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A necessary condition to test the minimality of generalized linear sequential machines using the theory of near-semirings

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Abstract. In this work first we study the properties of near-semirings to introduce an α-radical. Then we observe the role of near-semirings in generalized linear sequential machines, and we test the minimality through the radical.

Introduction

Holcombe used the theory of near-rings to study linear sequential machines of Eilenberg [1, 4]. Though the picture of near-rings in linear sequential machines is a natural extension of the syntactic semigroups, the decomposition of linear sequential machines, which is different from the one given by Eilenberg, enabled Holcombe to study these machines thoroughly using near-rings [4]. Holcombe established several properties of machines using near-rings. Indeed, he has introduced an α-radical of affine near-rings which plays an important role to test the minimality of linear sequential machines [5]. The construction of the radical is motivated by Theorem 4.6 of [4], which can be read as:

Let $\mathcal{M} = (Q, A, B, F, G)$ be a linear sequential machine. If $\mathcal{M}$ is minimal then there is no proper nonzero N-submodule K of Q such that

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$G_0(K) = \{0\}$, *where* N *is the syntactic near-ring of* $\mathcal{M}$ *and* $G_0(q) = G(q, 0)\ \forall q \in Q$.

However, with the hypothesis of the above result, one can observe that there is no proper nonzero N-submodule of Q (cf. Theorem 4). This observation enables us to obtain an improvement in the α-radical. Indeed we are able to introduce an improved α-radical for near-semirings in a more general setup to study linear sequential machines. In order to extend the work for near-semirings, we formulated generalized linear sequential machines by replacing modules with semimodules in linear sequential machines. Even in this generalization, without losing much information, we could get all the results that have been obtained by Holcombe for linear sequential machines. Moreover, the radical obtained is much simpler.

This paper is organized in four sections. In Section 1, we introduce fundamental notions in the theory of near-semirings, and prepare the background to study sequential machines. The tools and techniques of universal algebra [3] have been used to extend the notions of near-rings. Following van Hoorn, we have extended some of the notions of ideals for near-semirings, which have been defined for zero-symmetric near-semirings [6]. Section 2 is dedicated to introduce and study the properties of α-radicals of near-semirings. In Section 3, we generalize the notion of linear sequential machines and study the role of near-semirings via Holcombe's decomposition of sequential machines, through minimization. Finally, in Section 4 we state how the α-radical provides a necessary condition to test the minimality of generalized linear sequential machines.

1. Near-semirings

An algebraic structure $(S, +, \cdot)$ is said to be a *near-semiring* if

1. $(S, +)$ is a semigroup with identity 0,
2. $(S, \cdot)$ is a semigroup,
3. $(x + y)z = xz + yz\ \forall x, y, z \in S$, and
4. $0s = 0\ \forall s \in S$.

Let $(\Gamma, +)$ be an additive semigroup with zero. The set of all functions $f : \Gamma \longrightarrow \Gamma$, denoted by $\mathfrak{M}(\Gamma)$, is a near-semiring with respect to point wise addition and composition of mappings. A near-semiring is a *semiring* if $+$ is commutative, $z(x + y) = zx + zy\ \forall x, y, z \in S$, and $s0 = 0\ \forall s \in S$ [2].

In what follows S always denotes a near-semiring and Γ denotes an additive semigroup with zero.

The notions of *homomorphism* and *subnear-semiring* can be defined in the usual way, respectively, as a mapping that preserves both the operations along with zero, and a subset with zero which is closed with respect to both the operations.

The set of all constant mappings on Γ and the set of all mappings on Γ which fixes zero have importance in this work. In fact, they are subnear-semirings of $\mathfrak{M}(\Gamma)$. Let us define these substructures in an arbitrary near-semiring S as follows.

Define $S_c = \{s \in S \mid s0 = s\}$ and $S_0 = \{s \in S \mid s0 = 0\}$. Note that S_c and S_0 are subnear-semiring of S. Moreover, $S_cS = S_c = SS_c$. In a near-semiring S, S_c and S_0 are said to be *constant* and *zero-symmetric* parts respectively. A near-semiring S is said to be a *constant near-semiring* (*zero-symmetric near-semiring*) if $S = S_c$ ($S = S_0$).

Remark 1. $S_c = \{s \in S \mid st = s, \ \forall t \in S\}$.

Define $S_d = \{x \in S \mid x(y+z) = xy + xz \ \forall y, z \in S\}$. Note that S_d is a semigroup with respect to multiplication and $0 \in S_d$. A subnear-semiring of S is said to be *distributively generated* if it is generated by a subsemigroup of S_d. Also, observe that $S_d + S_c$ is a semigroup with respect to multiplication and $0 \in S_d + S_c$.

Remark 2. If $+$ is commutative in $(S, +, \cdot)$ then S_d is closed with respect to $+$ and hence S_d is a subnear-semiring of S which is distributive. Also, in this case $S_d + S_c$ is a subnear-semiring of S.

