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Weak factorization systems and fibrewise regular injectivity for actions of pomonoids on posets

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Abstract. Let S be a pomonoid. In this paper, **Pos**-S, the category of S-posets and S-poset maps, is considered. One of the main aims of this paper is to draw attention to the notion of weak factorization systems in **Pos**-S. We show that if S is a pogroup, or the identity element of S is the bottom (or top) element, then $(\mathcal{DU}, SplitEpi)$ is a weak factorization system in **Pos**-S, where $\mathcal{DU}$ and $SplitEpi$ are the class of du-closed embedding S-poset maps and the class of all split S-poset epimorphisms, respectively. Among other things, we use a fibrewise notion of complete posets in the category **Pos**-S/B under a particular case that B has trivial action. We show that every regular injective object in **Pos**-S/B is topological functor. Finally, we characterize them under a special case, where S is a pogroup.

1. Introduction

A comma category (a special case being a slice category) is a construction in category theory. It provides another way of looking at morphisms: instead of simply relating objects of a category to one another, morphisms become objects in their own right. This notion was introduced in 1963 by F. W. Lawvere, although the technique did not become generally known until many years later.

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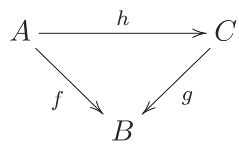
Injective objects with respect to a class $\mathcal{H}$ of morphisms have been investigated for a long time in various categories. Recently, injective objects in slice categories $(\mathbf{C}/B)$ have been investigated in detail (see [1, 6]), especially in relationship with weak factorization systems, a concept used in homotopy theory, in particular for model categories. More precisely, $\mathcal{H}$-injective objects in $\mathbf{C}/B$, for any B in $\mathbf{C}$, form the right part of a weak factorization system that has morphisms of $\mathcal{H}$ as the left part (see [1, 2]).

In this paper, the notion of weak factorization system in $\mathbf{Pos}$-S is investigated. After some introductory notions in section 1, we recall in section 2, the notion of weak factorization system and state some related basic theorems. Also, we give the guarantee about the existence of $(Emb, Emb^{\square})$ as a weak factorization system in $\mathbf{Pos}$-S, where Emb is the class of all order-embedding S-act maps. We then find that every Emb-injective object in $\mathbf{Pos}$-S/B is split epimorphism in $\mathbf{Pos}$-S. In section 3, we continue studying Emb-injectivity using a fibrewise notion of complete posets in the category $\mathbf{Pos}$-S/B in a particular case where B has the trivial action.

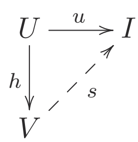
For the rest of this section, we give some preliminaries which we will need in the sequel.

Given a category $\mathbf{C}$ and an object B of $\mathbf{C}$, one can construct the *slice category* $\mathbf{C}/B$ (read: $\mathbf{C}$ over B): objects of $\mathbf{C}/B$ are morphisms of $\mathbf{C}$ with codomain B, and morphisms in $\mathbf{C}/B$ from one such object $f : A \to B$ to another $g : C \to B$ are commutative triangles in $\mathbf{C}$

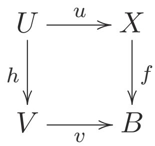


i.e, $gh = f$. The composition in $\mathbf{C}/B$ is defined from the composition in $\mathbf{C}$, in the obvious way (paste triangles side by side).

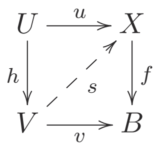
Let $\mathbf{C}$ be a category and $\mathcal{H}$ a class of its morphisms. An object I of $\mathbf{C}$ is called *$\mathcal{H}$-injective* if for each $\mathcal{H}$-morphism $h : U \to V$ and morphism $u : U \to I$ there exists a morphism $s : V \to I$ such that $sh = u$. That is, the following diagram is commutative:



In particular, in the slice category $\mathbf{C}/B$, where B is an object of $\mathbf{C}$, this means that, an object $f : X \to B$ is $\mathcal{H}$-injective if, for any commutative square



with $h \in \mathcal{H}$, there exists a diagonal morphism $s : V \to X$



such that $sh = u$ and $fs = v$. The category $\mathbf{C}$ is said to have *enough $\mathcal{H}$-injectives* if for every object A of $\mathbf{C}$ there exists a morphism $A \to C$ in $\mathcal{H}$ where C is an $\mathcal{H}$-injective object in $\mathbf{C}$.

