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CONVERGENCE RATES IN REGULARIZATION FOR THE CASE OF MONOTONE PERTURBATIONS

ШВИДКІСТЬ ЗБІЖНОСТІ В РЕГУЛЯРИЗАЦІЇ ДЛЯ ВИПАДКУ МОНОТОННИХ ЗБУРЕНЬ

Convergence rates are justified for regularized solutions of the Hammerstein operator equation of the form $x + F_2F_1(x) = f$ in the Banach space with monotone perturbations F_2^h and F_1^h.

Наведено обгрунтування швидкості збіжності регуляризованих розв'язків операторного рівняння Гаммерштейна вигляду $x + F_2F_1(x) = f$ в банахових просторах з монотонними збуреннями F_2^h, F_1^h.

1. Introduction. Let X be a real reflexive Banach space having the property: X and X^* are strictly convex and weak convergence and convergence of norms of any sequence in X follow its strong convergence, where X^* denotes the dual space of X. For the sake of simplicity the norms of X and X^* will be denoted by one symbol $\|\cdot\|$. We write $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ instead of $x^*(x)$ for $x^* \in X^*$ and $x \in X$. Let $F_1 : X \to X^*$ and $F_2 : X^* \to X$ be monotone, bounded (i. e. image of any bounded subset is bounded) and continuous operators.

Consider the operator equation of Hammerstein type

$$x + F_2F_1(x) = f, \quad f \in X. \tag{1}$$

Nonlinear operator equation in this form has been investigated in [1 – 3]. For solving (1) in [4] the approximate solution is constructed by the solution x_α of the approximate operator equation

$$x + F_{2\alpha}F_{1\alpha}(x) = f \tag{2}$$

with $F_{1\alpha} = F_1 + \alpha U_1$, U_1 is the standard dual mapping of X [5, p. 311], i. e.

$$\|U_1(x)\|^2 = \|x\|^2 = \langle U_1(x), x\rangle \qquad \forall x \in X,$$

$F_{2\alpha} = \alpha U_2$, U_2 is the standard dual mapping of X^*, and $\alpha > 0$ is a small parameter. For every $\alpha > 0$, equation (2) has a unique solution x_α, and if S_0, the set of solutions of (1), is not empty, then the sequence $\{x_\alpha\}$ converges to a solution x_0 of (1), as $\alpha \to 0$. Moreover, this solution x_α, for every fixed $\alpha > 0$, depends continuously on f, and the finite-dimensional problems

$$x + F_{2\alpha n}F_{1\alpha n}(x) = f_n, \quad x \in X_n, \tag{3}$$

where $F_{2\alpha n} = P_nF_{2\alpha}P_n^*$, $F_{1\alpha n} = P_n^*F_{1\alpha}P_n$, $f_n = P_nf$, P_n is a linear projection from X onto its finite-dimensional subspace X_n such that $X_n \subset X_{n+1}$, $P_nx \to x$, as $n \to \infty$ for every $x \in X$, and P_n^* is the dual of P_n with $\|P_n\| \le \tilde{c} =$ constant, for all n, have a unique solution $x_{\alpha n}$, and the sequence $\{x_{\alpha n}\}$ converges to x_α, as $n \to \infty$, without additional conditions on F_i, $i = 1, 2$. The convergence rates for the sequences $\{x_\alpha\}$ and $\{x_{\alpha n}\}$ are given in our recent paper [6] provided the linearity of F_2

together with the existence of bounded inversion $(I+F_2F_1'(x_0))^{-1}$, where I denotes the identity operator in X. It is not difficult to verify that this condition can be replaced by the bounded inversion of $\left(I+F_2'(x_0^*)F_1'(x_0)\right)^{-1}$, when F_2 also is nonlinear, where $x_0^*=F_1(x_0)$. The last requirement is equivalent to that -1 is not an eigenvalue of the operator $F_2'(x_0^*)F_1'(x_0)$ and is used in studying a method of collocation-type for nonlinear integral equations of Hammerstein type [7]. In general case, i. e. when both the operator F_i are nonlinear, it means that $\mathcal{R}$, the range of the operator $I+F_2'(x_0^*)F_1'(x_0)$, is the whole space X. It is natural to ask if we can estimate the convergence rates for the sequences $\{x_\alpha\}$, $\{x_{\alpha n}\}$, when $\mathcal{R}$ is not the whole space X. For this purpose, only requiring that $\mathcal{R}$ contains a necessary element of X, the convergence rates of $\{x_\alpha\}$ and $\{x_{\alpha n}\}$ are estimated in [8] on the base of the results and the technics in [9, 10].

