

Rapid communication

Synchronization and anchoring of two non-harmonic canonical-dissipative oscillators via Smorodinsky-Winternitz potentials

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Two non-harmonic canonical-dissipative limit cycle oscillators are considered that oscillate in one-dimensional Smorodinsky-Winternitz potentials. It is shown that the standard approach of the canonical-dissipative framework to introduce dissipative forces leads naturally to a coupling force between the oscillators that establishes synchronization. The non-harmonic character of the limit cycles in the context of anchoring, the phase difference between the synchronized oscillators, and the degree of synchronization are studied in detail.

Key words: *canonical-dissipative systems, Smorodinsky-Winternitz potentials, synchronization, anchoring*

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Mass-spring systems are a fundamental topic of classical mechanics and solid state physics. A sophisticated theoretical framework is available to explain the oscillatory dynamics of macroscopic particles connected by springs and the oscillatory vibration of molecules interacting by spring-like forces. Given the success of this field of physics concerned with the inanimate world, the question naturally arises whether its scope can be broadened to take the life science into account [1]. Since a key feature of living systems is their ability to move by themselves, the question can be asked from a slightly different perspective. Can the concepts of classical mechanics be generalized to self-mobile, so-called active [2–4], systems? A well-studied class of active systems both in the animate and inanimate world are self-oscillators [5]. A theoretical framework that bridges the research fields of classical mechanics and self-oscillators is the theory of canonical-dissipative (CD) systems [6–11]. The reason for this is that a CD system can exhibit attractors that, on the one hand, are stable and in doing so reflect non-conservative, dissipative system components but, on the other hand, are defined in their respective phase spaces by the dynamics of conservative systems. In fact, in a series of recent experimental studies it has been shown that the CD approach can be applied to human self-oscillators, that is, humans producing oscillatory single limb movements [12–14]. Importantly, this line of research has been generalized to the non-harmonic case [15]. In general, human rhythmic limb movements exhibit non-harmonic components and in particular can show a so-called anchoring phenomenon. Anchoring means that a limb movement slows down during a particular short period of the cyclic activity in a more pronounced way than in the harmonic case. Non-harmonic self-oscillator models are promising candidates to capture human non-harmonic rhythmic activity including the anchoring phenomenon. However, humans and animals are known to coordinate their activities, in general, and movement patterns, in particular [16, 17]. As far as the CD approach is concerned, for synchronization with zero phase lag and 180 degrees phase lag, a four-variable CD model has been proposed recently [18]. Here, we proposed a more general model for two active particles that is motivated by the so-called SET model for swarming [19] and assumes that active particles are coupled via their angular momentum values (see also [20]). In order to address the

non-harmonic case, we consider self-oscillations in Smorodinsky-Winternitz potentials [21–23]. These potentials play an important role in physics as confinement potentials [24, 25]. In the context of the CD approach, the four-dimensional, two particle Smorodinsky-Winternitz systems should be considered as benchmark systems because they feature three invariants rather than only two. The first two invariants are the particle Hamiltonian energy functions. The third invariant is an appropriately adjusted angular momentum [21].

Recall that the standard CD oscillator in a one-dimensional space with coordinate q and momentum p is defined by [8, 10, 11, 26]

$$\frac{d}{dt}q = \frac{p}{m}, \quad \frac{d}{dt}p = -kq - \frac{\partial g}{\partial p}, \quad g = \frac{\gamma}{2}(H - B)^2, \quad (1)$$

