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RESEARCH ARTICLE

On intersections of normal subgroups in groups

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Abstract. The paper is a generalization of [2]. For a group $H = \langle A|O\rangle$, conditions for the equality $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$ are given in terms of pictures, where $\bar{N}_i$ is the normal closure of a set $\bar{R}_i \subset H$ for $i = 1, 2$.

Introduction

The present paper is a generalization of [2]. So here we will use the definitions and notation from [2]. Moreover, in most of proofs in this paper we will refer to the proofs in [2] after necessary remarks.

Let F be a free group generated by an alphabet A.

Let H be a group given by presentation $\langle A|O\rangle$, where O is a set of words on A. By N denote the normal closure of O in F. We will suppose that O is the set of all words vanishing in H, i.e., O consists of all words from N.

Consider two sets of elements $\bar{R}_1, \bar{R}_2 \subset H$ such that $\bar{r}_1 \neq t\bar{r}_2^{\pm 1}t^{-1}$ in H for any $\bar{r}_1 \in \bar{R}_1$, $\bar{r}_2 \in \bar{R}_2$, and $t \in H$. By $\bar{N}_i$ denote the normal closure of $\bar{R}_i$ in H for $i = 1, 2$. The aim of this paper is to find out necessary and sufficient conditions for

$$\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2] \text{ in } H. \tag{1}$$

These conditions will be expressed in terms of certain geometric objects called pictures (see, for example [1] or [2]).

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One can reformulate the above problem as follows.

Let $\Phi : F \to H$ be the canonical homomorphism. For each $\bar{r} \in \bar{R}_i$ let r be its reduced lift to the free group F and R_i be the symmetrized set of $\{r | \bar{r} \in \bar{R}_i\}$ for $i = 1, 2$. By N_1 and N_2 denote the normal closures in F of R_1 and R_2 respectively. Then equality (1) holds iff

$$N_1 N \cap N_2 N = [N_1, N_2] \cdot N. \tag{2}$$

Generally, we will consider the latter situation and conditions for (1) will be gotten as consequences. In particular, we will show that if a presentation $\langle A \mid R_1, R_2, O\rangle$ is aspherical (the definition of it will be given below) then equalities (1) and (2) hold.

The paper is divided into two sections, each of which is further subdivided. In the first section we give main definitions, formulate the main result (Theorem 1), prove corollaries of it and consider some examples. The second section is devoted to the proof of Theorem $1'$ which is equivalent to Theorem 1.

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1. Formulation of theorems and corollaries

1.1. Definitions. Relations between definitions

In the beginning we give some definitions generalizing notions of [2].

By H we denote a group given by $\langle A|O\rangle$, by $\mathbf{1}$ the identity in H. Let

$$G = \langle A|O \cup R\rangle$$

be a presentation of a group, where R is a symmetrized set of cyclically reduced words on A.

Since the set of defining relations is the union of O and R, a picture P over G contains vertices of two types. Vertices of the first type correspond to defining relations of the set O. Such vertices will be called *0-vertices.* Vertices of the second type correspond to the set R. They will be called *R-vertices.* Note that this situation is dual to one considered in [4, 6].

Note that in this paper R generally will consist of two sets R_1 and R_2.

For two words u and v on A, $u \equiv v$ means that u is equal to v letter by letter.

A *dipole* in a picture P over $G = \langle A \mid O, R\rangle$ is two R-vertices V_1 and V_2 of P if there is a simple path ψ connecting points p_1 and p_2 lying on the circles C_1 and C_2 around these vertices such that

$$Lab^{-1}(\psi) Lab_{p_1}{}^{+}(C_1) Lab(\psi) Lab_{p_2}{}^{+}(C_2) = \mathbf{1} \text{ in } H.$$

A picture over $G = \langle A \mid O, R\rangle$ is *reduced* if it does not contain a dipole.

A presentation $G = \langle A \mid O, R\rangle$ is *aspherical* if every connected spherical picture over $G = \langle A \mid O, R\rangle$ contains a dipole.

Further we will need the following analogue of a well-known result (use Theorem 11.1 of [4] and dualise).

Lemma 1. *Let W be a non-empty word on the alphabet A. Then W belongs to $R^F \cdot N$ if and only if there is a disk picture over the presentation $G = \langle A \mid O, R\rangle$ with the boundary label W.*

Note that if R is empty, the formulation of Lemma 1 is standard. In particular, $\bar{W} \in \Phi(R)^H$ (where $\Phi : F \to H$ is the canonical homomorphism) iff there is a disk picture over the presentation $G = \langle A \mid O, R\rangle$ with the boundary label equal to a lift of $\bar{W}$ to the free group on A.

We will use *0-transformations* of a picture P over G defined as follows.