If instead of F_2 and F_1 we know only their monotone continuous approximations F_2^h and F_1^h, respectively, such that

$$\left\|F_2^h x^* - F_2 x^*\right\| \le hg_2\left(\left\|x^*\right\|\right) \qquad \forall x^* \in X^*,$$

$$\left\|F_1^h(x) - F_1(x)\right\| \le hg_1(\|x\|) \qquad \forall x \in X,$$

where $g_i(t)$, $i=1,2$, are some real nonegative bounded function with $g_i(0)=0$, the approximate solutions of (1) can be found by solving the following operator equation [11]

$$x + F_{2\alpha}^h F_{1\alpha}^h(x) = f, \tag{4}$$

where $F_{2\alpha}^h = F_2^h + \alpha V$, $F_{1\alpha}^h = F_1^h + \alpha U$. Equation (4) has, for each fixed $\alpha>0$, $h>$ >0, a unique solution, henthforth denoted by $x_{h\alpha}$. If h/α, and $\alpha\to 0$, then the sequence $\{x_{h\alpha}\}$ converges to the solution x_0 of (1). As $F_{2\alpha}^h$ and $F_{1\alpha}^h$ are both monotone and hemicontinuous, then equation

$$x + F_{2\alpha}^{hn} F_{1\alpha}^{hn}(x) = f_n, \tag{5}$$

where $F_{2\alpha}^{hn} = P_n F_{2\alpha}^h P_n^*$ and $F_{1\alpha}^{hn} = P_n^* F_{2\alpha}^h P_n$, has a unique solution (denoted by $x_{\alpha n}^h$), and the sequence $\left\{x_{\alpha n}^h\right\}$ converges to $x_{h\alpha}$, for each fixed $\alpha>0$, $h>0$, as $n\to+\infty$. The similar aspects with the similar conditions as in [6] for the sequences $\{x_{h\alpha}\}$ and $\left\{x_{\alpha n}^h\right\}$ are considered in [12]. In this paper, for the case of monotone perturbations F_2^h, F_1^h the convergence rates of $\{x_{h\alpha}\}$ and $\left\{x_{\alpha n}^h\right\}$ are investigated under another condition consisting of that the range of $F_2'(x_0^*)^* F_1'(x_0)^*$ contains some element of X.

Below, by "$a \sim b$" we mean "$a=O(b)$ and $b=O(a)$", and the symbols $\rightharpoonup$ and $\to$ denote weak convergence and convergence in norm, respectively.

2. Main results. Assume that the dual mappings U_i, $i=1,2$, of the spaces X and X^* satisfy the following conditions

$$\left\langle U_i(y_1^i) - U_i(y_2^i),\, y_1^i - y_2^i \right\rangle \ge m_i\left\|y_1^i - y_2^i\right\|^{s_i}, \qquad m_i > 0, \qquad s_i \ge 0, \tag{6}$$

$$\left\|U_i(y_1^i) - U_i(y_2^i)\right\| \le c_i(R_i)\left\|y_1^i - y_2^i\right\|^{\nu_i}, \qquad 0 < \nu_i \le 1, \tag{7}$$

where $y_1^i,\, y_2^i \in X$ or X^* on dependence of $i = 1$ or 2, respectively, and $c_i(R_i)$, $R_i > 0$, are the positive increasing functions on $R_i = \max\{\|y_1^i\|, \|y_2^i\|\}$.

The following theorem answers the question on convergence rates for $\{x_{h\alpha}\}$.