where m denotes mass, k is the spring constant and $H = p^2/(2m) + kq^2/2$ corresponds to the Hamiltonian energy in the conservative case in which the function g is neglected. The function g describes the dissipative mechanism. The parameter $\gamma \geq 0$ is the coupling parameter of oscillator with the dissipative mechanism. Note that $-\partial g/\partial p = -\gamma p(H - B)/m$ such that the dissipative mechanism is composed of a negative friction term (i.e., pumping mechanism) $+\gamma Bp/m$ with pumping parameter $B \geq 0$ and a nonlinear friction term $-\gamma pH/m$. The amplitude dynamics can be obtained using standard techniques. We put $q = A(t) \exp(i\omega t) + \text{c.c.}$ with $\omega^2 = k/m$, where A denotes the complex-valued oscillator amplitude that is related to the real-valued amplitude $r(t)$ and the oscillator phase $\phi(t)$ like $A(t) = r(t) \exp[i\phi(t)]/2$. Here and in what follows, c.c. denotes the complex conjugate expression. Assuming that γ is a small perturbation parameter, by means of the slowly varying amplitude approximation and the rotating wave approximation [27], we obtain, in lowest order of γ , the following amplitude dynamics

$$\frac{d}{dt}A = -\gamma\omega^2 A \left(|A|^2 - \frac{B}{2m\omega^2} \right). \quad (2)$$

Note that higher order correction terms in γ can be obtained using alternative techniques (see e.g., [28]). From equation (2) it follows that $dr/dt = -\gamma\omega^2 r[r^2 - 2B/(m\omega^2)]/4$ and $d\phi/dt = 0$. The solution reads $r(t) = \sqrt{2Br_0^2/[m\omega^2 r_0^2 + (2B - m\omega^2 r_0^2) \exp(-\gamma Bt/m)]}$ with $r_0 = r(t=0)$. Figure 1 (a) shows a simulation of the self-oscillator (2) and the analytical solution $r(t)$. From equations (2) and the analytical solution $r(t)$ it follows that γ (in combination with the factor B/m) determines the time scale of the amplitude dynamics $A(t)$ and $r(t)$, respectively. Importantly, in the long time limit, r approaches $r_{\text{st}} = \sqrt{2B/(m\omega^2)}$ and H converges to the pumping parameter B . That is, B acts as a fixed point value or target value for the energy dynamics $H(t)$.

Let us generalize the single-oscillator case to a model of two coupled self-oscillators. Each self-oscillator oscillates in a one-dimensional space and is subjected to the force of a Smorodinsky-Winternitz potential. Let us describe the oscillator coordinates q_1 and q_2 and momenta p_1 and p_2 by means of the vectors $\mathbf{q} = (q_1, q_2)$ and $\mathbf{p} = (p_1, p_2)$. The Smorodinsky-Winternitz potentials read [21]

$$V_1(q_1) = k \frac{q_1^2}{2} + \frac{\alpha}{2q_1^2}, \quad V_2(q_2) = k \frac{q_2^2}{2} + \frac{\beta}{2q_2^2} \quad (3)$$

with $k > 0$ and $\alpha, \beta \geq 0$. For $\alpha = \beta = 0$, the potentials correspond to parabolic potentials and k can be interpreted as a spring constant. For $\alpha, \beta > 0$, the potentials exhibit minima at $q_1 = \pm(\alpha/k)^{1/4}$ and $q_2 = \pm(\beta/k)^{1/4}$ and exhibit repulsive singularities at $q_1 = 0$ and $q_2 = 0$. By contrast, for $q_1 \rightarrow \pm\infty$ and $q_2 \rightarrow \pm\infty$ they increase like parabolic potentials. Therefore, for $\alpha, \beta > 0$, the potentials are asymmetric with respect to their minima. Let us define the dynamics of the two oscillators in the conservative case by means of the Hamiltonian dynamics

$$\frac{d}{dt}\mathbf{q} = \frac{\partial H_{\text{tot}}}{\partial \mathbf{p}}, \quad \frac{d}{dt}\mathbf{p} = -\frac{\partial H_{\text{tot}}}{\partial \mathbf{q}}, \quad H_{\text{tot}} = H_1 + H_2, \quad H_1 = \frac{p_1^2}{2m} + V_1, \quad H_2 = \frac{p_2^2}{2m} + V_2. \quad (4)$$

In equation (4), the functions H_1 , H_2 and H_{tot} correspond to the Hamiltonian energy functions of the individual oscillators and the total energy of the two-oscillators system. It can be shown that the