Let S be a monoid with identity 1. A (*right*) *S-act* or *S-set* is a set A equipped with an action $\mu : A \times S \to A$, $(a, s) \mapsto as$, such that $a1 = a$ and $a(st) = (as)t$, for all $a \in A$ and $s, t \in S$. Let **Act-**S denote the category of all S-acts with action-preserving maps or S-maps. Clearly S itself is an S-act with its operation as the action. For instance, take any monoid S and a non-empty set A. Then A becomes a right S-act by defining $as = a$ for all $a \in A$, $s \in S$, we call that A an S-act with *trivial action* (see [10] or [11]).

Recall that a *pomonoid* S is a monoid with a partial order $\leqslant$ which is compatible with the monoid operation: for $s, t, s', t' \in S$, $s \leqslant t$, $s' \leqslant t'$ imply $ss' \leqslant tt'$. A (*right*) *S-poset* is a poset A which is also an S-act whose action $\mu : A \times S \to A$ is order-preserving, where $A \times S$ is considered as a poset with componentwise order. The category of all S-posets with action preserving monotone maps is denoted by **Pos**-S. Clearly S itself is an S-poset with its operation as the action. Also, if B is a non-empty subposet of A, then B is called a *sub S-poset* of A if $bs \in B$, for all $s \in S$ and $b \in B$. Throughout this paper we deal with the pomonoid S and the category **Pos**-S, unless otherwise stated. For more information on S-posets see [5] or [8].

2. Weak factorization system

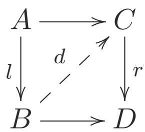
The concept of weak factorization systems plays an important role in the theory of model categories. Formally, this notion generalizes factorization systems by weakening the unique diagonalization property to the diagonalization property without uniqueness. However, the basic examples of weak factorization systems are fundamentally different from the basic examples of factorization systems.

Here, we introduce from [1] the notion which we deal with in the paper.

Notation. We denote by $\Box$ the relation *diagonalization property* on the class of all morphisms of a category $\mathbf{C}$: given morphisms $l : A \to B$ and $r : C \to D$ then

$$l \Box r$$

means that in every commutative square



there exists a diagonal $d : B \to C$ rendering both triangles commutative. In this case, l is also said to have the *left lifting property* with respect to r (and r to have the *right lifting property* with respect to l).

Let $\mathcal{H}$ be a class of morphisms. We denote by

$$\mathcal{H}^{\Box} = \{r |\ r \text{ has the right lifting property with respect to each } l \in \mathcal{H}\}$$

and

$$^{\Box}\mathcal{H} = \{l |\ l \text{ has the left lifting property with respect to each } r \in \mathcal{H}\}.$$

Let D be an object in $\mathbf{C}$ and $\mathcal{H}_D$ be the class of those morphisms in $\mathbf{C}/D$ whose underlying morphism in $\mathbf{C}$ lies in $\mathcal{H}$. Now, $r : C \to D \in \mathcal{H}^{\Box}$ if and only if r is an $\mathcal{H}_D$-injective object in $\mathbf{C}/D$. Dually, all morphisms in $^{\Box}\mathcal{H}$ are characterized by a projectivity condition in $\mathcal{H}_D$.

Recall from [1] that a *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of morphism classes such that:
(1) every morphism has a factorization as an $\mathcal{L}$-morphism followed by an $\mathcal{R}$-morphism,
(2) $\mathcal{R} = \mathcal{L}^{\Box}$ and $\mathcal{L} = {}^{\Box}\mathcal{R}$.

Remark 2.1. If we replace "$\Box$" by "$\perp$" where "$\perp$" is defined via the *unique diagonalization property* (i.e., by insisting that there exists precisely one diagonal), we arrive at the familiar notion of a factorization system in a category. Factorization systems are weak factorization systems. For instance, let $\mathcal{E}$ be the class of all S-poset epimorphisms. Then, by Theorem 1 of [5] one can easily seen that $(\mathcal{E}, Emb)$ in **Pos**-S is a factorization system.

Now, consider a functor $G : \mathbf{A} \to \mathbf{X}$. Recall from [1] that a source $(A \xrightarrow{f_i} A_i)_{i\in I}$ in $\mathbf{A}$ is called *G-initial* provided that for each source $(B \xrightarrow{g_i} A_i)_{i\in I}$ in $\mathbf{A}$ and each $\mathbf{X}$-morphism $h : GB \to GA$ with $Gg_i = Gf_i \circ h$ for each $i \in I$, there exists a unique $\mathbf{A}$-morphism $\bar{h} : B \to A$ in $\mathbf{A}$ with $G\bar{h} = h$ and $g_i = f_i \circ \bar{h}$ for each $i \in I$.

