціальному виборі $\varphi(z)$ і $\psi(z)$ одержані результати не тільки узагальнюють загальновідомі, а й дають можливість для встановлення нових результатів.

Introduction. Let A denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in the unit disc $U = \{z:\ |z| < 1\}$. Denote by $S(\alpha, \beta)$ the class of functions $f \in A$ that satisfy the condition

$$\left| z\frac{f'(z)}{f(z)} - 1 \right| < \beta \left| z\frac{f'(z)}{f(z)} + 1 - 2\alpha \right|$$

for some α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and for all $z \in U$ and let $C(\alpha, \beta)$ be the class of functions $f(z)$ for which $z f'(z) \in S(\alpha, \beta)$. The functions in $S(\alpha, \beta)$ and $C(\alpha, \beta)$ are called, respectively, starlike of order α and type β and convex of order α and type β in U.

Let $R(\alpha, \beta)$ be the class of functions $f \in A$ that satisfy the condition

$$|f'(z) - 1| < \beta\, |f'(z) + 1 - 2\alpha|$$

for some α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ and all $z \in U$.

The classes $S(\alpha, \beta)$, $C(\alpha, \beta)$, and $R(\alpha, \beta)$ were introduced and studied in [1, 2].

Given two functions $f, g \in A$, $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, their Hadamard product is defined by

$$f(z) * g(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

A function $f \in A$ is said to be prestarlike of order α $(0 \leq \alpha < 1)$ and type β $(0 <$

$< \beta \leq 1)$ and denoted by $D(\alpha, \beta)$ if $f(z) * z(1-z)^{2\alpha-2} \in S(\alpha, \beta)$. The class $D(\alpha, 1) = D(\alpha)$ was studied by Ruscheweyh [3], Al-Amiri [4], and others.

Let T denote the class of functions of the form $z - \sum_{n=2}^{\infty} |a_n| z^n$ that are analytic and univalent in U, and let $S_T(\alpha, \beta) = S(\alpha, \beta) \cap T$, $C_T(\alpha, \beta) = C(\alpha, \beta) \cap T$, $R_T(\alpha, \beta) = R(\alpha, \beta) \cap T$, and $D_T(\alpha, \beta) = D(\alpha, \beta) \cap T$. Several properties of $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$ and allied classes were obtained by Gupta and Jain [5, 6], Jain and Ahuja [7], and Bhoosnurmath and Swamy [8].

Silverman and Silvia [9] investigated several properties of the classes $S_T(\alpha, 1)$ and $C_T(\alpha, 1)$ with fixed second coefficient. To determine properties analogous to those obtained by Silverman and Silvia [9] and in [10] for the above-mentioned subclasses of T, we introduce the following new class:

Definition. *Given the functions* $\varphi(z) = z + \sum_{n=2}^{\infty} \lambda_n z^n$, $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ *analytic in* U *and such that* $\lambda_n \geq 0$, $\mu_n \geq 0$, *and* $\lambda_n \geq \mu_n$ *for* $n = 2, 3, \ldots$, *we say that* $f \in A$ *is in* $E(\varphi, \psi; \alpha, \beta)$ *if* $f(z) * \psi(z) \neq 0$ *and the inequality*

$$\left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} - 1 \right| < \beta \left| \frac{f(z) * \varphi(z)}{f(z) * \psi(z)} + 1 - 2\alpha \right| \tag{1}$$

holds for some α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$ *and for all* $z \in U$.

We shall denote by $E_T(\varphi, \psi; \alpha, \beta)$ the subclass of functions in $E(\varphi, \psi; \alpha, \beta)$ that have their nonzero coefficients, from second onwards, all negative. Thus, $E_T(\varphi, \psi; \alpha, \beta) = E(\varphi, \psi; \alpha, \beta) \cap T$.