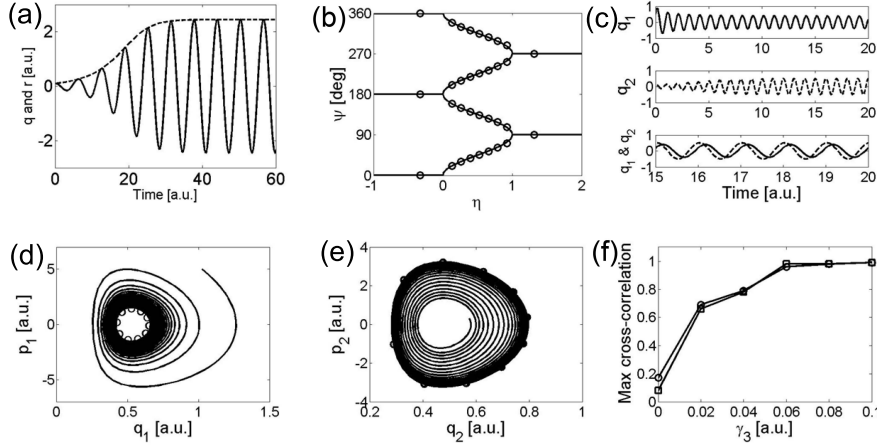


Figure 1. Panel (a): Solution $q(t)$ (solid line) of equation (1) obtained by a numerical Euler forward (EF) solution method. The analytical solution $r(t)$ is shown as well (dashed line). Parameters in a.u.: $m = k = 1$, $\gamma = 0.1$, $B = 3$. EF time step: $\tau = 0.01$. Initial value: $q(0) = 0.1$, $p(0) = 0 \Rightarrow r(0) = 0.1$. Panel (b): Phase difference ψ as function of η (solid lines) as predicted by our theoretical considerations for the harmonic case (see text). Circles denote simulation results obtained by solving equation (6) numerically (EF). Simulation parameters in a.u.: $m = 1$, $k = (2\pi)^2 \Rightarrow \omega = 2\pi$ (i.e., oscillator period equal to 1 time unit), $\alpha = \beta = 0$, $B_1 = 3$, $B_2 = 5$, B_3 was varied in the range $[-0.5, 1.5]$, $\tau = 0.001$. Various initial conditions were used. Panel (c): Trajectories $q_1(t)$ and $q_2(t)$ of equation (6) for a representative simulation trial used to generate the numerical results in panel (b). Top and middle sub-panels show transient and long term dynamics. Bottom sub-panel shows the synchronized state with a fixed phase difference. Here: $B_1 = 1.0 \Rightarrow \eta = 0.66$, $\psi = 54^\circ$. Panels (d) and (e): Phase portraits p_1 versus q_1 [panel (d)] and p_2 versus q_2 [panel (e)] obtained in the non-harmonic case by solving equation (6) numerically (EF). Parameters in a.u.: $m = 1$, $k = (2\pi)^2 \Rightarrow \omega = 4\pi$ (i.e. oscillator period equal to 0.5 time units), $\alpha = 3$, $\beta = 2$, $\gamma_{1,2} = 0.1$, $\gamma_3 = 0.2$, $B_1 = H_{1,\min} + 1$, $B_2 = H_{2,\min} + 5$, $B_3 = S_{3,\min} + 3$, $\tau = 0.001$. Initial conditions: $p_1 = 5.0$, $p_2 = 0.3$, $q_1 = (\alpha/k)^{0.25} + 0.5$, $q_2 = (\beta/k)^{0.25} + 0.1$. The circles show the predicted limit cycles obtained by solving numerically (EF) the evolution equations of the corresponding isolated, conservative oscillators. Panel (f): Maximal cross-correlation as function of γ_3 obtained by solving equation (14) numerically (stochastic EF [29]). Averages of 10 trials are shown. Trajectories of 10000 (circles) and 30000 (squares) time units were used in each trial. The maximal cross correlation scores for $\gamma_3 = 0$ are by-chance values that decay to zero when trajectory length goes to infinity. Parameters in a.u.: $D = 0.02$, all other parameters except for γ_3 as in panels (d) and (e). $\tau = 0.001$.