(1) Let ψ be a simple closed path not crossing through vertices of P and dividing P into two parts one of which does not contain R-vertices and hence forms a disk subpicture $\tilde{P}$ over H. Assume that another picture $\bar{P}$ can be constructed over H with $Lab(\partial\bar{P}) \equiv Lab(\partial\tilde{P})$. Then replacing of $\tilde{P}$ by $\bar{P}$ in P is *a 0-transformation of the first type.*

(2) Let γ be a simple path not passing through any vertex of P and intersecting some edges such that $Lab(\gamma) = \mathbf{1}$ in H, i.e. $Lab(\gamma) \in N$. We can assume that γ is not closed, otherwise cut out from γ a small interval not intersecting edges and denote the obtained path by γ. It is possible to draw a simple closed path ψ by-passing near γ in the both directions so that ψ intersects the same edges as γ does. Hence $Lab(\psi) \equiv Lab(\gamma)Lab^{-1}(\gamma)$ and ψ divides P into two parts one of which does not contain R-vertices and forms a disk subpicture $\tilde{P}$ over H. To use a 0-transformation of the first type, let us construct a new disk picture over H with the boundary label equal to $Lab(\partial\tilde{P})$.

There are a disk picture P' over H with $Lab(\partial P') \equiv Lab(\gamma)$ and a disk picture P'' over H with $Lab(\partial P'') \equiv Lab^{-1}(\gamma)$ (by Lemma 1 in the case of empty R). It is clear that it is possible to divide the boundary of P' into two non-trivial arcs ς' and $\bar{\varsigma}'$ one of which (say ς') is met by edges and $Lab(\varsigma') \equiv Lab(\partial P')$, and the other one ($\bar{\varsigma}'$) is not met by edges. Similarly the boundary of P'' can be divided into two non trivial arcs ς'' and $\bar{\varsigma}''$ such that $Lab(\varsigma'') \equiv Lab(\partial P'')$ and $\bar{\varsigma}''$ is not met by edges. Pasting together the disk pictures P' and P'' by the arcs $\bar{\varsigma}'$ and $\bar{\varsigma}''$ gives a disk picture $\bar{P}$ with the boundary label equal to $Lab(\gamma)Lab^{-1}(\gamma)$. Replacing of $\tilde{P}$ by $\bar{P}$ in P (that is the 0-transformation of the first type) gives rise to "cutting" of the edges intersected by γ. This transformation of P is *a 0-transformation of the second type.* It is denoted by $T^0(\gamma)$.

Notice that 0-transformations change neither the number of R-vertices in P nor the boundary label of P.

Definition 1. *Let R_1 and R_2 be two sets of words on A. We say that a presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ is $(R_1, R_2)_O$-separable or satisfies the condition of $(R_1, R_2)_O$-separability if for every reduced spherical picture P containing both R_1-vertices and R_2-vertices, there is a simple closed path γ dividing the sphere into two disks such that the following conditions hold:*

1) *the both disks contain R-vertices;*

2) *$Lab(\gamma) = \mathbf{1}$ in H.*

Remark. In this paper the words "to contain R-vertices" mean "to contain R_1-vertices or R_2-vertices or both R_1- and R_2- vertices". The words "to contain only R_1-vertices" mean "to contain no R_2-vertex". In any case there may be 0-vertices in P or may not be.

Assertion 1. *If in Definition 1 "every reduced spherical picture" is replaced by "every spherical picture", then the set of presentations satisfying $(R_1, R_2)_O$-separability is not changed.*

Proof. Let P be a non-reduced spherical picture over $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ containing both R_1-vertices and R_2-vertices. Since P is not reduced, there is a dipole, i.e., there are two R-vertices V_1 and V_2 and a simple path ψ connecting points p_1 and p_2 lying on the circles C_1 and C_2 around these vertices such that $Lab^{-1}(\psi)Lab_{p_1}{}^{+}(C_1)Lab(\psi)Lab_{p_2}{}^{+}(C_2) = \mathbf{1}$ in H. It is easily seen that a simple closed path γ from Definition 1 may be obtained going around V_1 and V_2 and by-passing near ψ in the both directions. □

Assertion 2. *If every spherical picture over a presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ containing both R_1-vertices and R_2-vertices is not reduced (this condition will be called $(R_1, R_2)_O$-asphericity), then the presentation is $(R_1, R_2)_O$-separable.*

Proof. By Assertion 1 the presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ is $(R_1, R_2)_O$-separable if for every spherical picture P containing both R_1-vertices and R_2-vertices, there is a simple closed path γ dividing the sphere into two disks such that the following conditions hold:

1) the both disks contain R-vertices;

2) $Lab(\gamma) = \mathbf{1}$ in H.