Proposition 1. *Suppose $T = T_d + T_c$ is a subsemigroup of $S_d + S_c$ such that $dt \in T$ for all $d \in T_d$ and $t \in T$, where $T_d \subseteq S_d$ and $T_c \subseteq S_c$. If $0 \in T$ then the subnear-semiring of S generated by T,*

$$\langle T \rangle = \{\sum_{i=1}^{n} t_i \mid t_i \in T, n \geq 1\}.$$

Proof. Since $\langle T \rangle$ is contained in any subnear-semiring of S that contains T, and is closed with respect to addition, it is enough to observe that $\langle T \rangle$ is closed with respect to multiplication. For $d_i, d'_j \in T_d$; $c_i, c'_j \in T_c$ with

$1 \leq i \leq n, 1 \leq j \leq m$, consider

$$\begin{aligned}
&\left(\sum_{i=1}^{n}(d_i + c_i)\right)\left(\sum_{j=1}^{m}(d'_j + c'_j)\right) \\
= \quad & \sum_{i=1}^{n}\left(d_i \sum_{j=1}^{m}(d'_j + c'_j) \quad + c_i\right) \\
= \quad & \sum_{i=1}^{n}\left(\sum_{j=1}^{m-1}(d_i(d'_j + c'_j)) \quad + d_i(d'_m + c'_m) + c_i\right) \\
= \quad & \sum_{i=1}^{n}\left(\sum_{j=1}^{m-1}(d_i(d'_j + c'_j)) \quad + (d_i + c_i)(d'_m + c'_m)\right)
\end{aligned}$$

Since T is a subsemigroup and $T_d T \subseteq T$, for each $1 \leq i \leq n, 1 \leq j \leq m - 1$; $(d_i + c_i)(d'_m + c'_m)$ and $d_i(d'_j + c'_j)$ are in T. Thus the above expression is a finite sum of elements of T, so that $\langle T \rangle$ is closed with respect to multiplication. Hence the result. □

Corollary 1. *The subnear-semiring of S generated by $S_d + S_c$,*

$$\langle S_d + S_c \rangle = \{\sum_{i=1}^{n} s_i \mid s_i \in S_d + S_c, n \geq 1\}.$$

Let us denote the set of endomorphisms of Γ by $End\Gamma$. It is clear that $End\Gamma$ is a semigroup. Note that each element of $End\Gamma$ is a distributive element of $\mathfrak{M}(\Gamma)$. Moreover, $End\Gamma = \mathfrak{M}(\Gamma)_d$. Indeed, if $f \in \mathfrak{M}(\Gamma)$ is distributive then for $\gamma_1, \gamma_2 \in \Gamma$,

$$\begin{aligned}
f(\gamma_1 + \gamma_2) = f(\hat{\gamma}_1(\gamma) + \hat{\gamma}_2(\gamma)) \quad &= \quad f(\hat{\gamma}_1 + \hat{\gamma}_2)(\gamma) \\
&= \quad (f\hat{\gamma}_1 + f\hat{\gamma}_2)(\gamma) = f(\gamma_1) + f(\gamma_2),
\end{aligned}$$

where $\hat{\gamma}_1$, $\hat{\gamma}_2$ are constant maps on Γ which take the values γ_1, γ_2 respectively and $\gamma \in \Gamma$ is arbitrary.

Let $Con\Gamma$ be the set of all constant functions of $\mathfrak{M}(\Gamma)$. $Con\Gamma$ is a subnear-semiring of $\mathfrak{M}(\Gamma)$ and

$$\mathfrak{M}(\Gamma)Con\Gamma = Con\Gamma = Con\Gamma\mathfrak{M}(\Gamma).$$

Furthermore, $Con\Gamma = \mathfrak{M}(\Gamma)_c$ (cf. Remark 1).

An element of $\mathfrak{M}(\Gamma)$ is said to be an *affine mapping* if it is a sum of an endomorphism and a constant map on Γ. The set of affine mappings on

Γ, denoted by $\mathfrak{M}_{\text{aff}}(\Gamma)$, i.e. $\mathfrak{M}_{\text{aff}}(\Gamma) = End\Gamma + Con\Gamma$, is a subsemigroup of $\mathfrak{M}(\Gamma)$. The near-semiring generated by the semigroup $End\Gamma + Con\Gamma$, denoted by $EC(\Gamma)$, is defined as the *affinely generated near-semiring.* It is clear from the Corollary 1 that a typical element of $EC(\Gamma)$ is of the form

$$\sum_{i=1}^{n}(e_i + c_i) \ \text{ for } \ e_i \in End\Gamma, \ c_i \in Con\Gamma.$$

Remark 3. If $(\Gamma, +)$ is commutative, then the set of affine mappings on Γ, $\mathfrak{M}_{\text{aff}}(\Gamma)$, is subnear-semiring of $\mathfrak{M}(\Gamma)$. That is

$$\mathfrak{M}_{\text{aff}}(\Gamma) = EC(\Gamma).$$

In this case we call $\mathfrak{M}_{\text{aff}}(\Gamma)$ as *affine near-semiring.*

In general, a near-semiring S is said to be *affine near-semiring* if there exist a semiring $E \subseteq S$, and a constant near-semiring $C \subseteq S$ such that $S = E + C$.

A semigroup $(\Gamma, +)$ is said to be an *S-semigroup* if there exists a composition $(x, \gamma) \mapsto x\gamma$ of $S \times \Gamma \longrightarrow \Gamma$ such that for all $x, y \in S, \gamma \in \Gamma$,

1. $(x + y)\gamma = x\gamma + y\gamma$,

2. $(xy)\gamma = x(y\gamma)$, and

3. $0\gamma = 0_\Gamma$, where 0_Γ is the zero of Γ.

