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RESEARCH ARTICLE

A note on simplicity of contact Lie algebras over GF(2)

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Abstract. In this note we investigate the structure of contact Lie algebras when the ground field is of characteristic 2. In order to describe the simple constituent of contact Lie algebras, by using computer algebra system, GAP, we make a conjecture which says that the quotient algebra of contact Lie algebra by its Nilradical is simple and there exists an isomorphism among this constituents and Witt Lie algebras and Hamilton Lie algebras.

Introduction

We have four infinite families of Cartan type Lie algebras which are deformations of the Witt algebras W, special algebras S, Hamilton algebras H and contact Algebras K such that their structure over fields of characteristic $p > 3$ has been studied and their simplicity have been determined in [7], [9] and [10]. For instance, contact Lie algebras are simple over finite fields of characteristic at least three. The Lie algebras of Cartan type are not necessarily simple in characteristic 2, but the simple constituent of these algebras have been determined computationally by Eick[1]. It is worth pointing out that the simplicity of the classical Lie algebras over GF(2) have been determined by Hogeweij [5] and Hiss [4]. In this paper we intend to determine (up to isomorphism) the simple constituent of

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contact lie algebras in characteristic 2 and make an isomorphism between this Lie algebras and Witt and Hamiltonian Lie algebras.

1. Preliminaries

Let $n = 2r + 1$ be a positive integer and n-tuple $m = (m_1, \ldots, m_n)$ of positive integers. Let $X_1, \ldots, X_n$ be n positive pairwise commuting indeterminate over GF(2).

For $a \in \mathbb{N}^n$ we write X^a for ${X_1}^{a_1} \ldots {X_n}^{a_n}$. Denote by $A(n)$ the commutative algebra consisting of all formal sums over GF(2) of the form $\sum_{a\in\mathbb{N}}^n \alpha_a X^a$ with usual addition and the multiplication

$$(\sum_a \alpha_a X^a)(\sum_b \beta_b X^b) = \sum_c (\sum_{a+b=c} \alpha_a \beta_b \binom{b}{a} X^c)$$

where the multi-binomial coefficient is evaluated modulo 2.

Let $m \in \mathbb{N}^n$ and $\tau = (2^{m_1} - 1, \ldots, 2^{m_n} - 1)$ then

$$A(n, m) = \langle X^a | 0 \leqslant a \leqslant \tau \rangle \ \leqslant A(n)$$

is subalgebra and D_j is the jth partial derivation on $A(n, m)$. Witt algebras $W(n, m)$ is defined as the set of elements

$$\{\sum_{j=1}^n f_j D_j \mid f_j \in A(n, m)\}$$

with the usual addition and the Lie bracket

$$[fD_i, gD_j] = fD_i(g)D_j - gD_j(f)D_i + fg[D_i, D_j].$$

In addition, we consider the following notations

$$X^\alpha = 0 \ \ if \ \ 0 \nleq \alpha \nleq \tau;$$

$$X^{\varepsilon_i} \ \ \text{for} \ \ \varepsilon_i = (\delta_{1,i}, \ldots, \delta_{n,i})$$

and for $n \in \mathbb{N}$, let r be the largest integer less than or equal to $n/2$ then

$$j' = \begin{cases} j + r & \text{if} \ \ 1 \leqslant j \leqslant r \\ j - r & \text{if} \ \ r + 1 \leqslant j \leqslant 2r. \end{cases}$$

where $j \in \{1, \ldots, 2r\}$.

Definition 1. The contact operator D_K from $A(n,m)$ to $W(n,m)$ defines as:

$$D_K : A(n,m) \to W(n,m), \quad f \mapsto D_K(f) = \sum_{j=1}^{2r} d_j(f) d_{j'}$$

where $d_j = D_j + X^{\varepsilon_{j'}} D_n$ and $[d_j, d_i] = 0$ for $1 \leqslant i,j \leqslant 2r$.

2. $K(n,m)$ over GF(2)

According to the definition of Hamilton operator

$$D_H : A(n,m) \to W(n,m),$$

$$D_H(f) = \sum_{j=1}^{2r} D_j(f) D_{j'},$$

we put

$$\{f,g\} = D_H(f)(g) \tag{1}$$

for all $f,g \in A(n,m)$. We compute contact operator $D_K(X^{(a)})$ over field of characteristic 2, then we obtain

$$D_K(X^{(a)}) = D_H(X^{(a)}) + X^{(a-\varepsilon_n)} \sum_{j=1}^{2r} X^{\varepsilon_j} D_j + \left(\sum_{j=1}^{2r} a_j \right) X^{(a)} D_n \tag{2}$$

Definition 2. The contact Lie algebra is defined as the derived subalgebra of the image of D_K and it is generated by $\{D_K(X^a) \mid 0 \leqslant a \leqslant \tau\}$.