So if every spherical picture over the presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ containing both R_1-vertices and R_2-vertices is not reduced, then such path can be found similar as in the proof of Assertion 1. Hence $(R_1, R_2)_O$-aspherical presentations are $(R_1, R_2)_O$-separable. $\square$

Definition 2. *Let R_1 and R_2 be two sets of words on A. We say that a presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ is weakly $(R_1, R_2)_O$-separable or satisfies the condition of weak $(R_1, R_2)_O$-separability if for every reduced spherical picture P containing both R_1-vertices and R_2-vertices, there is a simple closed path γ dividing the sphere into two disks such that the following three conditions hold:*

1) *the both disks contain R-vertices;*

2) *$Lab(\gamma) \in [N_1, N_2] \cdot N$;*

3) *if one of the disks does not contain R_2-vertices, then the other one does not contain R_1-vertices.*

Remark. If the condition 3) in Definition 2 is omitted, then the set of presentations satisfying only 1) and 2) of Definition 2 is wider as is shown in [2] for $H = F(A)$.

Remark. Weak $(R_1, R_2)_O$-separability is not equivalent to the condition of $(R_1, R_2)_O$-separability as is shown in [2] for $H = F(A)$.

Definition 3. *Let R_1 and R_2 be two sets of words in $F(A)$. We say that a presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ is strictly $(R_1, R_2)_O$-separable or satisfies the condition of strict $(R_1, R_2)_O$-separability if for every spherical picture P containing both R_1-vertices and R_2-vertices, there is a simple closed path γ dividing the sphere into two disks such that the following three conditions hold:*

1) *the both disks contain R-vertices;*

2) *$Lab(\gamma) \in [N_1, N_2] \cdot N$;*

3) *one of the disks does not contain R_1-vertices and the other one does not contain R_2-vertices.*

1.2. Formulation of theorems and corollaries

Now consider the normal closures $\bar{N}_1$ and $\bar{N}_2$ of sets $\bar{R}_1$ and $\bar{R}_2$ in a group $H = \langle A | O \rangle$ and the canonical homomorphism $\Phi : F = F(A) \to H$. We assume that $\bar{r}_1 \neq t\bar{r}_2^{\pm 1}t^{-1}$ in H for any $\bar{r}_1 \in \bar{R}_1$, $\bar{r}_2 \in \bar{R}_2$, and $t \in H$. For each $\bar{r} \in \bar{R}_i$ let r be any of its reduced lifts to the free group F and R_i be the symmetrized set of $\{r | \bar{r} \in \bar{R}_i\}$ for $i = 1, 2$.

Theorem 1. *A presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ is weakly $(R_1, R_2)_O$-separable if and only if $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$ in H.*

Let N_i be the normal closure of R_i in F for $i = 1, 2$. Since $N_1N \cap N_2N = [N_1, N_2] \cdot N$ if and only if $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$, Theorem 1 is equivalent to the following Theorem $1'$ which will be proved in Section 2.

Theorem $1'$. *A presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ is weakly $(R_1, R_2)_O$-separable if and only if $N_1N \cap N_2N = [N_1, N_2] \cdot N$ in F.*

In particular, all conditions sufficient for $N_1N \cap N_2N = [N_1, N_2] \cdot N$ are sufficient for $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$. Thus we have the following statements.

Corollary 1. *The conditions of weak $(R_1, R_2)_O$-separability and strict $(R_1, R_2)_O$-separability are equivalent.*

Proof. If a presentation is strictly $(R_1, R_2)_O$-separable, then it is weakly $(R_1, R_2)_O$-separable since the only difference is in the third condition of Definitions and it is clear that the third condition of weak $(R_1, R_2)_O$-separability follows from one of strict $(R_1, R_2)_O$-separability.

It remains to prove the converse statement. Let P be a spherical picture over a weakly $(R_1, R_2)_O$-separable presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ containing both R_1-vertices and R_2-vertices. It is evident that there is a simple closed path γ not passing through any vertex of P and dividing the sphere into two disks one of which does not contain R_1-vertices and the other one does not contain R_2-vertices. Hence by Lemma 1, $Lab(\gamma) \in N_1N \cap N_2N$. Since $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ is weakly $(R_1, R_2)_O$-separable, Theorem $1'$ leads to $Lab(\gamma) \in [N_1, N_2] \cdot N$. Consequently γ is desired. □

Corollary 2. *The conditions of weak $(R_1, R_2)_O$-separability and weak (R_2, R_1)-separability are equivalent. Moreover, we get the equivalent condition if in Definition 2 of weak $(R_1, R_2)_O$-separability the item 3) is replaced by*
$3'$) *if one disk contains both R_1- and R_2-vertices, then the other one contains also both R_1- and R_2-vertices.*

Corollary 3. *Let a presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ satisfy one of the following conditions:*

(i) $(R_1, R_2)_O$*-separability;*

(ii) $(R_1, R_2)_O$*-asphericity;*

(iii) asphericity.