Theorem 1. *Let the following conditions hold*:

(i) *F_1 is Frechet differentiable at some neighbourhood $\mathcal{U}_0$ of x_0 $s_1 - 1$-times if $s_1 = [s_1]$, the integer part of s_1, $[s_1]$-times if $s_1 \neq [s_1]$, and F_2 is Frechet differentiable at some neighbourhood $\mathcal{V}_0$ of x_0^* $s_2 - 1$-times, if $s_2 = [s_2]$, $[s_2]$-times if $s_2 \neq [s_2]$;*

(ii) *there exists a constant $\tilde{L} > 0$ such that*

$$\left\| F_1^{(k)}(x_0) - F_1^{(k)}(y) \right\| \le \tilde{L} \| x_0 - y \| \qquad \forall y \in \mathcal{U}_0,$$

$$\left\| F_2^{(k)}(x_0^*) - F_2^{(k)}(y^*) \right\| \le \tilde{L} \left\| x_0^* - y^* \right\| \qquad \forall y^* \in \mathcal{V}_0,$$

for F_i: $k = s_i - 1$ if $s_i = [s_i]$, $k = [s_i]$ if $s_i \neq [s_i]$, and if $[s_i] \ge 3$, then $F_1^{(2)}(x_0) = \ldots = F_1^{(k)}(x_0)$, and $F_2^{(2)}\left(x_0^*\right) = \ldots = F_1^{(k)}\left(x_0^*\right) = 0$.

(iii) *there exists an element $x^1 \in X$ such that*

$$\left(I + F_2'\left(x_0^*\right)^* F_2'(x_0)^*\right) x^1 = F_2'\left(x_0^*\right)^* U_1(x_0) - U_2\left(x_0^*\right),$$

if $s_1 = [s_1]$ then $\tilde{L}\left\| x^1 \right\| < m_1 s_1!$, and if $s_2 = [s_2]$ then $\tilde{L}\left\| F_1'(x_0)^* x^1 - U_1(x_0) \right\| < m_2 s_2!$.

Then, if α is chosen such that $\alpha \sim h^{\rho}$, $0 < \rho < 1$, then

$$\| x_{h\alpha} - x_0 \| \le O\left(h^{\theta/s_1}\right), \qquad \theta = \min\{\rho, 1 - \rho\}.$$

Proof. We set

$$A = m_1 \| x_{h\alpha} - x_0 \|^{s_1} + m_2 \left\| x_{h\alpha}^* - x_0^* \right\|^{s_2}, \qquad x_{h\alpha}^* = F_{1\alpha}^h(x_{h\alpha}).$$

It is easy to see that x_0 is a solution of (1) iff $z_0 = \left[x_0, x_0^*\right]$ is a solution of the following system of two operator equations

$$F_1(x) - x^* = 0,$$

$$F_2(x^*) + x - f = 0.$$

Similarily, $x_{h\alpha}$ is a regularized solution of the operator equation (4) iff $z_\alpha = \left[x_{h\alpha}, x_{h\alpha}^*\right]$ is a solution of the following system of equations

$$F_1^h(x) + \alpha U_1(x) - x^* = 0,$$

$$F_2^h(x^*) + \alpha U_2(x^*) + x - f = 0.$$

Basing on the properties (6) of U_1, U_2 and the above two systems of equations we have got

$$A \le \langle U_1(x_0), x_0 - x_{h\alpha} \rangle + \left\langle U_2\left(x_0^*\right), x_0^* - x_{h\alpha}^* \right\rangle +$$

$$+ \frac{1}{\alpha}\left[\left\langle x_{h\alpha}^* - F_1'(x_{h\alpha}), x_{h\alpha} - x_0 \right\rangle + \left\langle f - x_{h\alpha} - F_2^h\left(x_{h\alpha}^*\right), x_{h\alpha}^* - x_0 \right\rangle\right]. \tag{8}$$