Also, a source $(A \xrightarrow{\bar{f}_i} A_i)_{i\in I}$ *lifts* a G-structured source $(X \xrightarrow{f_i} GA_i)_{i\in I}$ provided that $G\bar{f}_i = f_i$ for each $i \in I$.

Definition 2.2. (cf. [1]) A functor $G : \mathbf{A} \to \mathbf{X}$ in the category **Cat** (of all categories and functors) is *topological* if every G-structured source $(X \xrightarrow{f_i} GA_i)_{i\in I}$ has a unique G-initial lift $(A \xrightarrow{\bar{f}_i} A_i)_{i\in I}$.

Example 2.3. (1) In the category **Set** (of all sets and functions between them) pair $(Mono, Epi)$ is a weak factorization system. But $(Epi, Mono)$ is a factorization system in this category, where $Mono$ is the class of all one-to-one maps and Epi is the class of all surjective maps.
(2) The pair $(Full, Top)$ is a weak factorization system in the category **Cat**, where $Full$ is the class of those morphisms in **Cat** that are full and Top is the class of those morphisms in **Cat** that are topological.
(3) In the category **Pos** of all posets with monotone maps, the pair (Emb, Top) is a weak factorization system, where Emb is the class of all order-embeddings; that is, maps $f : A \to B$ for which $f(a) \leqslant f(a')$ if and only if $a \leqslant a'$, for all $a, a' \in A$ and Top is the class of all topological monotone maps. For more details of the proof see [1].

We record the following two results from [1], that will be used later on.

Proposition 2.4. *Let* $\mathbf{C}$ *be a category and* $\mathcal{H}$ *a class of morphisms closed under retracts in arrow-category* $\mathbf{C}^{\to}$. *Then the following conditions are equivalent:*
1. $(\mathcal{H}, \mathcal{H}^{\square})$ *is a weak factorization system;*
2. *for all objects* B *of* $\mathbf{C}$, *the slice category* $\mathbf{C}/B$ *has enough* $\mathcal{H}_B$*-injectives.*

Proposition 2.5. *Let* $\mathbf{C}$ *be a category. Then* $(\mathcal{L}, \mathcal{R})$ *is a weak factorization system if and only if*

(1) *Any morphism* $h \in \mathbf{C}$ *has a factorization* $h = gf$ *with* $f \in \mathcal{L}$ *and* $g \in \mathcal{R}$.
(2) *For all* $f \in \mathcal{L}$ *and* $g \in \mathcal{R}$, f *has the left lifting property with respect to* g.
(3) *If* $f : A \to B$ *and* $f' : X \to Y$ *are such that there exist morphisms* $\alpha : B \to Y$ *and* $\beta : A \to X$ *then*
  (a) *If* $\alpha f \in \mathcal{L}$ *and if* α *is a split monomorphism then* $f \in \mathcal{L}$.
  (b) *If* $f'\beta \in \mathcal{R}$ *and if* β *is split epimorphism then* $f' \in \mathcal{R}$.

If $f : A \to B$ and $g : A \to C$ are morphisms in a category $\mathbf{C}$ such that there exist morphisms $\alpha : C \to B$ and $\beta : B \to C$ with $\beta\alpha = 1_C$, $\alpha g = f$ and $\beta f = g$ then we say that g is a *retract* of f. In categorical terms, g is a retract of f in the coslice category $A/\mathbf{C}$.

Notice that in $3(a)$ above, f is a retract of αf and that all retracts can be written in this way. So this result is simply saying that $\mathcal{L}$ is closed under retracts. Similarly $3(b)$ is equivalent to $\mathcal{R}$ being closed under retracts.

Recently, Bailey and Renshaw in [2], provide a number of examples of weak factorization systems for S-acts such as the following theorem. But first we need a definition. Following [2] an S-act (S-poset) monomorphism $f : X \to Y$ is *unitary* if $y \in \mathrm{im}(f)$ whenever $ys \in \mathrm{im}(f)$ and $s \in S$. Notice that in the case of S-acts, it is clear this is equivalent to saying that there exists an S-act Z such that $Y \cong X \dot{\cup} Z$ (the disjoint union of X and Z) or in other words, $\mathrm{im}(f)$ is a direct summand of Y, while in the other case this is not true (see Remark 3.3 below).