Then $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$ *in* H *and* $N_1N \cap N_2N = [N_1, N_2] \cdot N$ *in* F.

Proof.
(i) The proof is similar to the proof of Theorem 1 (Theorem 1′).
(ii) It follows directly from (i) and Assertion 2.
(iii) It follows from (ii). □

Remark. Theorem 1 and Corollary 3 lead that each condition of Corollary 3 implies weak $(R_1, R_2)_O$-separability.

Remark. It is clear that there is no difference which preimage of $\bar{r} \in \bar{R}_i$ we took forming R_i, since the conditions of weak $(R_1, R_2)_O$-separability, $(R_1, R_2)_O$-separability, $(R_1, R_2)_O$-asphericity, asphericity are defined up to elements from N.

1.3. Some applications

(1) In [1], W.A.Bogley and S.J.Pride consider the following situation.

In the notation of [1], let H be a group; adjoin a set of generators $\mathbf{x}$ to H; then factoring of the resulting free product $H * \langle \mathbf{x} \rangle$ by the normal closure N of a set $\mathbf{r}$ of cyclically reduced elements of $H * \langle \mathbf{x} \rangle \setminus H$ gives rise to a new group G. It is be said that G is defined by the relative presentation $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r} \rangle$.

W.A.Bogley and S.J.Pride introduce pictures over relative presentations, give the definitions of a dipole, asphericity and weak asphericity for relative presentations. Then they give a "wight test" for asphericity and introduce small cancellation conditions $C(p), T(q)$ for relative presentations in a slightly non-standard way (part of the definition makes use of star complexes). They prove that a relative presentation satisfying $C(p), T(q)$ with $1/p + 1/q = 1/2$ is aspherical (Theorem 2.2 [1]).

In addition, W.A.Bogley and S.J.Pride discuss the interplay between pictures over $\mathbf{P}$ and pictures over an ordinary presentation $\tilde{\mathbf{P}}$ defining the same group G which is considered in the present paper.

Let $Q = \langle \mathbf{a}; \mathbf{s} \rangle$ be an ordinary presentation of H. For each $R \in \mathbf{r}$, let $\tilde{\mathbf{R}}$ be its reduced lift to the free group on $\mathbf{a} \cup \mathbf{x}$. Then $\tilde{\mathbf{P}} = \langle \mathbf{a}, \mathbf{x}; \mathbf{s}, \tilde{\mathbf{r}} \rangle$ where $\tilde{\mathbf{r}} = \{\tilde{\mathbf{R}} : \mathbf{R} \in \mathbf{r}\}$. It was proved in [1] that if $\mathbf{P}$ is orientable (that is, $\mathbf{P}$ is slender and no element of $\mathbf{r}$ is a cyclic permutation of its inverse) and aspherical, then every picture over $\tilde{\mathbf{P}}$ having at least one $\tilde{\mathbf{r}}$-disk and having no $\mathbf{x}$-arcs meeting the boundary of the picture, contains an $\tilde{\mathbf{r}}$-dipole (Lemma 1.5 [1]). Also it was shown that without the orientability assumption, this result is false; and that there is a passage from pictures over $\tilde{\mathbf{P}}$ to pictures over $\mathbf{P}$ and a reverse passage. It is easy to see that weak asphericity of relative presentation $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r} \rangle$ considered in [1] is equivalent to asphericity (that is a particular case of asphericity in the sense of the present paper when two $\tilde{\mathbf{r}}$-vertices of a dipole are joined by an $\mathbf{x}$-edge) of ordinary presentation $\tilde{\mathbf{P}}$.

Let $\mathbf{r_1}, \mathbf{r_2}$ be two sets of cyclically reduced elements of $H * \langle \mathbf{x} \rangle \setminus H$ such that $\mathbf{R_1} \neq \mathbf{R_2}$ in $H * \langle \mathbf{x} \rangle$ for any $\mathbf{R_1} \in \mathbf{r_1^*}$ and any $\mathbf{R_2} \in \mathbf{r_2^*}$. Thus it is easy to see that by Corollary 3 we have the following.

If a relative presentation $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r} \rangle$ is orientable and aspherical, and $\mathbf{r} = \mathbf{r_1} \bigsqcup \mathbf{r_2}$, then the intersection of the normal closures of $\mathbf{r_1}$ and $\mathbf{r_2}$ in $H * \langle \mathbf{x} \rangle$ is equal to the mutual commutant of these normal closures. For example, it is sufficient for a relative presentation $\mathbf{P} = \langle H, \mathbf{x}; \mathbf{r} \rangle$ to be orientable and satisfy $C(p), T(q)$ with $1/p + 1/q = 1/2$.