It is easy to check that various subclasses of T referred to above can be represented as $E_T(\varphi, \psi; \alpha, \beta)$ for suitable choices of φ and ψ. For example,

$$E_T\left(\frac{z}{(1-z)^2}, \frac{z}{1-z}; \alpha, \beta\right) = S_T(\alpha, \beta),$$

$$E_T\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}; \alpha, \beta\right) = C_T(\alpha, \beta),$$

$$E_T\left(\frac{z}{(1-z)^2}, z; \alpha, \beta\right) = R_T(\alpha, \beta),$$

$$E_T\left(\frac{z + (1-2\alpha) z^2}{(1-z)^{3-2\alpha}}, \frac{z}{(1-z)^{2-2\alpha}}; \alpha, \beta\right) = D_T(\alpha, \beta),$$

etc. In fact, many new subclasses of T can be defined by suitably choosing $\varphi(z)$ and $\psi(z)$. It may by noted that $f \in E_T(\varphi, \psi; \alpha, \beta)$ if and only if

$$z f'(z) \in E_T\left(\int_0^z \frac{\varphi(t)}{t} dt, \int_0^z \frac{\psi(t)}{t} dt; \alpha, \beta\right)$$

or

$$\int_0^z \frac{f(t)}{t} dt \in E_T(z\varphi', z\psi'; \alpha, \beta).$$

We shall now make a systematic study of the class $E_T(\varphi, \psi; \alpha, \beta)$. It will be assumed throughout that φ and ψ satisfy the conditions stated in the definition and that

$f(z) * \psi(z) \neq 0$ for $z \in U$. The results thus obtained not only generalize the corresponding results of Silverman and Silvia [9] and those obtained in [10] but also give an analogous study for the classes $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, and $D_T(\alpha, \beta)$.

1. Coefficient estimates. In this section, we find a necessary and sufficient condition for a function to be in $E_T(\varphi, \psi; \alpha, \beta)$.

Theorem 1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* A. *If for some* α, β $(0 \leq \alpha < 1,\ 0 < \beta \leq 1)$

$$\sum_{n=2}^{\infty} \sigma_n |a_n| \leq 2\beta(1-\alpha), \tag{2}$$

where $\sigma_n = (1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n$, *then* $f \in E(\varphi, \psi; \alpha, \beta)$.

Proof. Suppose that (2) holds for all admissible values of α and β. Then for $z \in U$,

$$|f(z) * \varphi(z) - f(z) * \psi(z)| - \beta\,|f(z) * \varphi(z) + (1-2\alpha) f(z) * \psi(z)| =$$

$$= \left|\sum_{n=2}^{\infty} a_n(\lambda_n - \mu_n) z^n\right| - \left|2\beta(1-\alpha) z + \sum_{n=2}^{\infty} \{\beta\lambda_n + (1-2\alpha)\beta\mu_n\} a_n z^n\right| <$$

$$< \left[\sum_{n=2}^{\infty} (\lambda_n - \mu_n)|a_n| - 2\beta(1-\alpha) + \sum_{n=2}^{\infty} \{\beta\lambda_n + (1-2\alpha)\beta\mu_n\}|a_n|\right]|z| =$$

$$= \left[\sum_{n=2}^{\infty} \sigma_n |a_n| - 2\beta(1-\alpha)\right]|z| \leq 0,$$

in view of (2). Thus, it follows that

$$\left|\frac{f(z) * \varphi(z)}{f(z) * \psi(z)} - 1\right| < \beta \left|\frac{f(z) * \varphi(z)}{f(z) * \psi(z)} + 1 - 2\alpha\right|$$

and, hence, $f \in E(\varphi, \psi; \alpha, \beta)$.