Theorem 2.6. *Let* S *be a monoid and let* $\mathcal{U}$ *be the class of all unitary* S*-monomorphisms and SplitEpi be the class of all split* S*-epimorphisms. Then* $(\mathcal{U}, \mathit{SplitEpi})$ *is a weak factorization system in* **Act-**S.

3. Weak factorization systems via down and up-closed embeddings

Now, consider *Emb* as the class of all embedding S-poset maps. We try to provide a weak factorization system for **Pos**-S with *Emb* as the left part. In this section, we consider down and up-closed embeddings, briefly du-closed embeddings, as a subclass of *Emb* and find a weak factorization system in **Pos**-S with some conditions on pomonoid S.

Let $\mathbf{C}$ be a category with binary coproducts, Sum the class of all coproduct injections, and *SplitEpi* the class of all split epimorphisms in $\mathbf{C}$. The following is a particular case of [13, Theorem 2.7].

Proposition 3.1. *If Sum is stable under pullback in* **C**, *then* $(Sum, SplitEpi)$ *is a weak factorization system in* **C**.

Proposition 3.2. *Let S be an arbitrary pomonoid. Then the class of all unitary monomorphisms in* **Pos**-S *is stable under pullback in* **Pos**-S.

Proof. Let

$$\begin{array}{ccc} Z & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

be a pullback in **Pos**-S, where $f : X \to Y$ is unitary and $g : Y' \to Y$ is an S-poset map. Suppose that $y' \in Y'$ is such that $y's \in \mathrm{im}(f')$ for some $s \in S$. Then there is an element $z \in Z$ with $f'(z) = y's$, and we have:

$$f(g'(z)) = (fg')(z) = (gf')(z) = g(f'(z)) = g(y's) = g(y')s.$$

Thus $g(y')s \in \mathrm{im}f$ and since f is assumed to be unitary, one concludes that $g(y') \in \mathrm{im}f$. Thus $g(y') = f(x)$ for some $x \in X$ and since Z is a pullback of f and g, it follows that there is a unique element $z_0 \in Z$ such that $f'(z_0) = y'$ and $g'(z_0) = x$. In particular, $y' \in \mathrm{im}f'$. Thus f' is unitary. □

Remark 3.3. Note that the assertion 'any unitary monomorphism in **Pos**-S is just coproduct injection' is not true, in general. For instance, consider the trivial case $S = \{1\}$, then we have **Pos**-S=**Pos** and so any injective monotone map trivially is unitary. Next, let $A = \{0, 1\}$ be a poset with two elements 0 and 1 and with $0 \leqslant 1$ and let $\{0\}$ be a discrete poset with only one element 0. Then the map $\{0\} \to A$ that takes 0 to 0 is monotone and injective. But $A = \{0 \leqslant 1\}$ is not a coproduct of $\{0\}$ and $\{1\}$, since $\{0\}\dot{\cup}\{1\} = \{0, 1\}$ with discrete ordering, while $A = \{0 \leqslant 1\}$ has a nontrivial ordering on it.

Definition 3.4. A possibly empty sub S-poset A of an S-poset B is said to be *down-closed* (*up-closed*) in B if for each $a \in A$ and $b \in B$ with $b \leqslant a$ ($a \leqslant b$) we have $b \in A$. By a *du-closed embedding*, we mean an embedding S-poset map $f : A \to B$ such that $\mathrm{im}f$ is both down-closed and up-closed sub S-poset of B.

Proposition 3.5. *Let S be an arbitrary pomonoid. Then the class $\mathcal{D}$ (resp. $\mathcal{U}$) of all down-closed (resp. up-closed) embeddings is stable under pullback in* **Pos**-S.

Proof. Let

$$\begin{CD} Z @>{f'}>> Y' \\ @V{g'}VV @VV{g}V \\ X @>>{f}> Y \end{CD}$$

be a pullback in **Pos**-S, where $f \in \mathcal{D}$ (resp. $f \in \mathcal{U}$) and $g : Y' \to Y$ is an S-poset map. Suppose that $y' \in Y'$ is such that $y' \leqslant f'(x)$ (resp. $y' \geqslant f'(x)$) for some $x \in \mathrm{im}f'$. Then $g(y') \leqslant gf'(x) = fg'(x)$ (resp. $g(y') \geqslant gf'(x) = fg'(x)$), and since f is assumed to be down-closed (resp. up-closed), we conclude that $g(y') \in \mathrm{im}f$. Thus there is an element $x' \in X$ with $f(x') = g(y')$. But since Z is a pullback of f and g, it follows that there is a unique element $z \in Z$ such that $f'(z) = y'$ and $g'(z) = x$. In particular, $y' \in \mathrm{im}f'$. Thus f' is down-closed (resp. up-closed). □

A corollary follows immediately:

Corollary 3.6. *Let S be an arbitrary pomonoid. Then the class $\mathcal{DU}$ of all du-closed embeddings is stable under pullback in* **Pos**-S.