(2) In [6] A.Yu. Ol'shanskii considers a hyperbolic group G given by $\langle A|O \rangle$ and a group $G_1 = \langle A|O \cup R \rangle$ where R is some symmetrized system of additional relations. He investigates properties of G_1 depending on R. He introduces the $C(\varepsilon, \mu, \lambda, c, \rho)-$condition to construct torsion and many other kinds of quotient groups of hyperbolic groups.

Let us show (using the notation and definitions from [6]) that under the conditions stated in Lemma 6.6 [6] the symmetrized presentation $G_1 = \langle A|O \cup R \rangle$ is aspherical (in the sense of the present paper). Consider a spherical diagram Δ over G_1 containing R-faces. Suppose, contrary to our claim, that Δ is reduced. Then similar to the proof (a) of Lemma 6.5 [6], we have that Δ is tame (this notion of [6] is naturally generalized to spherical diagrams), since the subdiagrms in (a) of Lemma 6.5 [6] are circular and hence, Lemma 6.6 [6] is true for them. Lemmas 6.1-6.3 [6] are true for spherical diagrams, it follows from Lemmas 6.3 [6] for the spherical tame diagram Δ that $0 > 1 - 21\mu$, i.e. $\mu > 1/21$. This contradicts the assumption that $\mu < 1/30$. Hence Δ is not reduced. Therefore the presentation $G_1 = \langle A|O \cup R \rangle$ is aspherical (in the sense of the present paper), since pictures are dual objects to diagrams.

So by Corollary 3 we have the following.

Let R_1, R_2 be two symmetrized sets of cyclically reduced elements of the free group $F(A)$ such that $r_1 \neq tr_2t^{-1}$ in a hyperbolic group $H = \langle A|O \rangle$ for any $r_1 \in R_1$, any $r_2 \in R_2$ and any $t \in F$. Put $R = R_1 \cup R_2$. For any $\lambda > 0$, $\mu \in (0, 1/30)$, $c \geq 0$ there are $\varepsilon \geq 0$ and $\rho > 0$ such that if the symmetrized presentation $G = \langle A|O \cup R \rangle$ satisfies the $C(\varepsilon, \mu, \lambda, c, \rho)-$condition then $N_1N \cap N_2N = [N_1, N_2] \cdot N$ and $\bar{N}_1 \cap \bar{N}_2 = [\bar{N}_1, \bar{N}_2]$ in H (using the notations of Introduction).

It is mentioned in [6] that the standard $C'(\mu)$-condition (see [3]) is a particular case of the $C(\varepsilon, \mu, \lambda, c, \rho)-$condition, where H is a free group, $\varepsilon = 0, \lambda = 1, c = 0, \rho = 1$.

2. Proof of Theorem 1′

The proof of Theorem 1′ is almost similar to one of Theorem 1 in [2]. In the beginning we list definitions from [2] needed for the proof and describe necessary generalizations of them.

2.1. Additional definitions

1) *Picture P with equator Equ. Subpictures of P.*
The difference of this definition from one of [2] is the following. P is a picture on the sphere S^2 over a presentation $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$. The equator Equ is a fixed simple closed path on S^2 not passing through any vertex of P and dividing S^2 into two parts so that one part does not contain R_1-vertices and the other one does not contain R_2-vertices. 0-vertices may lie in both hemispheres. $Lab(Equ)$ is denoted by W or W^{-1} (the sign depends on the direction of reading). By the choice of the equator, it follows from Lemma 1 that $W \in N_1N \cap N_2N$.

2) Boundaries of vertices. North and south vertices. 0-vertices.
The difference of this definition from one of [2] is the following. There are vertices of three types: R_1-, R_2- and 0-vertices. An R-vertex is called *north* (respectively, *south*) if it lies in the north (respectively, south) hemisphere. As in [2], suppose that R_1-vertices are north, R_2-vertices are south.

3) Admissible transformations.
The difference of this definition from one of [2] is that admissible transformations can replace $W \equiv Lab(Equ)$ by a word W' equal to W to within an element of $[N_1, N_2] \cdot N$ (for simplicity of notation, we will use the same letter W for the notation W'). Admissible transformations also preserve the subdivision of P by Equ into the north and south vertices. For example, a 0-transformation is admissible if it is performed inside of any hemisphere.

4) Maps. This definition is the same as one in [2].

5) Components.
The difference of this definition from one of [2] is the following. The definitions of reduced and non-reduced components here are formally the same as in [2], but one should note that the definition of a dipole in the present paper differs slightly from one in [2]. A component is called north (respectively, south) if the corresponding subpicture contains only north (respectively, only south) vertices and possibly some 0-vertices. A component is called a 0-component if it contains only 0-vertices. South and north components and 0-components are called *uniform*. A component is called *mixed* if the corresponding subpicture contains both south and

north vertices.