dynamics (4) exhibits three invariants S_j with $j = 1, 2, 3$ given by [21]

$$S_1 = H_1, \quad S_2 = H_2, \quad S_3 = \frac{L^2}{m} + (q_1^2 + q_2^2) \left(\frac{\alpha}{q_1^2} + \frac{\beta}{q_2^2} \right), \quad (5)$$

where L denotes the angular momentum $L = p_2 q_1 - p_1 q_2$. In line with the CD oscillator (1), we define the CD case like

$$\frac{d}{dt} \mathbf{q} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d}{dt} \mathbf{p} = -\frac{\partial H}{\partial \mathbf{q}} - \frac{\partial g_{\text{tot}}}{\partial \mathbf{p}}, \quad g_{\text{tot}} = \sum_{j=1}^3 g_j, \quad g_j = \gamma_j \frac{1}{2} (S_j - B_j)^2, \quad (6)$$

where $\gamma_j \geq 0$ are the coupling constants. $B_{1,2} \geq 0$ are the pumping parameters. B_3 is a target value for S_3 . Importantly, since S_j are invariants of the conservative dynamics (4), for the dissipative dynamics (6) it follows that $dg_{\text{tot}}/dt = -(\partial g_{\text{tot}}/\partial \mathbf{p})^2 \geq 0$. In view of the boundedness of g_{tot} (i.e., $g_{\text{tot}} \geq 0$), g_{tot} is a Lyapunov function that becomes stationary in the long term limit. This in turn implies that $\partial g_{\text{tot}}/\partial \mathbf{p} = 0$ such that equation (6) reduces to equation (4). In total, for $t \rightarrow \infty$, the system (6) converges to an attractor that corresponds to a solution of the conservative system (4).

Let us consider the harmonic case defined by $\alpha = \beta = 0$. Using $q_k = A_k(t) \exp(i\omega t) + \text{c.c.}$ with $\omega^2 = k/m$ again and $A_k(t) = r_k(t) \exp[i\phi_k(t)]/2$ and assuming that γ_j are small perturbation parameters, we obtain

$$\begin{aligned}\frac{d}{dt}A_1 &= -\gamma_1\omega^2 A_1 \left(|A_1|^2 - \frac{B_1}{2m\omega^2} \right) - i\frac{\gamma_3}{m^2\omega} A_2 U(A_1, A_2), \\ \frac{d}{dt}A_2 &= -\gamma_2\omega^2 A_2 \left(|A_2|^2 - \frac{B_2}{2m\omega^2} \right) + i\frac{\gamma_3}{m^2\omega} A_1 U(A_1, A_2)\end{aligned}\quad (7)$$

with $U(A_1, A_2) = L(L^2/m - B_3)$ and $L = 2mi\omega(A_1^*A_2 - A_1A_2^*)$. In terms of the real-valued amplitudes r_1 and r_2 and the phase difference $\psi = \phi_1 - \phi_2$, we obtain

$$\frac{d}{dt}r_1 = -\frac{\gamma_1\omega^2}{4}r_1 \left(r_1^2 - \frac{2B_1}{m\omega^2} \right) - \frac{\gamma_3 r_2}{m^2\omega} \sin(\psi) U(\psi, r_1, r_2), \quad (8)$$

$$\frac{d}{dt}r_2 = -\frac{\gamma_2\omega^2}{4}r_2 \left(r_2^2 - \frac{2B_2}{m\omega^2} \right) - \frac{\gamma_3 r_1}{m^2\omega^2} \sin(\psi) U(\psi, r_1, r_2) \quad (9)$$

with $U(\psi, r_1, r_2) = m\omega r_1 r_2 \sin(\psi)(L^2/m - B_3)$, $L = m\omega r_1 r_2 \sin(\psi)$, and

$$\frac{d}{dt}\psi = -\frac{\gamma_3}{m^2\omega} \left(\frac{r_1^2 + r_2^2}{r_1 r_2} \right) \cos(\psi) U(\psi, r_1, r_2). \quad (10)$$