Now, we prove the following crucial lemma in the category **Pos**-S.

Lemma 3.7. *Let $f : A \to B$ be a du-closed S-poset embedding. Then f is a unitary if and only if* $\mathrm{im}f$ *is a direct summand of B.*

Proof. Let f be a unitary monomorphism. First we show that $\mathrm{im}f$ and $B \backslash \mathrm{im}f$ are two S-posets. These are S-acts because of the unitary property of f. Now, consider $a, b \in B$ with $a \leqslant b$. If $a \in \mathrm{im}f$, since $\mathrm{im}f$ is an up-closed subset in B so $b \in \mathrm{im}f$. This gives $a \leqslant b$ in $\mathrm{im}f$. Also, if $a \notin \mathrm{im}f$ then we have $b \notin \mathrm{im}f$; in fact $b \in \mathrm{im}f$ implies that $a \in \mathrm{im}f$ as $\mathrm{im}f$ is down-closed, which is a contradiction. So $a \leqslant b$ in $B \setminus \mathrm{im}f$. Hence if $a \leqslant b$ in B, then $a \leqslant b$ in $\mathrm{im}f \dot{\cup} (B \setminus \mathrm{im}f)$. Now if $a \leqslant b$ in $\mathrm{im}f \dot{\cup} (B \setminus \mathrm{im}f)$, then by definition of order on the coproduct, we have $a \leqslant b$ in $\mathrm{im}f$ or $a \leqslant b$ in $B \setminus \mathrm{im}f$. Since the order on $\mathrm{im}f$ and $B \setminus \mathrm{im}f$ is inherent from B, then $a \leqslant b$ in B. Consequently we get

$$a \leqslant b \text{ in } B \quad \Longleftrightarrow \quad a \leqslant b \text{ in } \mathrm{im}f \dot{\cup} (B \setminus \mathrm{im}f).$$

Hence $B = \mathrm{im}f \dot{\cup} (B \setminus \mathrm{im}f)$. The converse is trivially true by definition of unitary monomorphism. □

Let $\mathcal{DUU}$ denote the class of all unitary du-closed embedding S-poset maps.

Proposition 3.8. *Let S be an arbitrary pomonoid. Then $\mathcal{DUU} = Sum$.*

Combining Propositions 3.2, 3.8 and Corollary 3.6 gives

Proposition 3.9. *Let S be an arbitrary pomonoid. Then* Sum *is stable under pullback in* **Pos**-*S.*

It then follows from Propositions 3.1 and 3.9 that

Theorem 3.10. *Let S be a pomonoid. Then $(Sum, SplitEpi)$ is a weak factorization system in* **Pos**-*S.*

Next we state two useful results.

Lemma 3.11. *Let S be a pomonoid whose identity element e is the bottom element (resp. the top element) and $f : X \to Y$ be an S-poset map. If $\mathrm{im} f$ is a down-closed (resp. up-closed) subset of Y, then f is unitary.*

Proof. Suppose e is the bottom element of S. We show that if $y \in \mathrm{im} f$ whenever $ys \in \mathrm{im} f$ and $s \in S$. In fact, by hypothesis we have $e \leqslant s$ and we get $y \leqslant ys$, for every $y \in Y$ and $s \in S$. Now, as $\mathrm{im} f$ is down-closed and $ys \in \mathrm{im} f$ then have $y \in \mathrm{im} f$. The other case is similar. □

By Proposition 3.8 and Lemma 3.11, we deduce

Proposition 3.12. *Let S be a pomonoid such that its identity element is either the bottom or top element. Then $\mathcal{DU} = \mathcal{DUU}$, and hence $Sum = \mathcal{DU}$.*

Proposition 3.13. *If S is a pogroup, then any morphism in* **Pos**-*S is unitary.*