6) Countries. Their territories and boundaries. North and south countries, 0-countries.
This definition is the same as one in [2]. One should only add that a country is called *a 0-country* if the corresponding uniform component is a 0-component.

7) Pieces of equator Equ. This definition is the same as one in [2].

8) Regions of north and south countries. South, north and 0-regions. This definition is the same as one in [2]. One should only add that a region is called *a 0-region* if it doesn't contain R-vertices.

9) *σ-countries.* This definition is the same as one in [2].

10) *σ-pictures.* This definition is the same as one in [2].

2.2. Proof of Theorem $1'$ modulo Propositions 1 and 2

The proof of the statement that weak $(R_1, R_2)_O$-separability follows from the equality $N_1N \cap N_2N = [N_1, N_2] \cdot N$ is similar to the proof of Corollary 1.

Therefore it remains to prove the converse statement. Let a presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ satisfy weak $(R_1, R_2)_O$-separability.

Since the inclusion $[N_1, N_2] \cdot N \subset N_1N \cap N_2N$ always holds, it is sufficient to prove the inverse inclusion.

Let W be an arbitrary word of $N_1N \cap N_2N$. To show that $W \in [N_1, N_2] \cdot N$, let us construct two disk pictures. Since $W \in N_1 \cdot N$, by Lemma 1, there is a disk picture over $\langle A|R_1 \cup O\rangle$ with the boundary label equal to the word W. This picture contains only R_1-vertices and 0-vertices. Since $W \in N_2 \cdot N$, similarly there is a disk picture over $\langle A|R_2 \cup O\rangle$ with the boundary label equal to the word W^{-1}. This picture contains only R_2-vertices and 0-vertices. Pasting together the disk pictures by their boundaries gives a picture P on S^2 with a fixed equator Equ. $Lab(Equ)$ is equal to W or W^{-1} depending on the direction of moving along Equ.

It is obvious that the proof of Theorem $1'$ follows from the following Propositions 1 and 2, which will be proved in the subsections 2.4 and 2.5 below:

Proposition 1. *Let a picture P with a fixed equator Equ be over a presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$. If the presentation satisfies weak $(R_1, R_2)_O$-separability, then P is a σ-picture.*

Proposition 2. *Let a picture P with a fixed equator Equ be over a presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$. If P is a σ-picture, then the word W*

along Equ can be reduced to the identity element in the free group by a finite number of admissible transformations.

2.3. Some admissible transformations. Auxiliary lemmas

Below we will use the following lemma.

Lemma 2. *Let T be a country in P. Let I be an arbitrary piece of the equator belonging to the territory of T. Then $Lab(I) \in N_1 \cdot N$ if T is north, $Lab(I) \in N_2 \cdot N$ if T is south, and $Lab(I) \in N$ (i.e. $Lab(I) = \mathbf{1}$ in H) if T is a 0-country.*

Proof. Let T be north (the proof in the case of a south country is similar). The piece I divides the territory of T into two parts T' and T''. We consider one of them denoted by T'. The part T' may contain only north vertices (i.e., R_1-vertices), 0-vertices and edges labelled by letters of the alphabet A. Hence T' contains a disk picture over $\langle A \mid R_1 \cup O \rangle$. By Lemma 1, the word along the boundary of T' belongs to $N_1 \cdot N$. Since the edges intersecting the boundary of T' intersect it only in a part which coincides with I, Lemma 2 in the case of north T follows. The proof in the case of a 0-country is similar but we need to use Lemma 1 in the case of empty R. □

Further we list admissible transformations, which will be used in the proof of Propositions 1 and 2. Some of them are the same as in [2]. It is obvious that they are still admissible in the sense of the present paper. Also we describe admissible transformations generalizing some transformations from [2].

1) Isotopy. (It is the same transformation as in [2].)

2) *Bridge moves.* (It is the same transformation as in [2].)

3) *Removing components and edge-circles of P not intersecting Equ.* (It is the same transformation as in [2].)

4) *Removing regular 0-regions:*

Assume that Equ intersects the territory of a country T in two successive points such that the part ϕ of the boundary of T between these two points and an appropriate part $\bar{\phi}$ of Equ between these two points bound a 0-region of T. This 0-region (denote it by P^0) forms a disk picture over H. Hence $Lab(\partial P^0) = \mathbf{1}$ in H. Since edges in P^0 intersecting ∂P^0 intersect it only in $\bar{\phi}$, $Lab(\bar{\phi}) \equiv Lab(\partial P^0)$ and $Lab(\phi) \equiv 1$. Change Equ by $(Equ \setminus \bar{\phi}) \cup \phi$. Then slightly decrease the territory of T to separate Equ and the part ϕ of the boundary of T. This transformation corresponds to a cancellation of a word equal to $\mathbf{1}$ in H (i.e. belonging to N) in the equatorial label W. Hence it is admissible.