It is clear that Γ is an S-semigroup with $S = \mathfrak{M}(\Gamma)$. The semigroup $(S, +)$ of a near-semiring $(S, +, \cdot)$ is an S-semigroup. We denote this S-semigroup by S^+.

A subsemigroup Δ of an S-semigroup Γ is such that $S\Delta \subseteq \Delta$ then we say it is an *S-subsemigroup* of Γ. An *S-morphism* of an S-semigroup Γ is a semigroup morphism φ of Γ into an S-semigroup Γ' such that $\varphi(x\gamma) = x\varphi(\gamma)$ for all $\gamma \in \Gamma$ and $x \in S$. Note that $\varphi(0_\Gamma) = 0_{\Gamma'}$. Indeed, $\varphi(0_\Gamma) = \varphi(00_\Gamma) = 0\varphi(0_\Gamma) = 0_{\Gamma'}$.

Following van Hoorn, we extend some of the notions of ideals, which are appropriate in this context, from zero-symmetric near-semirings to near-semirings. For details on ideals of zero-symmetric near-semiring one may refer [6].

An *ideal* of a near-semiring is defined as the kernel of a near-semiring homomorphism. The kernel of an S-morphism is called as *S-kernel* of Γ. The S-kernels of the S-semigroup S^+ are called *left ideals* of S. A right invariant left ideal of S is called as a *weak ideal* of S. Note that every ideal of S is a weak ideal.

Theorem 1. *The annihilator $A(\Delta)$ of a non-void subset Δ of an S-semigroup Γ is a left ideal of S^+.*

Proof. Since $A(\Delta) = \bigcap\{A(\delta) \mid \delta \in \Delta\}$, it is enough to show that, for $\delta \in \Delta$, $A(\delta)$ is an S-kernel of S^+. Observe that the mapping $x \mapsto x\delta$: $S^+ \longrightarrow \Gamma$ is an S-morphism whose kernel is $A(\delta)$, so that $A(\delta)$ is a left ideal of S^+. Hence $A(\Delta)$ is a left ideal of S^+. □

Further, since $A(\Gamma)$ is right invariant in S we have $A(\Gamma)$ is a weak ideal of S. Indeed, for any $x \in S$, $s \in A(\Gamma)$, and $\gamma \in \Gamma$; $(sx)\gamma = s(x\gamma) = s\gamma' = 0_\Gamma$, where $\gamma' = x\gamma \in \Gamma$, so that $A(\Gamma)S \subseteq A(\Gamma)$.

Remark 4. For $\gamma \in \Gamma$, the relation $\overset{A(\gamma)}{\equiv}$ on S^+ defined by $x \overset{A(\gamma)}{\equiv} y$ if and only if $x\gamma = y\gamma$ is a congruence on S^+. Kernel of $\overset{A(\gamma)}{\equiv}$ is $A(\gamma)$.

Remark 5. The relation $\overset{A(\Gamma)}{\equiv}$ defined on the near-semiring S by $s \overset{A(\Gamma)}{\equiv} s'$ if and only if $s\gamma = s'\gamma$ for all $\gamma \in \Gamma$ is a congruence on S. Moreover, $\overset{A(\Gamma)}{\equiv} = \bigcap\limits_{\gamma\in\Gamma} \overset{A(\gamma)}{\equiv}$ so that the kernel of $\overset{A(\Gamma)}{\equiv}$ is $A(\Gamma)$.

Thus, by identifying $A(\Gamma)$ as the kernel of canonical homomorphism from S to $S/_{\overset{A(\Gamma)}{\equiv}}$ we have:

Theorem 2. *The annihilator $A(\Gamma)$ of an S-semigroup Γ is an ideal of S.* □

2. The α-radical

Let $(B, +)$ be a commutative semigroup and let $S = E + C$ be an affine near-semiring in which $+$ is commutative, where E is a semiring and C is a near-semiring of constants. A pair (S, α) is called a *B-pair* if

1. $\alpha : (S, +) \longrightarrow (B, +)$ is a semigroup morphism, and
2. $E \subseteq Ker\ \alpha$.

In the following by $\equiv_\varphi \leq \equiv_\psi$ we mean, whenever $\varphi(x) = \varphi(y)$ then $\psi(x) = \psi(y)$. Consequently, $\equiv_\varphi \leq \equiv_\psi \Longrightarrow Ker\ \varphi \subseteq Ker\ \psi$.

An S-semigroup Γ is said to be an *(S, α)-semigroup* if

$$\overset{A(0_\Gamma)}{\equiv} \leq \equiv_\alpha,$$

where $\equiv_\alpha$ is the congruence induced by the morphism α.

Theorem 3. *Let Γ be an (S,α)-semigroup and $\equiv$ be a congruence of S such that $\equiv\ \leq\ \overset{A(\Gamma)}{\equiv}$. Then there exists an $\bar{\alpha} : S/_{\equiv} \longrightarrow B$ such that Γ is an $(S/_{\equiv},\bar{\alpha})$-semigroup.*

Proof. Since $\equiv\ \leq\ \overset{A(\Gamma)}{\equiv}$, by setting $[x]\gamma = x\gamma\ \forall [x] \in S/_{\equiv},\ \gamma \in \Gamma$, one can observe that it is well-defined, and consequently, Γ is $S/_{\equiv}$-semigroup.