Theorem 2. *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* A, *then* $f \in E_T(\varphi, \psi; \alpha, \beta)$ *if and only if* (2) *is satisfied.*

Proof. In view of Theorem 1, it is sufficient to show the "only if" part. Thus, let $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ be in $E_T(\varphi, \psi; \alpha, \beta)$. Then

$$\left|\frac{(f(z) * \varphi(z))/(f(z) * \psi(z)) - 1}{(f(z) * \varphi(z))/(f(z) * \psi(z)) + 1 - 2\alpha}\right| =$$

$$= \left|\frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n)|a_n| z^n}{2(1-\alpha) z - \sum_{n=2}^{\infty} \{\lambda_n + (1-2\alpha)\mu_n\}|a_n| z^n}\right| < \beta$$

for all $z \in U$. Using the fact that $\operatorname{Re} z \leq |z|$ for all z, it follows that

$$\operatorname{Re}\left(\frac{\sum_{n=2}^{\infty} (\lambda_n - \mu_n)|a_n| z^{n-1}}{2(1-\alpha) - \sum_{n=2}^{\infty} \{\lambda_n + (1-2\alpha)\mu_n\}|a_n| z^{n-1}}\right) < \beta, \quad z \in U. \tag{3}$$

Now consider the values of $z \in U$ on the real axis such that $(f(z) * \varphi(z))/(f(z) *$

$* \psi(z))$ is real. Upon clearing the denominator in (3) and letting $z \to 1$ through positive values, we obtain

$$\sum_{n=2}^{\infty} (\lambda_n - \mu_n)|a_n| \le \beta \left[2(1-\alpha) - \sum_{n=2}^{\infty} \{\lambda_n + (1-2\alpha)\mu_n\}|a_n| \right]$$

or

$$\sum_{n=2}^{\infty} \sigma_n |a_n| \le 2\beta(1-\alpha)$$

and, hence, the result follows.

***Corollary* 1.** *If* $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$ *is in* $E_T(\varphi, \psi; \alpha, \beta)$, *then*

$$|a_n| \le \frac{2\beta(1-\alpha)}{\sigma_n}, \quad n = 2, 3, \ldots .$$

The result being sharp, with equality for each n, for functions of the form

$$f_n(z) = z - \frac{2\beta(1-\alpha)}{\sigma_n} z^n .$$

***Remark* 1.** Taking different choices of $\varphi(z)$, $\psi(z)$ as stated in Sec. 1, the above theorem leads to necessary and sufficient conditions for a function f to be in $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$ etc.

2. Extreme points. Since by Theorem 2 for $f \in E_T(\varphi, \psi; \alpha, \beta)$,

$$|a_n| \le \frac{2\beta(1-\alpha)}{\sigma_n}, \quad n = 2, 3, \ldots ,$$

we may write

$$|a_2| = \frac{2\beta(1-\alpha)\, p}{\sigma_2}, \quad \text{so that} \quad 0 \le p \le 1 .$$

We denote by $E_T^P(\varphi, \psi; \alpha, \beta)$ functions in $E_T(\varphi, \psi; \alpha, \beta)$ of the form

$$z - \frac{2\beta(1-\alpha)\, p}{\sigma_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n ,$$

where p satisfying $0 \le p \le 1$ is kept fixed. From Theorem 2, it is easily seen that the family $E_T(\varphi, \psi; \alpha, \beta)$ is closed under convex linear combinations and, hence, the family $E_T^P(\varphi, \psi; \alpha, \beta)$ and, consequently, the closed convex hull of $E_T^P(\varphi, \psi; \alpha, \beta)$ is $E_T^P(\varphi, \psi; \alpha, \beta)$ itself. We now determine the extreme points of $E_T^P(\varphi, \psi; \alpha, \beta)$.

Theorem 3. *Let*

$$f_2(z) = z - [2\beta(1-\alpha)p/\sigma_2]\, z^2$$

and

$$f_n(z) = z - \frac{2\beta(1-\alpha)\, p}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_n} z^n , \qquad (3)$$

$n = 3, 4, \ldots$. *Then* $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z), \quad \textit{where} \quad \rho_n \ge 0 \quad \textit{and} \quad \sum_{n=2}^{\infty} \rho_n = 1 .$$

Proof. Suppose that

$$f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2} z^2 - \sum_{n=3}^{\infty} \frac{2\beta(1-\alpha)(1-p)\rho_n}{\sigma_n} z^n.$$