Proof. Consider any morphism $f : X \to Y$ in **Pos**-S and suppose that $y \in Y$ is such that $ys \in \mathrm{im} f$ for some $s \in S$. Then $ys = f(x)$ for some $x \in X$ and we have:

$$y = (ys)s^{-1} = f(x)s^{-1} = f(xs^{-1})$$

proving that $y \in \mathrm{im} f$. Thus f is unitary. □

Proposition 3.14. *If S is a pogroup, then $\mathcal{DU} = \mathcal{DUU}$, and hence $Sum = \mathcal{DU}$.*

Proof. By propositions 3.8 and 3.13, it is obvious. □

Next, combining Propositions 3.12 and 3.14 with Theorem 3.10, we get the following result.

Theorem 3.15. *Suppose that*

(i) *S is a pogroup, or*

(ii) *the identity element of S is the bottom element, or*

(iii) *the identity element of S is the top element,*

Then $(\mathcal{DU}, SplitEpi)$ is a weak factorization system in **Pos**-*S.*

Recall that each poset can be embedded (via an order-embedding) into a complete poset, called the Dedekind-MacNeille completion. In fact, given a poset P, its MacNeille completion is the poset $\bar{P}$ consisting of all subsets A of P for which $LU(A) = A$, where

$$U(A) = \{x \in P : x \geqslant a, \ \forall a \in A\}$$

and

$$LU(A) = \{y \in P : y \leqslant x, \forall x \in U(A)\},$$

and the embedding $\downarrow (-) : P \to \bar{P}$ is given by

$$a \mapsto \downarrow a = \{x \in P : x \leqslant a\}$$

for every $a \in P$ (see [3]).

Notice that in the category **Pos**-S/B, regular monomorphisms correspond to regular monomorphisms in **Pos**-S and these are exactly order-embeddings (in **Pos**-S) (see [5, 9]). We state the following theorem which gives us enough *Emb*-injectivity property in **Pos**-S/B. For details of the proof see [9].

Theorem 3.16. *For an arbitrary S-poset B, the category* **Pos**-*S/B has enough regular injectives. More precisely, each object $f : A \to B$ in* **Pos**-*S/B can be regularly embedded into a regular injective object $\pi_B^{\bar{A}^{(S)}} : \bar{A}^{(S)} \times B \to B$ in* **Pos**-*S/B in which $\bar{A}^{(S)}$ is the set of all monotone maps from S into $\bar{A}$, with pointwise order and the action is given by $(fs)(t) = f(st)$ for $s, t \in S$ and $f \in \bar{A}^{(S)}$ and $\pi_B^{\bar{A}^{(S)}} : \bar{A}^{(S)} \times B \to B$ is the second projection.*

It is easy to show that the class *Emb* is closed under retracts in **Pos**-S/B. So by Proposition 2.4 and the theorem above, we can say that $(Emb, Emb^{\square})$ is a weak factorization system for **Pos**-S. This implies that *Emb* is saturated, that is, *Emb* is closed under pushouts, transfinite compositions and retracts (see [2]).

Up to now, we can not succeed to determine if there is a class $\mathcal{R}$ such that $(Emb, \mathcal{R})$ is a weak factorization system in **Pos**-S. However we do have:

Proposition 3.17. *Let S be a pomonoid. Suppose $(Emb, \mathcal{R})$ is a weak factorization system in* **Pos**-S*. Then $\mathcal{R} \subseteq SplitEpi$.*

Proof. Since $Sum \subset Emb$, it follows that $Emb^{\square} \subset Sum^{\square}$. But since $(Sum, SplitEpi)$ is a weak factorization system in **Pos**-S by Theorem 3.10, one has that $Sum^{\square} = SplitEpi$. Thus

$$\mathcal{R} = Emb^{\square} \subseteq SplitEpi.$$

□

4. Fibrewise regular injectivity of S-poset maps

In the previous section, we deduced that every Emb-injective object in **Pos**-S/B is a split epimorphism in **Pos**-S (see Proposition 3.17). In this section, we are going to characterize them using a fibrewise notion of complete posets. (A poset is said to be *complete* if each of its subsets has an infimum and a supremum.)