Define $\bar{\alpha} : S/_{\equiv} \longrightarrow B$ by

$$\bar{\alpha}([x]) = \alpha(x),$$

for all $[x] \in S/_{\equiv}$. Suppose $[x] = [x']$, i.e. $x \equiv x'$, then since $\equiv\ \leq\ \overset{A(\Gamma)}{\equiv}$ and $\overset{A(\Gamma)}{\equiv} = \bigcap_{\gamma\in\Gamma} \overset{A(\gamma)}{\equiv}$, we have $\equiv\ \leq\ \overset{A(0_\Gamma)}{\equiv}$. Thus $\equiv\ \leq\ \equiv_\alpha$, so that $\alpha(x) = \alpha(x')$. Hence, $\bar{\alpha}$ is well-defined. Moreover, $\bar{\alpha}$ is a semigroup morphism.

It is clear that $S/_{\equiv}$ is an affine near-semiring. To show the rest of the result, i.e. $(S/_{\equiv},\bar{\alpha})$ forms a B-pair, it is enough to show that semiring part of $S/_{\equiv}$ is in $Ker\ \bar{\alpha}$. If $[x]$ is in semiring part of $S/_{\equiv}$, then $[x][0] = [0]$, i.e. $[x0] = [0]$.

$$[x0] = [0] \Longrightarrow x0 \equiv 0 \Longrightarrow x0 \overset{A(\Gamma)}{\equiv} 0 \Longrightarrow (x0)\gamma = 0\gamma \quad \forall \gamma \in \Gamma.$$

In particular, $(x0)0_\Gamma = 00_\Gamma$, i.e. $x0_\Gamma = 00_\Gamma$. Which implies $x \overset{A(0_\Gamma)}{\equiv} 0$, so that $\alpha(x) = \alpha(0)$. Hence, $\bar{\alpha}([x]) = \bar{\alpha}([0])$ as desired. □

Corollary 2. *Let (S,α) be a B-pair and Γ an (S,α)-semigroup, then Γ is an $(S/_{\overset{A(\Gamma)}{\equiv}},\bar{\alpha})$-semigroup.*

Let S be a near-semiring. An S-semigroup Γ is said to be *zero-generated* if

$$\Gamma = S0_\Gamma.$$

Theorem 4. *If an S-semigroup Γ is zero-generated, then Γ has no proper S-subsemigroups.*

Proof. Suppose there exists a proper S-subsemigroup Δ of Γ. Note that $0_\Gamma \in \Delta$, as it is a subsemigroup of Γ. Since Γ is zero-generated, for $\gamma \in \Gamma \setminus \Delta$, there exists $s \in S$, such that $\gamma = s0_\Gamma$. But since Δ is an S-subsemigroup, $s0_\Gamma \in \Delta$ for all $s \in S$. A contradiction to $\gamma \notin \Delta$. □

Note that $\Gamma = S0_\Gamma$ if and only if $\Gamma = S_c0_\Gamma$. In particular, if $S = E+C$, an affine near-semiring, then $\Gamma = S0_\Gamma$ if and only if $\Gamma = C0_\Gamma$.

An (S,α)-semigroup is *zero-generated* if it is zero-generated as S-semigroup. Given a zero-generated (S,α)-semigroup Γ we define the function $\psi : \Gamma \longrightarrow B$ by $\psi(\gamma) = \alpha(c)$, where $\gamma = c0_\Gamma$, for $c \in C$. Thus we have:

Proposition 2. *$\psi : \Gamma \longrightarrow B$ is a semigroup morphism.*

Proof. Let $\gamma = c0_\Gamma = c'0_\Gamma$, for $c, c' \in C$. Then $c \overset{A(0_\Gamma)}{\equiv} c'$, which implies $\alpha(c) = \alpha(c')$, as $\overset{A(0_\Gamma)}{\equiv} \leq \equiv_\alpha$. Hence ψ is well-defined, and since α is a semigroup morphism, we have ψ is a semigroup morphism. □

An (S,α)-semigroup Γ is said to be *B-minimal* if Γ is zero-generated and $\equiv_\psi$, the congruence induced by ψ, is the identity relation.

Theorem 5. *Let (S,α) be a B-pair and Γ an (S,α)-semigroup. Suppose $\equiv$ is congruence of S such that $\equiv \leq \overset{A(\Gamma)}{\equiv}$. Then the $(S/_\equiv, \bar{\alpha})$-semigroup Γ is B-minimal if and only if the (S,α)-semigroup Γ is B-minimal.*

Proof. If the (S,α)-semigroup Γ is zero-generated then so is the $(S/_\equiv, \bar{\alpha})$-semigroup Γ and conversely. Also, the function $\psi' : \Gamma \longrightarrow B$ defined by $\bar{\alpha}$ equals the function $\psi : \Gamma \longrightarrow B$ defined by α. Thus, (S,α)-semigroup Γ is B-minimal if and only if it is B-minimal as an $(S/_\equiv, \bar{\alpha})$-semigroup. □

Now we define α-radical of a near-semiring as follows:

Given a B-pair (S,α) an *α-radical* of S, denoted by $R_\alpha(S)$, is defined as the intersection of annihilators of all B-minimal (S,α)-semigroups, i.e.

$$R_\alpha(S) = \bigcap_{\Gamma \in \mathcal{B}} A(\Gamma),$$

where $\mathcal{B}$ is the class of all B-minimal (S,α)-semigroups.