We set $x^2 = U_1(x_0) - F_1'(x_0)^* x^1$. From condition (iii) of the Theorem it follows that x^1 and x^2 ($\in X^*$) satisfy the system of following equalities

$$F_1'(x_0)^* x^1 + x^2 = U_1(x_0),$$
$$F_2'(x_0^*)^* x^2 - x^1 = U_2(x_0^*).$$

Therefore, it follows from (1), (8), and the monotonicity of F_i^h that

$$A \le \langle U_1(x_0), x_0 - x_{h\alpha} \rangle + \langle U_2(x_0^*), x_0^* - x_\alpha^* \rangle +$$
$$+ \frac{1}{\alpha}\Big[\langle x_{h\alpha}^* - x_0^*, x_{h\alpha} - x_0 \rangle + \langle x_0 - x_{h\alpha}, x_{h\alpha}^* - x_0^* \rangle +$$
$$+ \Big[\langle F_1(x_0) - F_1^h(x_{h\alpha}), x_{h\alpha} - x_0 \rangle + \langle F_2(x_0^*) - F_2^h(x_{h\alpha}^*), x_{h\alpha}^* - x_0^* \rangle\Big] \le$$
$$\le \frac{Ch}{\alpha}\left(\| x_{h\alpha} - x_0 \| + \| x_{h\alpha}^* - x_0^* \|\right) +$$
$$+ \langle x^2, x_0 - x_{h\alpha} \rangle + \langle x^1, F_1'(x_0)(x_0 - x_{h\alpha}) \rangle +$$
$$+ \langle -x^1, x_0^* - x_{h\alpha}^* \rangle + \langle x^2, F_2'(x_0^*)(x_0^* - x_{h\alpha}^*) \rangle, \tag{9}$$

where C is a some positive constant such that $g_1(\| x_{h\alpha} \|)$, $g_2(\| x_{h\alpha}^* \|) \le C$. First, consider the case $s_i = [s_i]$, $i = 1, 2$. As

$$F_1'(x_0)(x_0 - x_{h\alpha}) = F_1(x_0) - F_1(x_{h\alpha}) + r_{h\alpha},$$
$$F_2'(x_0^*)(x_0^* - x_\alpha^*) = F_2(x_0^*) - F_2(x_{h\alpha}^*) + \tilde{r}_{h\alpha},$$
$$\| r_{h\alpha} \| \le \frac{\tilde{L}}{s_1!} \| x_{h\alpha} - x_0 \|^{s_1}, \qquad \| \tilde{r}_{h\alpha} \| \le \frac{\tilde{L}}{s_2!} \| x_{h\alpha}^* - x_0^* \|^{s_2},$$

form the inequality (9) it follows

$$A \le \frac{Ch}{\alpha}\left(\| x_{h\alpha} - x_0 \| + \| x_{h\alpha}^* - x_0^* \|\right) +$$
$$+ \langle x^2, x_0 - x_{h\alpha} \rangle + \langle -x^1, x_0^* - x_{h\alpha}^* \rangle + \langle x^1, F_1(x_0) - F_1(x_{h\alpha}) \rangle +$$
$$+ \langle x^2, F_2(x_0^*) - F_2(x_{h\alpha}^*) \rangle + \frac{\tilde{L}\| x^1 \|}{s_1!} \| x_{h\alpha} - x_0 \|^{s_1} +$$
$$+ \frac{\tilde{L}\| x^2 \|}{s_1!} \| x_{h\alpha}^* - x_0^* \|^{s_2} \le \frac{Ch}{\alpha}\left(\| x_{h\alpha} - x_0 \| + \| x_{h\alpha}^* - x_0^* \|\right) +$$
$$+ \langle x^1, x_{h\alpha}^* - F_1(x_{h\alpha}) \rangle + \langle x^2, f - x_{h\alpha} - F_2(x_{h\alpha}^*) \rangle +$$
$$+ \frac{\tilde{L}\| x^1 \|}{s_1!} \| x_{h\alpha} - x_0 \|^{s_1} + \frac{\tilde{L}\| x^2 \|}{s_2!} \| x_{h\alpha}^* - x_0^* \|^{s_2} \le$$
$$\le \frac{Ch}{\alpha}\left(\| x_{h\alpha} - x_0 \| + \| x_{h\alpha}^* - x_0^* \|\right) +$$