We recall [14] that in the category **Pos** of partially ordered sets and monotone maps, an Emb-injective monotone map can be characterized as follows:

Theorem 4.1. *A monotone map $f : X \to B$ is Emb-injective in* **Pos**$/B$ *if and only if it satisfies the following conditions:*

(I) *$f^{-1}(b)$ is a complete poset, for every $b \in B$;*

(II) *f is a fibration (that is, for every $x \in X$ and $b \in B$ with $f(x) \leqslant b$, $\{x' \in f^{-1}(b) \mid x \leqslant x'\}$ has a minimum element) and a cofibration (=dual of fibration).*

By [6, Theorem 1.2] we have:

Theorem 4.2. *Let S be a pomonoid. Then $f : X \to B$ is a regular injective object in* **Pos**-S/B *if and only if the following two conditions hold:*

(1) *$\langle 1_X, f\rangle : f \to \pi_B^X$ is a section in* **Pos**-S/B *where $\pi_B^X : X \times B \to B$ is the second projection;*

(2) *the object $S(f)$ of sections of f is a regular injective object in* **Pos**-S*.*

Remark 4.3 ([9])**.** For a pomonoid S, the category **Pos**-S is cartesian closed (see [5]). Indeed, given two S-posets A and B, the exponential

B^A is given by $B^A = \mathrm{Hom}_{\mathbf{Pos}\text{-}S}(S \times A, B)$, the set of all S-poset maps from the product S-poset $S \times A$ to B. (Note that the action on $S \times A$ operates on both components.) This set is an S-poset, with pointwise order and the action is given by $(f \cdot s)(t, a) = f(st, a)$ for $f \in B^A$, $a \in A$ and $s, t \in S$ (see [5, 11]). Now, given $f : X \to B$ in $\mathbf{Pos}$-S/B we have

$$S(f) = \{h \in \mathrm{Hom}_{\mathbf{Pos}\text{-}S}(S \times B, X) \mid fh = \pi_B^S\}.$$

Moreover, if B has the trivial action, then we get the following embedding induced by fibres of f:

$$m : S(f) \rightarrowtail \prod_{b \in B} f^{-1}(b) \quad \text{given by} \quad m(h) = (h(s, b))_{s \in S, b \in B}$$

Now, we supply a partial answer to the characterization of regular injectivity in the category $\mathbf{Pos}$-S/B in a special case, when the S-poset B has the trivial action.

First recall the following result from [9].

Proposition 4.4. *Let S be a pomonoid. If $f : X \to B$ is a regular injective object in the category* $\mathbf{Pos}$-S/B*, then:*
(1) *$\langle 1_X, f \rangle : f \to \pi_B^X$ is a section in* $\mathbf{Pos}$-S/B*.*
(2) *for every $b \in B$, the sub S-poset $f^{-1}(b)$ of X is a regular injective object in* $\mathbf{Pos}$-S*, so it is a complete poset.*

In this section, we are going to give a new characterization of injective objects in $\mathbf{Pos}$-S/B that removes condition (1) of the above proposition.

Proposition 4.5. *Let S be a pomonoid and $f : X \to B$ be a S-poset map. If $\langle 1_X, f \rangle : f \to \pi_B^X$ is a section in* $\mathbf{Pos}$-S/B *then for every $x \in X$ and $b \in B$ with $f(x) \leqslant b$, the poset $\{x' \in f^{-1}(b) \mid x \leqslant x'\}$ has a minimum element.*

Proof. Let $r : X \times B \to X$ be a retraction of $\langle 1_X, f \rangle$ over B. For every $x \in X$ and $b \in B$ with $f(x) \leqslant b$, let $r(x, b) = x_b$. Then we have

$$x = r(x, f(x)) \leqslant r(x, b) = x_b,$$

so $x_b \in \{x' \in f^{-1}(b) \mid x \leqslant x'\}$. Also, take x' in $f^{-1}(b)$ with $x \leqslant x'$ then

$$x_b = r(x, b) \leqslant r(x', b) = x'.$$

This means that x_b is the minimum of $\{x' \in f^{-1}(b) \mid x \leqslant x'\}$. □

Corollary 4.6. *Let S be a pomonoid and $f : X \to B$ be a regular injective S-poset map. Then*

(i) *for every $b \in B$, the sub S-poset $f^{-1}(b)$ of X is regular injective object in* **Pos**-*S, so it is a complete poset;*

(ii) *for $x \in X$ and $b \in B$ with $f(x) \leqslant b$, $\{x' \in f^{-1}(b) \mid x \leqslant x'\}$ has a minimum element x_b (also we have the dual of this fact).*

Proof. Applying Proposition 4.4 and the above proposition we get the result. □

Now, we consider the category **Pos**-S as a sub category of **Cat**. On the other words, every S-poset is a category as a poset and all action-preserving monotone maps are functors. Further, by a *topological S-poset functor* we mean an S-poset map which is topological as a functor. So we get the following result.