Remark 6. $R_\alpha(S)$ is an ideal of the near-semiring S.

Define the congruence relation $\overset{R_\alpha(S)}{\equiv}$ on S by $x \overset{R_\alpha(S)}{\equiv} y$ if and only if $x \overset{A(\Gamma)}{\equiv} y$ for all $\Gamma \in \mathcal{B}$, so that the kernel of $\overset{R_\alpha(S)}{\equiv}$ is $R_\alpha(S)$.

Theorem 6. *Let (S,α) be a B-pair and $\equiv$ a congruence of S such that $\equiv \leq \overset{R_\alpha(S)}{\equiv}$. Then we have*

$$R_{\bar{\alpha}}(S/_\equiv) \subseteq (R_\alpha(S))/_\equiv.$$

Proof. Let Γ be a B-minimal (S, α)-semigroup, so that $R_\alpha(S) \subseteq A(\Gamma)$. We have seen that Γ is a B-minimal $(S/_\equiv, \bar{\alpha})$-semigroup. Let $[x] \in S/_\equiv$ be such that $[x]\Gamma = (0_\Gamma)$ then $x\Gamma = (0_\Gamma)$ and so $x \in A(\Gamma)$. Thus the annihilator of Γ in $S/_\equiv$ is contained in $A(\Gamma)/_\equiv$. If we write $A(\Gamma)^*$ to denote the annihilator of Γ in $S/_\equiv$ then

$$R_{\bar{\alpha}}(S/_\equiv) \subseteq \bigcap_{\Gamma \in \mathcal{B}} A(\Gamma)^*,$$

where $\mathcal{B}$ is the class of B-minimal (S, α)-semigroups, and each $A(\Gamma)^* \subseteq A(\Gamma)/_\equiv$ so

$$\begin{aligned} R_{\bar{\alpha}}(S/_\equiv) &\subseteq \bigcap_{\Gamma \in \mathcal{B}} (A(\Gamma)/_\equiv) \\ &= (\bigcap_{\Gamma \in \mathcal{B}} A(\Gamma))/_\equiv = (R_\alpha(S))/_\equiv. \end{aligned}$$

□

Corollary 3. $R_{\bar{\alpha}}(S/_{\underset{\equiv}{R_\alpha(S)}}) = [0]$, *the zero of* $S/_{\underset{\equiv}{R_\alpha(S)}}$.

This is a justification for calling $R_\alpha(S)$ an α-radical of S. In the following we observe that α-radical is not always zero. In order to observe this, first we extend the notion of the radical $J_{(2,0)}$ of zero-symmetric near-semirings, introduced by van Hoorn [7], to near-semirings, then we prove that $J_{(2,0)}(S) \subseteq R_\alpha(S)$ for any α. Since many examples are known where $J_{(2,0)}(S)$ is nonzero, it is clear that $R_\alpha(S)$ is not always zero.

An S-semigroup $\Gamma \neq \{0_\Gamma\}$ is said to be *essentially minimal* or *of type* $(2, 0)$ if $S\Gamma \neq \{0_\Gamma\}$ and the only S-subsemigroups of Γ are $S0_\Gamma$ and Γ. The radical $J_{(2,0)}(S)$ of a near-semiring S is defined as the intersection of the annihilators of all S-semigroups of type $(2, 0)$.

Theorem 7. *Let* (S, α) *be a* B*-pair. Then* $J_{(2,0)}(S) \subseteq R_\alpha(S)$ *for any* α.

Proof. From Theorem 4 one can ascertain that every B-minimal (S, α)-semigroup is essentially minimal, and so

$$J_{(2,0)}(S) = \bigcap_{\Gamma \in \mathcal{C}} A(\Gamma) \subseteq \bigcap_{\Gamma \in \mathcal{B}} A(\Gamma) = R_\alpha(S),$$

where $\mathcal{C}$ is the class of essentially minimal S-semigroups, and $\mathcal{B}$ is the class of B-minimal (S, α)-semigroups. □

3. Generalized linear sequential machines

Let R be a semiring. A *generalized linear sequential machine* over R is a quintuple $\mathcal{M} = (Q, A, B, F, G)$, where

Q, A, B are R-semimodules,

$F : Q \times A \longrightarrow Q$ and $G : Q \times A \longrightarrow B$ are R-homomorphisms.

In what follows $\mathcal{M}$ always stands for a generalized linear sequential machine (Q, A, B, F, G) over a semiring R.

We call Q as the set of states, A as the input alphabet and B as the output alphabet. Let A^*, B^* be the free monoids generated by the sets A, B, respectively. The empty word Λ will be regarded as a member of both A^* and B^*.