Then one can easily verify, by using Theorem 2, that $f \in E_T^P(\varphi, \psi; \alpha, \beta)$. Conversely, suppose that

$$f(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2} z^2 - \sum_{n=3}^{\infty} |a_n| z^n \in E_T^P(\varphi, \psi; \alpha, \beta).$$

Since, by (2), we get

$$|a_n| \le \frac{2\beta(1-\alpha)(1-p)}{\sigma_n}, \quad n = 3, 4, \ldots,$$

by setting $\rho_n = [\sigma_n |a_n|]/\{2\beta(1-\alpha)(1-p)\}$, $n = 3, 4, \ldots$, and $\rho_2 = 1 - \sum_{n=3}^{\infty} \rho_n$, we have $f(z) = \sum_{n=2}^{\infty} \rho_n f_n(z)$. This implies the statement of the theorem.

***Corollary* 2.** *The extreme points of* $E_T^P(\varphi, \psi; \alpha, \beta)$ *are the functions* $f_n(z)$, $n = 2, 3, \ldots$.

The same procedure as in the last theorem leads to the following characterization for extreme points of the class $E_T(\varphi, \psi; \alpha, \beta)$:

Theorem 4. *Let*

$$g_1(z) = z, \quad g_n(z) = z - [2\beta(1-\alpha)/\sigma_n] z^n, \quad n = 2, 3, \ldots.$$

Then $f \in E_T(\varphi, \psi; \alpha, \beta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \rho_n g_n(z),$$

where $\rho_n \ge 0$ *and* $\sum_{n=1}^{\infty} \rho_n = 1$.

It may be noted that whereas the function $g_1(z) = z$ is an extreme point of $E_T(\varphi, \psi; \alpha, \beta)$, it is not even a member of $E_T^P(\varphi, \psi; \alpha, \beta)$ unless $p = 0$.

3. Distortion and covering theorems. The extreme points found in Theorem 4 immediately yield the following distortion theorems for functions in $E_T(\varphi, \psi; \alpha, \beta)$:

Theorem 5. *Let* $f \in E_T(\varphi, \psi; \alpha, \beta)$. *If* $\{\sigma_n\}_{n=2}^{\infty}$ *is a nondecreasing sequence, then*

$$r - \frac{2\beta(1-\alpha)}{\sigma_2} r^2 \le |f(re^{i\theta})| \le r + \frac{2\beta(1-\alpha)}{\sigma_2} r^2. \tag{4}$$

Further, if $\{n/\sigma_n\}_{n=2}^{\infty}$ *forms a nondecreasing sequence, then*

$$1 - \frac{4\beta(1-\alpha)}{\sigma_2} r \le |f'(re^{i\theta})| \le 1 + \frac{4\beta(1-\alpha)}{\sigma_2} r. \tag{5}$$

The equality holds at both extremes in (4) *and* (5) *for* $g_2(z)$.

The sharp upper and lower bounds for $|f|$ as well as $|f'|$ for a function $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ will occur at one of the extreme points of $E_T^P(\varphi, \psi; \alpha, \beta)$. The bounds, however, are not straight forward. We need the following lemmas, which can be proved by using arguments similar to those of Silverman and Silvia [9] and, hence, their proofs are omitted:

Lemma 1. *Let*

$$\rho_0 = \frac{-\{\sigma_3 + 4\sigma_2 - 2\beta(1-\alpha)\} + \left[\{\sigma_3 + 4\sigma_2 - 2\beta(1-\alpha)\}^2 + 32\beta(1-\alpha)\sigma_2\right]^{1/2}}{4\beta(1-\alpha)},$$

$$r_0 = \frac{-2(1-p)\sigma_2 + [4(1-p)^2\sigma_2^2 + 2p^2\beta(1-\alpha)(1-p)\sigma_3]^{1/2}}{2\beta(1-\alpha)(1-p)\,p}$$

and

$$f_3(z) = z - \frac{2\beta(1-\alpha)\,p}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} z^3.$$