$$+ \alpha\left\langle x^1, U_1(x_{h\alpha})\right\rangle + \alpha^2\left\langle x^2, U_2\left(x^*_{h\alpha}\right)\right\rangle + \frac{\tilde{L}\left\|x^1\right\|}{s_1!}\left\| x_{h\alpha} - x_0 \right\|^{s_1} +$$

$$+ \frac{\tilde{L}\left\|x^2\right\|}{s_2!}\left\| x^*_{h\alpha} - x^*_0 \right\|^{s_2} \le \frac{Ch}{\alpha}\left(\| x_{h\alpha} - x_0 \| + \left\| x^*_{h\alpha} - x^*_0 \right\|\right) +$$

$$+ \alpha\left(\left\|x^1\right\|\| x_{h\alpha} \| + \left\|x^2\right\|\| x^*_{h\alpha} \|\right) + \frac{\tilde{L}\left\|x^1\right\|}{s_1!}\| x_{h\alpha} - x_0 \|^{s_1} +$$

$$+ \frac{\tilde{L}\left\|x^2\right\|}{s_2!}\left\| x^*_{h\alpha} - x^*_0 \right\|^{s_2}.$$

Hence,

$$m_1\left(1 - \frac{\tilde{L}\left\|x^1\right\|}{m_1 s_1!}\right)\| x_{h\alpha} - x_0 \|^{s_1} \le O\left(h^{\rho} + h^{1-\rho}\right). \tag{10}$$

Consequently,

$$\| x_{h\alpha} - x_0 \| \le O\left(h^{\theta/s_1}\right).$$

If $s_i \neq [s_i]$ for one or both the two numbers s_i, for example $s_1 \neq [s_1]$, then

$$\| r_{h\alpha} \| \le \frac{\tilde{L}}{([s_1]+1)!}\| x_{h\alpha} - x_0 \|^{[s_1]+1}$$

and the left-hand side of (10) will be replaced by

$$m_1\left(1 - \frac{\tilde{L}\left\|x^1\right\|}{m_1([s_1]+1)!}\right)\| x_{h\alpha} - x_0 \|^{[s_1]+1-s_1}\| x_{h\alpha} - x_0 \|^{s_1}.$$

Since $\| x_{h\alpha} - x_0 \| \to 0$, and $[s_1] + 1 - s_1 > 0$, we have

$$1 - \frac{\tilde{L}\left\|x^1\right\|}{m_1([s_1]+1)!}\| x_{h\alpha} - x_0 \|^{[s_1]+1-s_1} \ge 1/2$$

for sufficiently small α. The case where $s_2 \neq [s_2]$ and both numbers s_1, s_2 are not integer is considered analogously. This remark completes the proof of the theorem.

Now, we establish convergence rates for the sequence $\left\{ x^h_{\alpha n} \right\}$.

Theorem 2. *Assume that the conditions of Theorem* 1 *hold, and* α *is chosen such that* $\alpha \sim (h+\gamma_n)^{\rho}$, $1 < \rho < 1$, *where*

$$\gamma_n = \max\left\{\|(I-P_n)x_0\|, \|(I-P_n)f\|, \left\|(I-P_n)x^1\right\|, \left\|\left(I^*-P^*_n\right)x^*_0\right\|, \left\|\left(I^*-P^*_n\right)x^2\right\|\right\},$$

and I^* *denotes the identity operator in* X^*. *Then*

$$\left\| x^h_{\alpha n} - x_0 \right\| = O\left(h^{\eta} + \gamma_n^{\mu}\right),$$

$$\eta = \min\left\{\frac{1-\rho}{s_1}, \frac{\rho}{s_1}\right\},$$

$$\mu = \min\left\{\eta, \frac{\nu_1}{s_1 - 1}, \frac{\nu_2}{s_1}\right\}.$$

Proof. We set

$$B = m_1 \left\| x_{\alpha n}^h - x_{0n} \right\|^{s_1} + m_2 \left\| x_{\alpha n}^{h*} - x_{0n}^* \right\|^{s_2},$$