A detailed stability analysis based on equations (8)–(10) for the case that both oscillators are excited like $B_1, B_2 > 0$, shows that the target level B_3 defines the location of the aforementioned attractor. By rescaling B_3 we obtain the location parameter $\eta = m\omega^2 B_3 / (4B_1 B_2)$. It can be shown that for $B_3 \leq 0 \Rightarrow \eta \leq 0$, the two-oscillators system exhibits a stable attractor characterized by $S_3 = 0 \Rightarrow \psi = 0^\circ \vee \psi = 180^\circ$ and $H_{1,2} = B_{1,2}$. This implies that $g_{\text{tot}} \rightarrow \gamma_3 B_3^2 / 2 > 0$ for $t \rightarrow \infty$. For $B_3 \geq 0$, we distinguish between two cases. If $\eta \in [0, 1]$, then the limit cycle attractor is characterized by $H_{1,2} = B_{1,2}$ again but with $S_3 = B_3$. The latter relation implies that the phase difference is given by $\psi = \arcsin(\sqrt{\eta})$, $\psi = 180^\circ - \arcsin(\sqrt{\eta})$, $\psi = \arcsin(\sqrt{\eta}) + 180^\circ$, and $\psi = 360^\circ - \arcsin(\sqrt{\eta})$. Moreover, $g_{\text{tot}} \rightarrow 0$ for $t \rightarrow \infty$. For $B_3 > 0$ and $\eta > 1$, it is impossible to have $H_{1,2} = B_{1,2}$ and $S_3 = B_3$. Rather, the attractor is given by $\psi = 90^\circ \vee \psi = 270^\circ$, $H_{1,2} = B_{1,2}$ and $S_3 < B_3$. In addition, we have $g_{\text{tot}} \rightarrow \gamma_3(S_3 - B_3)^2 / 2 = 8\gamma_3 B_1^2 B_2^2 (1 - \eta)^2 / (m^2 \omega^4) > 0$ for $t \rightarrow \infty$. Figure 1 (b) illustrates the attractor location in terms of the phase difference ψ as a function of the location parameter η . Figure 1 (c) shows, for a representative simulation, the trajectories q_1 and q_2 . The trajectories demonstrate that the two-oscillators system converges to a stable periodic pattern (limit cycle).

Let us consider the non-harmonic case with $\alpha, \beta > 0$. First we note that the individual oscillators (i.e., $\gamma_{1,2,3} = 0$) for small amplitudes experience a linearized force of $f(q_j) = -k_{\text{lin}} q_j$ with $k_{\text{lin}} = 4k$ irrespective of α and β . Consequently, the oscillation frequency is two times the oscillation frequency of the harmonic case and the period is half the period of the harmonic case. This implies that when removing the singularities in the potentials V_j by putting $\alpha = \beta = 0$, then the oscillation frequency drops in a discontinuous fashion from $2\omega_0$ to ω_0 with $\omega_0^2 = k/m$. In general, the oscillation period for the oscillators $j = 1, 2$ can be computed from the integral $T_j = 2\sqrt{m} \int_{q_{\text{min}}}^{q_{\text{max}}} \{2[H_j(t=0) - V_j(q_j)]\}^{-1/2} dq_j$ with $H_j(t=0)$ being the initial energy of oscillator j . The integration limits $q_{\text{min,max}}$ are the turning points defined by $V_j = H_j(t=0)$. Numerical computations show that T_j is independent of $H_j(t=0)$ and α and β . In line with the small amplitude oscillation case, we obtain $T_j = T_0/2 \vee H_j(t=0) \geq H_{j,\text{min}}$, $\alpha, \beta > 0$ with $T_0 = 2\pi/\omega_0$, where $H_{j,\text{min}}$ denote the minimal energy values $H_{1,\text{min}} = \sqrt{\alpha k}$ and $H_{2,\text{min}} = \sqrt{\beta k}$.