5) *Joining σ-countries.* (It is the same transformation as in [2].)
6) *Pasting a map with a commutator subpicture in P:*
We will use this transformation in the subsection 2.5. In notation of this subsection, assume that there are two regular regions: one of them belongs to a north country T_1, the other one belongs to a south country T_2. By Lemma 2, the north region contains a picture with a word $w_1 \in N_1 \cdot N$ written along a piece I_2 of the equator, the south region contains a picture with a word $w_2 \in N_2 \cdot N$ written along a piece I_3 of the equator. Similar to the admissible transformation 6) of [2] construct a map M containing a picture with the word along the equator equal to $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$. Also similar to [2] a small map M_s in P containing nothing but a part $\{y = 0\}$ of Equ is replaced by the constructed map M. This transformation is admissible because it corresponds to an insertion of the commutator $w_2 w_1 {w_2}^{-1} {w_1}^{-1}$ of the elements from $N_1 \cdot N$ and $N_2 \cdot N$ in the equatorial label W (i.e. $w_2 w_1 {w_2}^{-1} {w_1}^{-1} \in [N_1 \cdot N, N_2 \cdot N] \subset [N_1, N_2] \cdot N$).

Lemma 3. *Let T be a country in P. If Equ intersects the boundary of T exactly two times, then a word along the piece of the equator lying inside the territory of T is equal to the identity element in H (i.e. it belongs to N).*

Proof. Lemma 3 in the case of a 0-country follows from Lemma 2. Assume that T is north (the proof in the case of a south country is similar). The given piece of the equator divides the territory of T into two regions: north and south. Moreover all R-vertices lie in the north region. The south region is a regular 0-region. Removing it (see the admissible transformation 4)) gives rise to the case when no edge of T intersects Equ. Since this transformation corresponds to the cancellation of a word equal to **1** in H, the original word along the piece of the equator lying inside the territory of T was equal to the identity element in H. □

2.4. Proof of Proposition 1

The proof of Proposition 1 will be divided into several steps (lemmas). In Step 1 we will show that the picture P with the fixed equator Equ can be divided into a finite number of uniform components. In Step 2 the uniform components should be transformed to countries and edges-circles not belonging to the countries. In Step 3 we should get rid of the edges-circles not belonging to the countries. In Step 4 the countries should be divided into σ-countries. In all steps a finite number of admissible transformations will be used. Therefore we will get that P is a σ-picture. Note that Steps 2-4 are similar to Steps 2-4 in [2] if one considers 0-components as north components.

All pictures obtained from P will be denoted by P again for simplicity of notation.

Step 1. *Reducing P to a picture containing only uniform components.*
In Step 1 we will use the following two admissible transformations.
Operation A: Transformations of reduced mixed components.
Let the presentation $G = \langle A \mid R_1 \cup R_2 \cup O\rangle$ be weakly $(R_1, R_2)_O$-separable. By K denote a reduced mixed component.

Since K is a reduced spherical picture, the condition of the weak $(R_1, R_2)_O$-separability leads to the existence of a simple closed path γ dividing the sphere into two parts so that

1) the both parts contain R-vertices;
2) $U \equiv Lab(\gamma) \in [N_1, N_2] \cdot N$;
3) if one part does not contain south vertices, then the other one does not contain north vertices.

By the property 3) of γ, the following three cases are possible.

The first case: the path γ divides K into two parts one of which does not contain north vertices, the other one does not contain south vertices. The second case: γ divides K into two parts one of which does not contain north vertices, the other one contains both south and north vertices. In these two cases we can assume that some segment ψ of the path γ lies on Equ. The complement of ψ to γ will be denoted by $\neg\psi$. One of the endpoints of ψ will be denoted by p.

The third case: the path γ divides K into two parts each of which contains both south and north vertices. Consequently, γ is intersected by Equ and divided by it into segments among which there are segments lying in the north hemisphere wholly. Fix one of them. By ψ denote its connected part not intersecting Equ. By p denote one of the endpoints of ψ. By $\neg\psi$ denote the complement of ψ to γ.

As in [2], in each of these three cases one can assume that all edges intersecting the path γ intersect it in the segment ψ, because otherwise all edges intersecting $\neg\psi$ can be moved by isotopy (the admissible transformation 1)) to ψ along the path γ in the direction of the point p starting successively at the nearest to p edge.