Then for $0 \le r < 1$, $0 \le p \le 1$,

$$|f_3(re^{i\theta})| \ge r - \frac{2\beta(1-\alpha)\,p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} r^3$$

with equality for $\theta = 0$. *For either* $0 \le p < p_0$ *and* $0 \le r \le r_0$ *or* $p_0 \le p \le 1$,

$$|f_3(re^{i\theta})| \le r + \frac{2\beta(1-\alpha)\,p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} r^3$$

with equality for $\theta = \pi$. *For* $0 \le p < p_0$ *and* $r_0 \le r < 1$,

$$|f_3(re^{i\theta})| \le r\left[\frac{2(1-p)\sigma_2^2 + \beta(1-\alpha)p^2\sigma_3}{2(1-p)\sigma_2^2} + \right.$$

$$+ \frac{2\beta^2(1-\alpha)^2p^2\sigma_3 + 4\beta(1-\alpha)(1-p)\sigma_2^2}{\sigma_2^2\sigma_3} r^2 +$$

$$\left. + \frac{(1-p)\beta^2(1-\alpha)^2\{4(1-p)\sigma_2^2 + 2\beta(1-\alpha)p^2\sigma_3\}}{\sigma_2^2\sigma_3^2} r^4\right]^{1/2}$$

with equality for

$$\theta = \cos^{-1}\left[\frac{2\beta(1-\alpha)(1-p)\,pr^2 - p\sigma_3}{4(1-p)\sigma_2 r}\right].$$

Lemma 2. *If* $\sigma_n = (1+\beta)\lambda_n - (1-\beta+2\alpha\beta)\mu_n$ *is a nondecreasing function of* n, *then for* $n \ge 4$, $|f_n(re^{i\theta})| \le |f_4(-r)|$.

Lemma 1 and Lemma 2 lead to the following statement:

Theorem 6. *If* $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ *and* σ_n *is a nondecreasing function of* n, *then*

$$|f(re^{i\theta})| \ge r - \frac{2\beta(1-\alpha)\,p}{\sigma_2} r^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} r^3, \quad 0 \le r < 1,$$

with equality for $f_3(z)$ *at* $z = r$ *and*

$$|f(re^{i\theta})| \le \max\left\{\max_\theta |f_3(re^{i\theta})|, -f_4(-r)\right\},$$

where $\max_\theta |f_3(re^{i\theta})|$ *is given by Lemma* 1.

Remarks. **2.** For $p=0$, the sharp upper bound for $|f|$ always occurs for

$$|f_3(re^{i\pi})| = -f_3(-r).$$

3. For

$$p > \frac{-(\lambda_3+4\lambda_2-1)+[(\lambda_3+4\lambda_2-1)^2+16\lambda_2]^{1/2}}{2},$$

the sharp upper bound for $|f|$ always occurs for

$$|f_4(re^{i\pi})| = -f_4(-r),$$

since

$$p_0 \le \frac{-(\lambda_3+4\lambda_2-1)+[(\lambda_3+4\lambda_2-1)^2+16\lambda_2]^{1/2}}{2},$$

for all (α, β).

In view of the fact that $f\in E_T(\varphi, \psi; \alpha, \beta)$ if and only if

$$zf'(z) \in E_T\left(\int_0^z \frac{\varphi(t)}{t}\,dt, \int_0^z \frac{\psi(t)}{t}\,dt; \alpha, \beta\right),$$

Lemma 3 and Theorem 7 for $f'(z)$ below easily follow from Lemma 1 and Theorem 6, respectively.