Let us consider the case $\gamma_{1,2,3} > 0$. For the sake of brevity, we consider only the case in which all three invariants of the conservative dynamics, $H_{1,2}$ and S_3 , converge to their respective target values (i.e., $H_{1,2} \rightarrow B_{1,2}$ and $S_3 \rightarrow B_3$) such that $g_{\text{tot}} \rightarrow 0$. Our first objective is to show that the shapes of the oscillator limit cycles are distorted compared to the harmonic case. Panels (d) and (e) of figure 1 show the phase portraits q_1, p_1 and q_2, p_2 obtained from a numerical simulation. The trajectories (solid lines) converge to “egg shaped” limit cycles. The limit cycles are defined by the limit cycles of the corresponding

conservative oscillators with $\gamma_{1,2,3} = 0$. For the example shown in panels (d) and (e), the limit cycles of the corresponding conservative oscillators are illustrated by circles. Importantly, the limit cycles reveal an anchoring phenomenon. The dynamics slows down (more pronounced as in the harmonic case) when the oscillators swing to the right of their potential minimum locations $q_1 = (\alpha/k)^{0.25}$ and $q_2 = (\beta/k)^{0.25}$, see figures 1 (d) and (e). By contrast, the dynamics speeds up when the oscillators swing to the left of their potential minimum locations. This is because the forces are relatively weak on the right-hand sides [where $V_j(q_j)$ are approximatively parabolic potentials] and relatively strong on the left-hand sides [where $V_j(q_j)$ exhibit singularities].

Our second objective is to address the synchronization of the oscillators. On the limit cycles, the oscillators oscillate with the same oscillation frequency of $\omega = 2\omega_0$, see above. From $H_{1,2} \rightarrow B_{1,2}$ and $S_3 \rightarrow B_3$ it follows that

$$\frac{p_1^2}{2m} = B_1 - V_1(q_1), \quad \frac{p_2^2}{2m} = B_2 - V_2(q_2), \quad B_3 = \underbrace{\frac{L^2}{m} + (q_1^2 + q_2^2) \left(\frac{\alpha}{q_1^2} + \frac{\beta}{q_2^2} \right)}_W. \quad (11)$$

Using two first equations of (11), the function L occurring in W (defined above) can be expressed as

$$L(q_1, q_2) = p_2 q_1 - p_1 q_2 = (-1)^n q_1 \sqrt{2m[B_2 - V_2(q_2)]} + (-1)^m q_2 \sqrt{2m[B_1 - V_1(q_1)]} \quad (12)$$

with $m, n \in \{0, 1\}$. This implies that the last equation of (11) can be written as $B_3 = W(q_1, q_2)$. The synchronized state is then described by

$$\left\{ m\ddot{q}_1 = -\frac{d}{dq_1} V_1(q_1) \Rightarrow q_1(t) \right\} \wedge \left\{ B_3 = W(q_1, q_2) \Rightarrow q_2(t) \right\}. \quad (13)$$

That is, the coordinate q_2 is given by a nonlinear (implicit) mapping from q_1 to q_2 . Importantly, this mapping is stable against perturbation because the two-oscillators system is attracted to the state with $g_{\text{tot}} = 0$ and $S_{1,2,3} = B_{1,2,3}$. Therefore, the two oscillators are synchronized. Note that the same argument holds in the opposite direction. Considering the second oscillator as independent oscillator, the coordinate q_1 of the first oscillator is given by a nonlinear (implicit) mapping from q_2 to q_1 .