Further, similar to Operation A of Step 1 in [2], in each of these three cases we select a map M on S^2. A new map M' is constructed as follows. Since $[N_1, N_2] \cdot N \subset N_1N \cap N_2N$, the word U along ψ belongs to the both groups $N_1 \cdot N$ and $N_2 \cdot N$. By Lemma 1, one can construct disk pictures P_1 and P_2 with the boundary labels respectively U and U^{-1}. Moreover in the first two cases P_1 is constructed over $\langle A \mid R_2 \cup O\rangle$ (using south vertices and 0-vertices) and P_2 is constructed over $\langle A \mid R_1 \cup O\rangle$ (using north vertices and 0-vertices); in the third case the both pictures P_1 and P_2 are constructed over $\langle A \mid R_1 \cup O\rangle$ (using north vertices and 0-vertices). Then these pictures are disposed on the new map M' similar to Operation A of Step 1 in [2].

The old map M is cut out from P and replaced by the new one M'.

By this replacing of maps, the equatorial label W is changed by the commutator from $[N_1, N_2] \cdot N$ in the first two cases and it is not changed in the third case. After such replacing all R_1-vertices lie in the north hemisphere and all R_2-vertices lie in the south one. Therefore this transformation of P is admissible.

With the help of such replacing of maps the component K falls into two components K_1 and K_2 separated from each other by the path γ. In the first case the components K_1 and K_2 are uniform; in the second case one of the components (denote it by K_1) is mixed, the other one (K_2) is south uniform; in the third case each of the components K_i is mixed.

Remark 1. The components K_1 and K_2 can be non-reduced. The number of south vertices in each mixed component K_i (K_1 in the second case; K_1 and K_2 in the third case) is strictly less than the number of south vertices in the original component K, since during Operation A we added only north vertices and 0-vertices to obtain the mixed components.

Operation B: Transformations of non-reduced mixed components. Let K be a non-reduced component. Then there is a dipole in K, i.e., there are two R-vertices V' and V'' such that there is a simple path ψ joining some points p_1 and p_2, which lie on the boundaries C_1 and C_2 of V' and V'', so that $Lab^{-1}(\psi)Lab_{p_1}{}^{+}(C_1)Lab(\psi)Lab_{p_2}{}^{+}(C_2) = \mathbf{1}$ in H.

Remark 2. The both vertices of a dipole can be either north or south, since $r_1 \neq tr_2t^{-1}$ in H for any $r_i \in R_i (i = 1, 2)$, $t \in F$.

Evidently, it is possible to surround the vertices of the dipole by a simple closed path γ passing along ψ and around V' and V'' such that $Lab(\gamma) = \mathbf{1}$ in H.

Similar to Operation A, one can assume that all edges intersecting the path γ intersect it in a segment $\bar{\gamma}$ not intersecting Equ. By $T^0(\bar{\gamma})$ the component K falls into two components K_1 and K_2 not connected with each other. The component K_1 contains only V' and V'' and some 0-vertices and edges. Hence K_1 is uniform. The component K_2 may be either uniform or mixed, either reduced or non-reduced.

Remark 3. Operation B does not increase the number of R-vertices. Therefore the number of R-vertices in each K_i is strictly less than the number of R-vertices in the original component K. In particular, the number of south vertices in K_2 is not more than the corresponding number in the original component K.

Lemma 4. *Let a presentation* $G = \langle A \mid R_1 \cup R_2 \cup O \rangle$ *be weakly* $(R_1, R_2)_O$*-separable. Then a picture* P *with a fixed equator Equ falls into a finite number of uniform components with the help of a finite number of Operations A and B (being admissible transformations).*

Proof. Proof of Lemma 4 is similar to the proof of Lemma 4 in [2]. □

The rest of the proof of Proposition 1 is similar to the proof of Proposition 1 of [2]. The only difference is the following. Firstly 0-components and 0-countries should be considered as north and called north (if a 0-component (a 0-country) does not have a part in the north hemisphere, it can be removed by the admissible transformation 3)). Secondly one should use Lemma 1-2 of the present paper instead of Lemma 1-2 of [2] for construction of disk pictures over $\langle A | R_1 \cup O \rangle$ in Step 4. □

2.5. Proof of Proposition 2

By admissible transformations, P is reduced to a picture containing only σ-countries.

One can suppose that every regular region of every σ-country contains R-vertices otherwise it will be so after a finite number of the admissible transformations 4). In addition if there is a σ-country containing only 0-vertices it can be removed by the admissible transformation 3). Hence all σ-countries are north or south.

The proof of Proposition 2 repeats the proof of Proposition 2 of [2]. The only difference is that one should use Lemmas 2-3 and corresponding admissible transformations of the present paper instead of Lemmas 2-3 and corresponding admissible transformations of [2] and take into account that $\mathbf{1}$ in [2] denotes the identity in the free group F, N_i in [2] corresponds to $N_i \cdot N$ here and $[N_1, N_2]$ in [2] corresponds to $[N_1, N_2] \cdot N$ here. □

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