Theorem 4.7. *Every regular injective object in* **Pos**-*S/B is a topological S-poset functor.*

Proof. First by Corollary 4.6 and Theorem 4.1, one concludes that every regular injective object in **Pos**-S/B is a regular injective object in **Pos**$/B$. Then, part (3) of Example 2.3, implies that in **Pos**, $(Emb)^{\square} = Top$, and we get the result. □

Our next goal is to prove the converse of the above fact in a special case where S is a pogroup. But first we record a well known result.

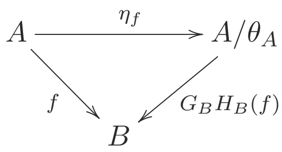
Theorem 4.8. *Let* $\mathcal{A} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{B}$ *with $F \dashv U$ be an adjunction. Then for any $B \in \mathcal{B}$, one has an adjunction $F_B \dashv U_B$, where $F_B : \mathcal{A}/U(B) \to \mathcal{B}/B$, and $U_B : \mathcal{B}/B \to \mathcal{A}/U(B)$, are such that*

- *$U_B(f : X \to B) = (U(f) : U(X) \to U(B))$, and*
- *$F_B(g : Y \to U(B)) = (F(Y) \overset{F(g)}{\to} FU(B) \overset{\epsilon_B}{\to} B)$*

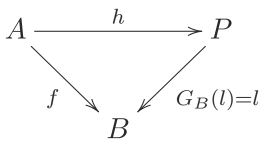
In light of [5, Theorem 12], the following result is a particular case of Theorem 4.8, but for the convenient of the reader we give a proof here.

Theorem 4.9. *The functor G_B :* **Pos**$/B \to$ **Pos**-S/B *which assigns every object in* **Pos**$/B$ *(i.e., a monotone map in* **Pos**) *to itself (which equips with the trivial action) has a left adjoint.*

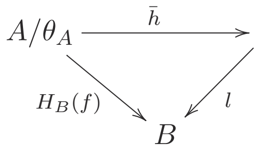
Proof. Define the functor $H_B : \mathbf{Pos}\text{-}S/B \to \mathbf{Pos}/B$ given by $H_B(h) = \bar{h} : A/\theta_A \to B$ with $\bar{h}([a]) = h(a)$ for every object $h : A \to B$ in $\mathbf{Pos}\text{-}S/B$, where the poset A/θ_A was introduced in [5, Theorem 12] and $\bar{h}$ is a monotone map. If $g : A \to C$ is an S-poset map over B, then $H_B(g) : A/\theta_A \to C/\theta_C$ defined by $H_B(g)([a]) = [g(a)]$, is a well-defined monotone map over B. The unit of this adjunction



for an object $f : A \to B$ in $\mathbf{Pos}\text{-}S/B$, is the canonical S-poset map over B, i.e, $\eta_f = \pi$. It is a universal arrow to G_B because for a given S-poset map h such that the diagram



commutes, where $l : P \to B$ is a monotone map, we have a unique S-poset map $\bar{h}$ as in the following diagram



(4.1)

given by $\bar{h}([a]) = h(a)$. By a similar proof as in [5, Theorem 12] one can prove that $\bar{h}$ is a well-defined S-poset map. The diagram (4.1) is commutative, since for every $[a] \in A/\theta_A$ we have:

$$l(\bar{h}[a]) = l(h(a)) = f(a) = \bar{f}[a] = H_B(f)[a]. \qquad \square$$

Theorem 4.10. *Let S be a pogroup. Then all topological S-poset functors are regular injective as S-poset maps with trivial action.*

Proof. By an analogue proof as in [7, Theorem 4.6], we can show that the functor $H_B : \mathbf{Pos}\text{-}S/B \to \mathbf{Pos}/B$ preserves order-embeddings, these are the regular monomorphisms in two categories $\mathbf{Pos}\text{-}S/B$ and $\mathbf{Pos}\text{-}S$. Therefore, by [9, Lemma 3.2] and the adjunction as mentioned in Proposition 4.9, the functor $G_B : \mathbf{Pos}/B \to \mathbf{Pos}\text{-}S/B$ preserves regular injective objects. Since, part (3) of Example 2.3 implies that $(Emb)^{\square} = Top$ so we get the result. $\square$

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