Let us illustrate the synchronization of the two oscillators by considering the CD oscillator model (6) under the impact of fluctuating forces. Using a standard approach for introducing noise terms into CD systems [6–8, 10], equation (6) becomes

$$\frac{d}{dt} \mathbf{q} = \frac{\partial H}{\partial \mathbf{p}}, \quad \frac{d}{dt} \mathbf{p} = -\frac{\partial H}{\partial \mathbf{q}} - \frac{\partial g}{\partial \mathbf{p}} + \sqrt{D} \begin{pmatrix} \Gamma_1(t) \\ \Gamma_2(t) \end{pmatrix}, \quad (14)$$

where $\Gamma_j(t)$ are independent Langevin forces [29] normalized to 2 units. The parameter $D \geq 0$ is the diffusion constant. The Langevin equation (14) exhibits a Fokker-Planck equation that can be cast into the form of a free energy Fokker-Planck equation [26]. The stationary probability density $P(q_1, q_2, p_1, p_2)$ can then be expressed in terms of a Boltzmann function of g_{tot} as $P = \exp(-g_{\text{tot}}/D)/Z_0$, where Z_0 is a normalization factor [19, 26]. Considering γ_j as small perturbation parameters, we may introduce the smallness parameter γ_0 and put $\gamma_j = c_j \gamma_0$ with $c_j \geq 0$. Then, $P = \exp(-\tilde{g}_{\text{tot}}/\theta)/Z_0$ with $\tilde{g}_{\text{tot}} = g_{\text{tot}}/\gamma_0$ holds, where $\theta = D/\gamma_0$ can be considered as a non-equilibrium temperature. In this form, the analogy to equilibrium systems becomes obvious [6]. Importantly, without coupling between the oscillators, that is, for $\gamma_3 = 0 \Rightarrow c_3 = 0$, we have $P(q_1, q_2, p_1, p_2) = P_1(q_1, p_1)P_2(q_2, p_2)$ with $P_j = \exp(-g_j/D)/Z_j$ for $j = 1, 2$, where Z_j are normalization factors again. That is, the probability density $P(q_1, q_2, p_1, p_2)$ factorizes. By analogy, the transition probability density (conditional probability density) factorizes. Therefore, for $\gamma_3 = 0$, there are no cross-correlations between the oscillators at any time lag.

Let us show with the help of a stochastic CD model (14) that for $\gamma_3 > 0$, the two-oscillators model exhibits a stable synchronized state. To this end, we solved numerically equation (14) for a fixed value of γ_3 and calculated the cross-correlation coefficients $\text{Corr}(q_1(t), q_2(t - \tau))$ for different time lags $\tau \in [0, T = T_0/2]$. We determined the maximal coefficient. Subsequently, we varied γ_3 . In doing so, we

obtain the maximal cross-correlation coefficient as a function of the coupling parameter γ_3 . Figure 1 (f) summarizes the simulation results. For $\gamma_3 = 0$, there was a finite by-chance value for the maximal cross-correlation coefficient that decayed when the simulation duration was increased. As far as the impact of γ_3 is concerned, figure 1 (f) demonstrates that the maximal cross-correlation increased as a function of γ_3 — as predicted. This increase of the maximal cross-correlation coefficient was taken as the evidence that for γ_3 , the two oscillators were to some degree synchronized.

Future studies may focus, in particular, on the stochastic aspects of the proposed CD two-oscillators model. For example, it has been suggested to use the analytical solution for the short-time propagator to define maximum likelihood estimators that can be used to estimate the model parameters of CD systems [30]. In fact, for single CD oscillator models in a series of studies, the CD theory has been applied to the experiments on human rhythmic motor behavior and model parameters have been estimated from experimental data both in the harmonic [12–14] and non-harmonic case [15].

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Синхронізація та анкерування двох негармонічних канонічно-дисипативних осциляторів за допомогою потенціалів Смородинського-Вінтерніца

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Розглянуто канонічно-дисипативні граничні цикли для двох негармонічних осциляторів, що коливаються в одномірних потенціалах Смородинського-Вінтерніца. Показано, що стандартний канонічно-дисипативний підхід із введенням дисипативних сил природно приводить до появи взаємодії між осциляторами, яка синхронізує їх рух. Детально досліджено негармонічний характер граничних циклів у контексті анкерування, різницю фаз між синхронізованими осциляторами та ступінь їх синхронізації.

Ключові слова: канонічно-дисипативні системи, потенціали Смородинського-Вінтерніца, синхронізація, анкерування