Definition 3. Let $X^{(a)}, X^{(b)} \in A(n,m)$. We define a multiplication on $A(n,m)$ as $\langle X^{(a)}, X^{(b)} \rangle = D_K(X^{(a)})(X^{(b)})$. By applying (1) and (2) we obtain:

$$\begin{aligned}\langle X^{(a)}, X^{(b)} \rangle =& \{X^{(a)}, X^{(b)}\} \\ &+ (|b| \binom{a+b-\varepsilon_n}{a-\varepsilon_n} + |a| \binom{a+b-\varepsilon_n}{a}) X^{(a+b-\varepsilon_n)})\end{aligned}$$

where $\mid a \mid = \sum_{j=1}^{2r} a_j$.

Remark 1. We note that $D_K(\langle f,g \rangle) = [D_K(f), D_K(g)]$ which implies Lie multiplication on $A(n,m)$. The multiplication $\langle\ ,\ \rangle$ has a key role in proof of theorem 1 through defining a gradation for contact Lie algebra.

For $m \in \mathbb{N}$ and $0 \leqslant i \leqslant 2^{m_n} - 1$, let $\sigma(i) = (2^{m_1} - 2, \dots, 2^{m_{n-1}} - 2, i)$ for $n = 2r + 1$ and let $\delta_i = \varepsilon_i + \varepsilon_{i'}$ such that $1 \leqslant i \leqslant r - 1$.

Theorem 1. *Let $K = K(n, m)$ and let*

$$S = \{\sigma(j) + \sum_{k=1}^{r} \lambda_k \delta_k \mid 1 \leqslant i \leqslant 2^{(m_n)} - 2,\ \lambda_k \in \{0, 1\}\}$$

where at least one λ_k is non-zero. Then K' has a basis

$$\{X^{(a)} \mid 0 \leqslant i < \tau\} \cup \{X^{(a)} + X^{(a - \delta_j + \varepsilon_n)} \mid a \in S, j \in \Omega(a)\}$$

where

$$\Omega(a) = \{j \mid 1 \leqslant j \leqslant r, a_j = 2^{m_j} - 1\}.$$

Proof. According to the gradation of contact Lie algebras over field of characteristic $p > 2$ we have $K = \oplus_{j > -2} K_j$. In the case of contact Lie algebras over GF(2) we use the subalgebra K'_j to write Lie derived subalgebra as a $\mathbb{Z}$-graded Lie algebra. We know

$$degX^a = \sum_{j=1}^{2r} a_j + a_n + a_n$$

and define

$$K_j = \langle X^{(a)} \mid \deg X^{(a)} = j + 2 \rangle.$$

For $Y \in K_0$, $\langle X_n, Y \rangle = 0$ then $K'_0 = \langle X_i X_j, X_j^{(2)} \rangle$ for $i, j \leqslant 2r$.

In addition, we have $K'_{-2} = K_{-2}$, $K'_{-1} = K_{-1}$ and $K'_0 = I$ such that I is a subalgebra of K_0 and for $i > 0$ we have

$$K'_i = \{X^{(a)} \in K_i \mid \langle X^{(a)}, K_{-1} \rangle\} \subseteq K'_{i-1}.$$

Defined set for K'_i implies that $X_n \notin K'_i$ so $X_n \notin K'$. Then K' has codimension at least 1 in K.

Let $X_n^2 \in K_2$, then for each $X^{(a)} \in K_2$, $\langle X^{(a)}, X_n^2 \rangle = 0$. We also consider $\langle X^{(a)}, X_i \rangle \in K'_1$, since $X_n^2 \notin K'_2$ then $\langle \langle X^{(a)}, X_i \rangle, X_j \rangle$ implies that $X_n^2 \notin K'_1$. Then $X_n^2 \notin K'_i$ for $i > 2$ and $X_n^2 \notin K'$, consequently $X_n^{(a_n)} \notin K'$. Therefore we obtain $\dim K' = \dim K - (2^{m_n} - 1)$.