Lemma 3. *Let*

$$p_1 = \frac{-(\sigma_3+6\sigma_2-6\beta(1-\alpha))+[\{\sigma_3+6\sigma_2-6\beta(1-\alpha)\}^2+144\beta(1-\alpha)\sigma_2]^{1/2}}{12\beta(1-\alpha)},$$

let

$$r_1 = \frac{-3(1-p)\sigma_2+[\{9(1-p)^2\sigma_2^2+6\beta(1-\alpha)(1-p)p^2\sigma_3\}]^{1/2}}{6\beta(1-\alpha)(1-p)p},$$

and let

$$f_3(z) = z - \frac{2\beta(1-\alpha)p}{\sigma_2}z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3}z^3.$$

Then for $0\le r<1$, $0\le p<1$,

$$|f_3'(re^{i\theta})| \ge 1-\frac{4\beta(1-\alpha)p}{\sigma_2}r-\frac{6\beta(1-\alpha)(1-p)}{\sigma_3}r^2$$

with equality for $\theta=0$. *For either* $0\le p<p_1$ *and* $0\le r\le r_1$ *or* $p_1\le p\le 1$,

$$|f_3'(re^{i\theta})| \le 1+\frac{4\beta(1-\alpha)p}{\sigma_2}r-\frac{6\beta(1-\alpha)(1-p)}{\sigma_3}r^2$$

with equality for $\theta=\pi$. *For* $0\le p<p_1$ *and* $r_1\le r<1$,

$$|f_3'(re^{i\theta})| \le \left[\frac{3(1-p)\sigma_2^2+2\beta(1-\alpha)p^2\sigma_3}{3(1-p)\sigma_2^2} + \right.$$

$$+\frac{4\beta(1-\alpha)\{2\beta(1-\alpha)p^2\sigma_3+3(1-p)\sigma_2^2\}}{\sigma_2^2\sigma_3}r^2 +$$

$$\left. + \frac{12\beta^2(1-\alpha)^2(1-p)\{3(1-p)\sigma_2^2+2\beta(1-\alpha)p^2\sigma_2\}}{\sigma_2^2\sigma_3^2}r^4\right]^{1/2}$$

with equality for

$$\theta = \cos^{-1}\left[\frac{6\beta(1-\alpha)(1-p)\,pr^2 - p\sigma_3}{6(1-p)\,\sigma_2 r}\right].$$

Theorem 7. *If* $f \in E_T^P(\varphi, \psi; \alpha, \beta)$, *then*

$$|f'(re^{i\theta})| \geq 1 - \frac{4\beta(1-\alpha)\,p}{\sigma_2} r - \frac{6\beta(1-\alpha)(1-p)}{\sigma_3} r^2, \quad 0 \leq r < 1,$$

with equality for $f_3'(z)$ *at* $z = r$ *and*

$$|f'(re^{i\theta})| \leq \max\left\{\max_\theta |f_3'(re^{i\theta})|, -f_4'(-r)\right\},$$

where $\max|f_3'(re^{i\theta})|$ *is given by Lemma* 3.

***Corollary* 3.** *If* $f \in E_T^P(\varphi, \psi; \alpha, \beta)$, *then* $f(U)$ *contains a disc of radius*

$$1 - \frac{2\beta(1-\alpha)\,p}{\sigma_2} - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3}, \quad 0 \leq p \leq 1, \quad 0 \leq \alpha < 1, \quad 0 < \beta \leq 1.$$

***Remark* 4.** For suitable choices of $\varphi(z)$ and $\psi(z)$ as mentioned in Sec. 1, the results of this section lead to distortion and covering theorems for functions in $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$, etc. when their second coefficient is kept fixed.

Finally, we prove

Theorem 8. *Let* $f \in E_T^P(\varphi, \psi; \alpha, \beta)$ *and let* $g(z) = z + \sum_{n=2}^\infty \lambda_n' z^n$, $h(z) = z + \sum_{n=2}^\infty \mu_n' z^n$, *where* $\lambda_n' \geq 0$, $\mu_n' \geq 0$, *and* $\lambda_n' \geq \mu_n'$. *Let*

$$\delta = \frac{\sigma_2\sigma_3 - (1-\alpha)[(1+\beta)\{p\lambda_2'\sigma_3 + (1-p)\,\lambda_3'\sigma_2\} - (1-\beta)\{p\mu_2'\sigma_3 + (1-p)\,\mu_3'\sigma_2\}]}{\sigma_2\sigma_3 - 2\beta(1+\alpha)[p\mu_2'\sigma_3 + (1-p)\,\mu_3'\sigma_3]}.$$