For $x \in A^*$, we define the function $F_x : Q \longrightarrow Q$, called the *next state function induced by* x, by

$$\begin{aligned} F_\Lambda(q) &= q, \\ F_{xa}(q) &= F(F_x(q), a) \quad \text{for } x \in A^*, a \in A. \end{aligned}$$

Proposition 3. *For* $x = a_1 a_2 ... a_n \in A^*$,

$$F_x = F_0^n + (F_0^{n-1}\bar{q}_{a_1} + F_0^{n-2}\bar{q}_{a_2} + ... + F_0\bar{q}_{a_{n-1}} + \bar{q}_{a_n}),$$

where $\bar{q_a} : Q \longrightarrow Q$ *is the constant map given by* $\bar{q_a}(p) = F_a(0_Q)$ $\forall p \in Q$.

Proof. We prove this result by induction on the length of the string x. Let $a \in A$ and $q \in Q$.

$$\begin{aligned} F_a(q) = F(q, a) &= F(q, 0) + F(0_Q, a) \\ &= F_0(q) + F_a(0_Q). \end{aligned}$$

Therefore $F_a = F_0 + \bar{q_a}$, so that the result is true for $n = 1$. Assume the result is true for $n = k - 1$, i.e.

$$F_{a_1 a_2 ... a_{k-1}} = F_0^{k-1} + (F_0^{k-2}\bar{q}_{a_1} + F_0^{k-3}\bar{q}_{a_2} + ... + F_0\bar{q}_{a_{k-2}} + \bar{q}_{a_{k-1}}).$$

Now,

$$\begin{aligned} F_{a_1 a_2 ... a_k} &= F_{a_k} F_{a_1 a_2 ... a_{k-1}} \\ &= (F_0 + \bar{q}_{a_k}) F_{a_1 a_2 ... a_{k-1}} \\ &= F_0 F_{a_1 a_2 ... a_{k-1}} + \bar{q}_{a_k} F_{a_1 a_2 ... a_{k-1}} \\ &= F_0(F_0^{k-1} + (F_0^{k-2}\bar{q}_{a_1} + F_0^{k-3}\bar{q}_{a_2} + ... + F_0\bar{q}_{a_{k-2}} + \bar{q}_{a_{k-1}})) + \bar{q}_{a_k} \\ &= F_0^k + F_0^{k-1}\bar{q}_{a_1} + F_0^{k-2}\bar{q}_{a_2} + ... + F_0\bar{q}_{a_{k-1}} + \bar{q}_{a_k}. \end{aligned}$$

Hence the result by induction. □

Note that the function $F_0 : Q \longrightarrow Q$ is an R-endomorphism of Q. Thus we have:

Corollary 4. *For $x \in A^*$ the function F_x is an affine function of Q.*

The set of all affine functions of Q, $\mathfrak{M}_{\text{aff}}(Q)$, is a near-semiring under pointwise addition and composition of mappings (cf. Remark 3). Consider the syntactic monoid M of $\mathcal{M}$, i.e. $M = \{F_x \mid x \in A^*\}$, a submonoid of $\mathfrak{M}_{\text{aff}}(Q)$. Note that $M_d = \{F_0^n \mid n \geq 1\}$, the endomorphism part of M.

Proposition 4. $M_d M \subseteq M$.

Proof. Let $F_0^n \in M_d$ and $F_x \in M$ with $x = a_1 a_2 \dots a_k$. Choose $y = a_1 a_2 \dots a_k \overbrace{00\dots0}^{n \text{ times}} \in A^*$. Then by Proposition 3,

$$\begin{aligned} F_y &= F_0^{n+k} + (F_0^{n+k-1}\bar{q}_{a_1} + F_0^{n+k-2}\bar{q}_{a_2} + \dots + F_0^{n+1}\bar{q}_{a_{n-1}} + F_0^n \bar{q}_{a_n} \\ &\quad + F_0^{n-1}\bar{q}_0 + F_0^{n-2}\bar{q}_0 + \dots + \bar{q}_0) \\ &= F_0^{n+k} + (F_0^{n+k-1}\bar{q}_{a_1} + F_0^{n+k-2}\bar{q}_{a_2} + \dots + F_0^{n+1}\bar{q}_{a_{n-1}} + F_0^n \bar{q}_{a_n}) \\ &= F_0^n (F_0^k + (F_0^{k-1}\bar{q}_{a_1} + F_0^{k-2}\bar{q}_{a_2} + \dots + F_0 \bar{q}_{a_{n-1}} + \bar{q}_{a_n})) \\ &= F_0^n F_x. \end{aligned}$$

Thus, for any $n \geq 1$ and $x \in A^*$, $F_0^n F_x \in M$. □

The subnear-semiring of $\mathfrak{M}_{\text{aff}}(Q)$, generated by M is defined as the *syntactic near-semiring* of $\mathcal{M}$, denoted by $S_{\mathcal{M}}$.

Remark 7.

1. Every nonzero element of $S_{\mathcal{M}}$ can be written as $\sum_{i=1}^{n} m_i$ for $m_i \in M$ (cf. Propositions 1 and 4).
2. The state set Q of $\mathcal{M}$ is an S-semigroup with $S = S_{\mathcal{M}}$.
3. $S_{\mathcal{M}}$ is an affine near-semiring.