Let a, b be n-tuples such that $0 \leqslant a, b \leqslant (2^{m_1} - 1, \dots, 2^{m_n} - 1)$. For $0 \leqslant i \leqslant n$ the i-th term of coefficient denoted by $\binom{a_i + b_i - 1}{a_i - 1}$ shows that $0 \leqslant a_i + b_i - 1 \leqslant 2^{m_i} - 2$. Therefore $\{X^{(a)} \mid 0 \leqslant a \leqslant (2^{m_1} - 2, \dots, 2^{m_n} - 2)\}$ generates K'.

For elements that are not generated by $\{X^{(a)} | 0 \leqslant a < \tau\}$, we fix $a = (2^{m_1} - 2, \dots, 2^{m_n} - 2)$ and $b = (b_1, \dots, b_n)$ and compute $\langle X^{(a)}, X^{(b)} \rangle$ for different cases as follows:

- $\langle X^{(a)}, X_i \rangle = X^{(a)-\varepsilon_{i'}} + X^{(a)+\varepsilon_i-\varepsilon_n}$
- $\langle X^{(a)}, X_i X_{i'} \rangle = X^{(a)}$
- $\langle X^{(a)}, X_i X_j \rangle = 0$
- $\langle X^{(a)}, X_i^2 X_{i'} \rangle = X^{(a)+\varepsilon_i}$
- $\langle X^{(a)}, X_i X_n \rangle = X^{(a)-\varepsilon i'+\varepsilon_n} + X^{(a)+\varepsilon_i}$
- $\langle X^{(a)}, X_i^2 X_n \rangle = X^{(a)+\varepsilon_i-\varepsilon_{i'}+\varepsilon_n}$
- $\langle X^{(a)}, X_i^2 X_j^2 \rangle = 0$
- Let $b_n = 0, b_i = 1$ where $i \leqslant 2r$ then

$$\langle X^{(a)}, X^{(b)} \rangle = \sum_{j=1}^{2r} X^{(a)+\sum_{j=1}^{r} \delta_j}.$$

- Let $b = (1, \dots, 1)$ then $\langle X^{(a)}, X^{(b)} \rangle = \sum_{j=1}^{2r} X^{(a)+\varepsilon_n+\sum_{j=1}^{r} \delta_j}$.
- If $(b_1, \dots, b_n) > (1, \dots, 1)$ then $\langle X^{(a)}, X^{(b)} \rangle = 0$.

We define $\sigma(i) = (2^{m_1} - 2, \dots, 2^{m_{2r}} - 2, i)$ such that $0 \leqslant i \leqslant 2^{m_n} - 1$. Let $\delta_i = \varepsilon_i + \varepsilon_{i'}$, then we consider

$$\{\sigma(j)+\sum_{k=1}^{r} \lambda_k \delta_k \mid 1 \leqslant i \leqslant 2^{(m_n)}-2, \lambda_k \in \{0,1\} \text{ at least one } \lambda_k \textit{ non}-\textit{zero}\}$$

then

$$\{X^{(a)} + X^{(a-\delta_j+\varepsilon_n)}\}$$

generates the rest elements of K'. □

Eick [3], classified simple Lie algebras over field of characteristic 2 up to dimension 20. According to this classification, the derived subalgebra of the Witt algebra denoted by $W'(m)$ is simple for $m \neq 1$. Furthermore, Brough and Eick [1] proved that the Hamilton Lie algebra $H(n,m)$ is simple for $n > 2$ and there exists an isomorphism $H \cong S(2,m)'$ where $n = 2$. Based on computer computation with computer algebra system, GAP [11], we consider these two cases:

- $K'(3,(1,2,1))/N(K') \cong H(2(1,2))/N(H) \cong W'(2)$,
- $K'(5,(2,2,2,2,2))/N(K') \cong H(4,(2,2,2,2))$,

and provide the following conjecture:

Conjecture 2. *Let* $\operatorname{char}(\mathbb{F}) = 2$ *and* $K = K(n,m)$ *for* $n \in \mathbb{N}$ *and* $m \in \mathbb{N}^n$. *Then we have:*

1) *If* $n = 3$ *and* $1 \in (m_1, m_2)$, *then* $K'/N(K') \cong W'(1,(m_i))$ *for* $m_i \neq 1$.
2) *If* $n > 3$, *then* $K'/N(K') \cong H$.

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