with $x_{0n} = P_n x_0$, $x_{\alpha n}^{h*} = F_{1\alpha}^n\left(x_{\alpha n}^h\right)$ and $x_{0n}^* = P_n^* x_0^*$. It is easy to see that $x_{\alpha n}^h$ is a solution of (5) iff $x_{\alpha n}^h$ and $x_{\alpha n}^{h*}$ are the solution of the following system of equations

$$F_{1n}^h(x) + \alpha U_1^n(x) - x^* = 0, \tag{11}$$

$$F_{2n}^h(x^*) + \alpha U_2^n(x^*) + x - f_n = 0 \tag{12}$$

with $U_1^n = P_n^* U_1 P_n$, $U_2^n = P_n U_2 P_n^*$, $F_{1n}^h = P_n^* F_1^h P_n$ and $F_{2n}^h = P_n F_2^h P_n^*$. From the monotone property of F_{in}^h, $i = 1, 2$, properties (6) of the dual mappings U_i, $i = 1, 2$, and (11), (12) it implies that

$$B \le \left\langle U_1^n(x_{0n}), x_{0n} - x_{\alpha n}^h \right\rangle + \left\langle U_2^n\left(x_{0n}^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle +$$

$$+ \frac{1}{\alpha}\left[\left\langle x_{\alpha n}^{h*} - F_{1n}^h\left(x_{\alpha n}^h\right), x_{\alpha n}^h - x_{0n} \right\rangle + \left\langle f_n - x_{\alpha n}^h - F_{2n}^h\left(x_{\alpha n}^{h*}\right), x_{\alpha n}^{h*} - x_{0n}^* \right\rangle\right] \le$$

$$\le \left\langle U_1(x_{0n}), x_{0n} - x_{\alpha n}^h \right\rangle + \left\langle U_2\left(x_{0n}^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle +$$

$$+ \frac{1}{\alpha}\left[\tilde{C}h\left(\left\| x_{\alpha n}^h - x_{0n} \right\| + \left\| x_{\alpha n}^{h*} - x_{0n}^* \right\|\right) + \left\langle x_{\alpha n}^{h*} - F_{1n}(x_{0n}), x_{\alpha n}^h - x_{0n} \right\rangle +\right.$$

$$\left. + \left\langle f_n - x_{\alpha n}^h - F_{2n}\left(x_{0n}^*\right), x_{\alpha n}^{h*} - x_{0n}^* \right\rangle \le\right.$$

$$\le \left\langle U_1(x_{0n}), x_{0n} - x_{\alpha n}^h \right\rangle + \left\langle U_2\left(x_{0n}^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle +$$

$$+ \frac{1}{\alpha}\left[\tilde{C}h\left(\left\| x_{\alpha n}^h - x_{0n} \right\| + \left\| x_{\alpha n}^{h*} - x_{0n}^* \right\|\right) + \left\langle F_1(x_0) - F_1(x_{0n}), x_{\alpha n}^h - x_{0n} \right\rangle +\right.$$

$$\left. + \left\langle F_2\left(x_0^*\right) - F_2\left(x_{0n}^*\right), x_{\alpha n}^{h*} - x_{0n}^* \right\rangle\right], \tag{13}$$

where $\tilde{C}$ is a some positive constant such that $g_1\left(\left\| x_{\alpha n}^h \right\|\right)$, $g_2\left(\left\| x_{\alpha n}^{h*} \right\|\right) \le \tilde{C}$. First, consider the case $s_i = [s_i]$, $i = 1, 2$. By virtue of the properties of F_i we can write

$$F_1(x_{0n}) - F_1(x_0) = F_1'(x_0)(x_{0n} - x_0) + r_n,$$

$$F_2\left(x_{0n}^*\right) - F_2\left(x_0^*\right) = F_2'\left(x_0^*\right)\left(x_{0n}^* - x_0^*\right) + \tilde{r}_n,$$

$$\| r_n \| \le \frac{\tilde{L}}{s_1!}\| (I - P_n)x_0 \|^{s_1}, \qquad \| \tilde{r}_n \| \le \frac{\tilde{L}}{s_2!}\| (I^* - P_n^*)x_0^* \|^{s_2}.$$