For $q \in Q$, the *sequential function defined by* q, $f_q : A^* \longrightarrow B^*$ is defined inductively as

$$\begin{aligned} f_q(\Lambda) &= \Lambda, \\ f_q(a) &= G(q, a), \\ f_q(xa) &= f_q(x) f_{F_x(q)}(a) \quad \text{for } x \in A^*, a \in A. \end{aligned}$$

For $a \in A$ and $q \in Q$ note that

$$\begin{aligned} f_q(a) = G(q,a) &= G(q,0) + G(0_Q, a) \\ &= G_0(q) + G_a(0_Q), \end{aligned}$$

by adapting the notation of the state function F_x.

Let $\mathcal{M} = (Q, A, B, F, G)$ and $\mathcal{M}' = (Q', A, B, F', G')$ be generalized linear sequential machines. A *generalized linear sequential machine morphism*, denoted by $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$, is an R-homomorphism $\varphi : Q \longrightarrow Q'$ such that

$$\begin{aligned} \varphi(F_a(q)) &= F'_a(\varphi(q)), \\ G_a(q) &= G'_a(\varphi(q)) \qquad \text{for } \; q \in Q, a \in A. \end{aligned}$$

Remark 8. For $x \in A^*$, $\varphi(F_x(q)) = F'_x(\varphi(q))$.

The following result establishes the interrelation between $\mathcal{M}$ and $S_{\mathcal{M}}$.

Theorem 8. *If $\varphi : \mathcal{M} \longrightarrow \mathcal{M}'$ is a surjective generalized linear sequential machine morphism, then there exists a near-semiring homomorphism $\psi : S_{\mathcal{M}} \longrightarrow S_{\mathcal{M}'}$ such that, for $q \in Q, s \in S_{\mathcal{M}}$,*

$$\varphi(sq) = \psi(s)(\varphi(q)).$$

Furthermore, for $q \in Q$, $f_q = f'_{\varphi(q)}$.

Proof. Let M and M' be the syntactic monoids of $\mathcal{M}$ and $\mathcal{M}'$ respectively. By a standard result in automata theory, there exists a monoid morphism $\eta : M \longrightarrow M'$ such that $\varphi(mq) = \eta(m)\varphi(q)$ for $q \in Q, m \in M$. Define a zero fixing mapping $\psi : S_{\mathcal{M}} \longrightarrow S_{\mathcal{M}'}$ by

$$\psi(\sum_{i=1}^{k} m_i) = \sum_{i=1}^{k} \eta(m_i), \quad \text{for } m_i \in M.$$

For $m_i, m'_j \in M$, assume $\sum_{i=1}^{k} m_i = \sum_{j=1}^{l} m'_j$. That means, they are equal at every point of Q. By applying the morphism φ, we will arrive at $(\sum_{i=1}^{k} \eta(m_i))(\varphi(q)) = (\sum_{j=1}^{l} \eta(m'_j))(\varphi(q))$ for all $q \in Q$. But since φ is surjective, we get $\sum_{i=1}^{k} \eta(m_i) = \sum_{j=1}^{l} \eta(m'_j)$. Thus, ψ is well-defined. Also, it can be easily observed that ψ is a near-semiring homomorphism.

For $q \in Q, s = \sum_{i=1}^{k} m_i \in S_{\mathcal{M}}$, we have

$$\begin{aligned} \varphi(sq) &= \varphi(\sum_{i=1}^{k} m_i q) = \sum_{i=1}^{k} \varphi(m_i q) \\ &= \sum_{i=1}^{k} \eta(m_i)(\varphi(q)) = \left(\sum_{i=1}^{k} \eta(m_i)\right)(\varphi(q)) \\ &= \psi(s)(\varphi(q)). \end{aligned}$$

Now we show by an induced argument that for $q \in Q$, $f'_{\varphi(q)} = f_q$. For $a \in A$, $f_q(a) = G_a(q) = G'_a(\varphi(q)) = f'_{\varphi(q)}(a)$. Further, for $x \in A^*$; $a \in A$,

$$\begin{aligned} f_q(xa) &= f_q(x) f_{F_x(q)}(a) \\ &= f'_{\varphi(q)}(x) f'_{\varphi(F_x(q))}(a) \\ &= f'_{\varphi(q)}(x) f'_{F'_x(\varphi(q))}(a), \quad \text{by Remark 8} \\ &= f'_{\varphi(q)}(xa). \end{aligned}$$

Thus $f_q(x) = f'_{\varphi(q)}(x)$ for all $x \in A^*$. □

In the following we discuss the role of syntactic near-semiring $S_{\mathcal{M}}$ in the minimization of $\mathcal{M}$.