If $\sigma_n' = (1-\beta)\,\lambda_n' - (1-\beta+2\delta\beta)\,\mu_n'$ *and* $w(n) = \sigma_n'/\sigma_n$ *for* $n \geq 3$ *is a decreasing function of* n, *then* $f \in E_T(g, h; \delta, \beta)$. *The result is sharp with*

$$f^*(z) = z - \frac{2\beta(1-\alpha)\,p}{\sigma_2} z^2 - \frac{2\beta(1-\alpha)(1-p)}{\sigma_3} z^3.$$

Proof. Since

$$\frac{\sigma_2'}{2\beta(1-\delta)}\frac{2\beta(1-\alpha)\,p}{\sigma_2} + \frac{\sigma_3'}{2\beta(1-\delta)}\frac{2\beta(1-\alpha)(1-p)}{\sigma_3} =$$

$$= p\frac{\sigma_2'}{\sigma_2}\frac{1-\alpha}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta}\frac{\sigma_3'}{\sigma_3} = 1, \tag{6}$$

we have $f^* \in E_T(g, h; \delta, \beta)$. To show that an arbitrary

$$f(z) = z - \frac{2\beta(1-\alpha)\,p}{\sigma_2} z^2 - \sum_{n=3}^\infty |a_n|\, z^n,$$

is in $E_T^P(g, h; \delta, \beta)$, we must show that

$$\frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)\,p}{1-\delta} + \sum_{n=3}^\infty \frac{\sigma_n'}{2\beta(1-\delta)}|a_n| \leq 1.$$

By Theorem 3, we may set, for $n \geq 3$, $|a_n| = \{2\beta(1-\alpha)(1-p)\rho_n\}/\sigma_n$, where $\sum_{n=3}^{\infty} \rho_n \leq 1$. Thus,

$$\frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)p}{1-\delta} + \sum_{n=3}^{\infty}\frac{\sigma_n'}{2\beta(1-\delta)}|a_n| =$$

$$= \frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)p}{1-\delta} + \sum_{n=3}^{\infty}\frac{\sigma_n'}{2\beta(1-\delta)}\frac{2\beta(1-\alpha)(1-p)\rho_n}{\sigma_n} =$$

$$= \frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta}\sum_{n=3}^{\infty}\rho_n\frac{\sigma_n'}{\sigma_n} \leq$$

$$\leq \frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta}\frac{\sigma_3'}{\sigma_3}\sum_{n=3}^{\infty}\rho_n \leq$$

$$\leq \frac{\sigma_2'}{\sigma_2}\frac{(1-\alpha)p}{1-\delta} + \frac{(1-\alpha)(1-p)}{1-\delta}\frac{\sigma_3'}{\sigma_3} = 1,$$

since σ_n'/σ_n is a decreasing function of n, and by (6) the result follows.

Remark 5. Choosing $\varphi(z) = (z+z^2)/(1-z)^3$, $\psi(z) = z/(1-z)^2$, and $g(z) = z/(1-z)^2$, $h(z) = z/(1-z)$ so that $\lambda_n = n^2$, $\mu_n = n$, $\lambda_n' = n$, and $\mu_n' = 1$ with $\beta = 1$, we get the corresponding result due to Silverman and Silvia [9]. The corresponding result for the classes $S_T(\alpha, \beta)$, $C_T(\alpha, \beta)$, $R_T(\alpha, \beta)$, $D_T(\alpha, \beta)$, etc., when their second coefficient is kept fixed, can be obtained for suitable choices of the functions $\varphi(z)$, $\psi(z)$, $g(z)$, and $h(z)$.

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