On the other hand, from (7) it implies that

$$\left\langle U_1(x_{0n}), x_{0n} - x_{\alpha n}^h \right\rangle \le c_1(R_1)\gamma_n^{\nu_1}\left\| x_{0n} - x_{\alpha n}^h \right\| + \left\langle U_1(x_0), x_{0n} - x_{\alpha n}^h \right\rangle,$$

where the second term of the right-hand side is estimated as follows

$$\left\langle U_1(x_{0n}), x_{0n} - x_{\alpha n}^h \right\rangle = \langle U_1(x_0), x_{0n} - x_0 \rangle + \left\langle U_1(x_0), x_0 - x_{\alpha n}^h \right\rangle \le$$

$$\le O(\gamma_n) + \left\langle x^1, F_1(x_0) - F_1\left(x_{\alpha n}^h\right)\right\rangle + \left\langle x^2, x_0 - x_{\alpha n}^h \right\rangle + \frac{\tilde{L}\left\| x^1 \right\|}{s_1!}\left\| x_0 - x_{\alpha n}^h \right\|^{s_1}.$$

In the similar way, we also have

$$\left\langle U_2\left(x_{0n}^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle \le c_2(R_2)\gamma_n^{\nu_2}\left\| x_{0n}^* - x_{\alpha n}^{h*} \right\| + \left\langle U_2\left(x_0^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle$$

with the estimation

$$\left\langle U_2\left(x_0^*\right), x_{0n}^* - x_{\alpha n}^{h*} \right\rangle \le O(\gamma_n) + \left\langle x^2, F_2\left(x_0^*\right) - F_2\left(x_{\alpha n}^{h*}\right) \right\rangle + \left\langle x^1, x_0^* - x_{\alpha n}^{h*} \right\rangle +$$
$$+ \frac{\tilde{L}\left\| x^2 \right\|}{s_2!}\left\| x_0^* - x_{\alpha n}^{h*} \right\|^{s_2}.$$

Since

$$\left\| x_0 - x_{\alpha n}^h \right\|^{s_1} \le O(\gamma_n) + \left\| x_{0n} - x_{\alpha n}^h \right\|^{s_1},$$

$$\left\| x_0^* - x_{\alpha n}^{h*} \right\|^{s_2} \le O(\gamma_n) + \left\| x_{0n}^* - x_{\alpha n}^{h*} \right\|^{s_2},$$

$$\left\langle x^1, F_1(x_0) - F_1\left(x_{\alpha n}^h\right) \right\rangle = \left\langle x^1, x_0^* - x_{\alpha n}^{h*} \right\rangle + \left\langle x^1, x_{\alpha n}^{h*} - F_{1n}^h\left(x_{\alpha n}^h\right) \right\rangle -$$
$$- \left\langle (I - P_n)x^1, F_1\left(x_{\alpha n}^h\right) \right\rangle \le$$
$$\le O(\gamma_n) + \left\langle x^1, x_0^* - x_{\alpha n}^{h*} \right\rangle + \alpha\left\| x^1 \right\|\left\| x_{\alpha n}^h \right\|,$$

$$\left\langle x^2, F_2\left(x_0^*\right) - F_2\left(x_{\alpha n}^{h*}\right) \right\rangle = \left\langle x^2, -x_0 - x_{\alpha n}^h \right\rangle + \left\langle x^2, f - f_n \right\rangle +$$
$$+ \left\langle x^2, f_n - x_{\alpha n}^h - F_{2n}^h\left(x_{\alpha n}^{h*}\right) \right\rangle - \left\langle \left(I_n^* - P_n^*\right)x^2, F_2\left(x_{\alpha n}^{h*}\right) \right\rangle \le$$
$$\le O(\gamma_n) - \left\langle x^2, x_0 - x_{\alpha n}^h \right\rangle + \alpha\left\| x^2 \right\|\left\| x_{\alpha n}^{h*} \right\|,$$