Define the relation $\sim$ on Q by

$$q \sim q' \text{ if and only if } G_0 F_0^n(q) = G_0 F_0^n(q') \text{ for all } n \geq 0.$$

Theorem 9. *For $q, q' \in Q$, if $q \sim q'$ then $f_q = f_{q'}$.*

Proof. We prove this by induction on length of $x \in A^*$. Since $f_q(a) = G_0(q) + G_a(0_Q)$; $f_{q'}(a) = G_0(q') + G_a(0_Q)$; and $G_0(q) = G_0(q')$ from the hypothesis, we have $f_q(a) = f_{q'}(a)$ for all $a \in A$. If $f_q(x) = f_{q'}(x)$, where $x = a_1 a_2 \dots a_n$, then $f_q(xa) = f_q(x) f_{F_x(q)}(a)$, and

$$\begin{aligned} f_{F_x(q)}(a) &= G(F_x(q), a) = G_0(F_x(q)) + G_a(0_Q) \\ &= G_0(F_0^n(q) + \sum_{i=1}^{n} F_0^{n-i} \bar{q}_{a_i}(q)) + G_a(0_Q) \\ &= G_0 F_0^n(q) + \sum_{i=1}^{n} G_0 F_0^{n-i}(q_{a_i}) + G_a(0_Q). \end{aligned}$$

Similarly,

$$f_{F_x(q)}(a) = G_0 F_0^n(q') + \sum_{i=1}^{n} G_0 F_0^{n-i}(q_{a_i}) + G_a(0_Q),$$

so that

$$f_q(xa) = f_q(x) f_{F_x(q)}(a) = f_{q'}(x) f_{F_x(q')}(a) = f_{q'}(xa).$$

$\square$

Theorem 10. *The relation $\sim$ is a congruence relation on Q.*

Proof. Since $F_0^n : Q \longrightarrow Q$ is a composition of R-homomorphism F_0 with itself for n times, we have F_0^n is an R-homomorphism $\forall n$. Also, since $G_0 : Q \longrightarrow B$ is an R-homomorphism, we get $G_0 F_0^n : Q \longrightarrow B$ an R-homomorphism $\forall n$. Hence the relation $\sim$ is a congruence on Q, as it is the intersection of induced congruences of $G_0 F_0^n$ for all n. $\square$

A machine $\mathcal{M}$ is said to be *reduced* if the relation $\sim$ defined on Q is trivial, i.e. $q \sim q' \Longrightarrow q = q'$ for all $q, q' \in Q$. Thus we have:

Corollary 5. *The generalized linear sequential machine $\mathcal{M}_r = (Q', A, B, F', G')$ is a reduced machine of $\mathcal{M}$, where $Q' = Q/\sim$, $F'([q], a) = [F(q, a)]$, and $G'([q], a) = [G(q, a)]$, for all $q \in Q$, $a \in A$.*

As in Eilenberg's work [1], we assume $0_Q \in Q$ to be the initial state of $\mathcal{M}$. A generalized linear sequential machine $\mathcal{M} = (Q, A, B, F, G)$ is called *accessible* if given any $q \in Q$ there exists $x \in A^*$ such that $F_x(0_Q) = q$, i.e. any state is reachable from the initial state 0_Q.

Remark 9. If $\mathcal{M}$ is accessible then the S-semigroup Q is zero generated, i.e.

$$S_{\mathcal{M}} 0_Q \supseteq M 0_Q = Q.$$

Consequently, the S-semigroup Q has no proper S-subsemigroups.

A generalized linear sequential machine $\mathcal{M}$ is called *minimal* if it is accessible and reduced. In the following section we obtain a necessary condition to test the minimality of $\mathcal{M}$ using an α-radical of $S_{\mathcal{M}}$.

4. The radical applied to machines

Assume that $\mathcal{M} = (Q, A, B, F, G)$ is a generalized linear sequential machine over a semiring R. Let $S = S_{\mathcal{M}}$ be the syntactic near-semiring of $\mathcal{M}$. We have observed that S is an affine near-semiring, say $S = E + C$, and Q an S-semigroup. Furthermore, G defines an R-homomorphism $G_0 : Q \longrightarrow B$. Define $\alpha : S \longrightarrow B$ by

$$\alpha(s) = G_0(s 0_Q) \;\; \forall s \in S.$$

Note that, α is a semigroup morphism and $\alpha(e) = G_0(e0_Q) = G_0(0_Q) = 0$ $\forall e \in E$, so that (S, α) is B-pair. Moreover, Q is an (S, α)-semigroup, because $\forall s, t \in S$,

$$\begin{aligned} s \overset{A(0_Q)}{\equiv} t &\implies s0_Q = t0_Q \implies G_0(s0_Q) = G_0(t0_Q) \\ &\implies \alpha(s) = \alpha(t) \implies s \equiv_\alpha t. \end{aligned}$$

Now we examine the α-radical $R_\alpha(S)$, which is the intersection of annihilators of all B-minimal (S, α)-semigroups.

If $\mathcal{M}$ is minimal, then the state set Q, which is an (S, α)-semigroup, is zero-generated and the equivalence relation $\sim$ is trivial. Consequently, Q is B-minimal. But the annihilator of Q,

$$\begin{aligned} A(Q) &= \{s \in S \mid sq = 0 \ \ \forall q \in Q\} \\ &= \{s \in S \mid s = 0\} = (0) \end{aligned}$$

so that the α-radical of S,

$$R_\alpha(S) = \bigcap_{\Gamma \in \mathcal{B}} A(\Gamma) \subseteq A(Q) = (0),$$

where $\mathcal{B}$ is the class of B-minimal (S, α)-semigroups. Thus $R_\alpha(S) = (0)$. This can be summarized as follows:

Theorem 11. *If $\mathcal{M}$ is minimal then $R_\alpha(S) = (0)$.* □

The converse of Theorem 11 is not necessarily true, as there exist non-minimal machines with a zero radical [5].

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