(13) can be written in the form

$$m_1\left(1 - \frac{\tilde{L}\left\| x^1 \right\|}{m_1 s_1!}\right)\left\| x_{\alpha n}^h - x_{0n} \right\|^{s_1} \le m_1\left(1 - \frac{\tilde{L}\left\| x^1 \right\|}{m_1 s_1!}\right)\left\| x_{\alpha n}^h - x_{0n} \right\|^{s_1} +$$
$$+ m_2\left(1 - \frac{\tilde{L}\left\| x^2 \right\|}{m_2 s_2!}\right)\left\| x_{\alpha n}^{h*} - x_{0n}^* \right\|^{s_2} \le O\left((h+\gamma_n)^{1-\rho} + \gamma_n^{\nu_1}\right)\left\| x_{\alpha n}^h - x_{0n} \right\| +$$
$$+ O\left((h+\gamma_n)^{1-\rho} + \gamma_n^{\nu_2}\right)\left\| x_{\alpha n}^{h*} - x_{0n}^* \right\| + O\left((h+\gamma_n)^{\rho}\right).$$

Because the sequence $\left\{ x_{\alpha n}^{h*} \right\}$ is bounded (see [4]) from here it follows

$$\left\| x_{\alpha n}^h - x_{0n} \right\|^{s_1} \le O\left((h+\gamma_n)^{1-\rho} + \gamma_n^{\nu_1}\right)\left\| x_{\alpha n}^h - x_{0n} \right\| +$$
$$+ O\left((h+\gamma_n)^{1-\rho} + (h+\gamma_n)^{\rho} + \gamma_n^{\nu_2}\right).$$

Applying the relation

$$a, b, c > 0, \quad p > q > 0, \quad a^p \le ba^q + c \Rightarrow a^p = O\left(b^p/(p-q) + c\right)$$

in [13] to the last inequality we obtain

$$\left\| x_{\alpha n}^h - x_{0n} \right\| = O\left(h^{\eta} + \gamma_n^{\mu}\right).$$

Therefore,

$$\left\| x_{\alpha n}^{h} - x_0 \right\| = O\left(h^{\eta} + \gamma_n^{\mu}\right).$$

The case $s_i \neq [s_i]$ is considered in the similar way as in the proof of Theorem 1.

Remarks. **1.** If S_0 contains more than one element, then F_1 and F_2 are affine on the sets S_0 and $F_1(S_0)$, respectively (see [2]). Therefore, the condition $F_1^{(2)}(x) = \ldots = F_1^{(k)}(x) = 0$, $x \in S_0$, and $F_2^{(2)}(x^*) = \ldots = F_2^{(k)}(x^*)$, $x \in F_1(S_0)$ is automatically satisfied. Moreover, $F_1'(x)$ and $F_2'(x^*)$ do not depend on x and x^*, respectively. Hence, condition (iii) of Theorem 1, in fact, is an existence condition of solution of a linear operator equation.

2. When X is the spaces of type $L_p(\Omega)$ or $W_p(\Omega)$, $1 < p < +\infty$: if $p = 2$ X is a Hilbert space and $U_i = I$, $s_i = 2$, $m_i = 1$, $\nu_i = 1$ and $c(R_i) \equiv 1$, and if $1 < p < 2$ we have $s_1 = 2$, $m_1 = p - 1$, $c(R_1) = p2^{2p-1}e^p L^{p-1}$, $e = \max\left\{2^p, 2R_1\right\}$, $1 < L < 3{,}18$, $\nu_1 = p - 1$, and $s_2 = q$, $m_2 = 2^{2-q}/q$, $c(R_2) = 2^q R_2^{q-2}\{q[q-1+\max\{R_2, L\}]\}^{-1}$, $\nu_2 = 1$, $p^{-1} + q^{-1} = 1$. The case $p > 2$ is considered analogously [14, 15].

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