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REGULARITY RESULTS FOR KOLMOGOROV EQUATIONS IN $L^2(H, \mu)$ SPACES AND APPLICATIONS

РЕЗУЛЬТАТИ ПРО РЕГУЛЯРНІСТЬ ДЛЯ РІВНЯННЯ КОЛМОГОРОВА В ПРОСТОРАХ $L^2(H, \mu)$ ТА ЇХ ЗАСТОСУВАННЯ

We consider the transition semigroup $R_t = e^{t\mathscr{A}}$ associated to an Ornstein–Uhlenbeck process in a Hilbert space H. We characterize, under suitable assumptions, the domain of $\mathscr{A}$ as a subspace of $W^{2,2}(H, \mu)$, where μ is the invariant measure associated to R_t. This characterization is then used to treat some Kolmogorov equations with variable coefficients.

Розглядається перехідна півгрупа $R_t = e^{t\mathscr{A}}$, що пов'язана з процесом Орнштейна–Уленбека в гільбертовому просторі H. При належних умовах наводиться характеризація області визначення $\mathscr{A}$ як підпростору $W^{2,2}(H, \mu)$, де μ — інваріантна міра, що асоціюється з R_t. Ця характеризація використовується для розгляду деяких рівнянь Колмогорова зі змінними коефіцієнтами.

1. Introduction. Let us consider the Ornstein–Uhlenbeck process X in a separable Hilbert space H defined as

$$dX = AX\,dt + K^{1/2}\,dW(t), \qquad X(0) = x \in H, \tag{1}$$

where $A: D(A) \subset H \to H$ is the infinitesimal generator of a C_0 semigroup in H and K is a strictly positive linear operator in H (for instance, $K = I$). Moreover $W(t)$, $t \geq 0$, is a cylindrical H-valued Wiener process defined in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Under suitable assumptions (see Sec. 2 below) equation (1) has a unique mild solution $X(t, x)$. Let R_t, $t \geq 0$, be the corresponding transition semigroup:

$$R_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))], \quad \varphi \in C_b(H)^*.$$

Then $u(t, x) = R_t\varphi(x)$ is, in a sense to be precise, a solution to the Kolmogorov equation

$$\begin{cases} u_t = \dfrac{1}{2}\mathrm{Tr}\left[KD^2u\right] + \langle Ax, Du\rangle, \\ u(0, x) = \varphi(x). \end{cases} \tag{2}$$

The Kolmogorov equations in infinite dimensions were extensively studied by many people, starting from the pioneering papers by Yu. Daletskii (see the monograph [1]).

Another approach, motivated by several problems of mathematical physics, is based on the Dirichlet form

$$a(\varphi, \psi) = \int_H D\varphi(x)D\psi(x)\mu(dx), \quad \varphi, \psi \in W^{1,2}(H; \mu),$$

where μ is the invariant measure associated to problem (1), and $W^{1,2}(H;\mu)$ is the

* $C_b(H)$ is the Banach space of all uniformly continuous and bounded mappings from H into $\mathbb{R}$ endowed with the norm $\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|$.

corresponding Sobolev space (see, e.g., the monograph by Z. M. Ma and M. Röckner [2]). Using this method, one can construct the transition semigroup R_t, $t \geq 0$, in the space $L^2(H; \mu)$. If $\mathcal{A}$ denotes its infinitesimal generator, one has that the domain of $\mathcal{A}$ is a subspace of $W^{1,2}(H; \mu)$.

In this paper, we present some new results about the characterization, under suitable assumptions, of the domain of $\mathcal{A}$ as a linear operator on $L^2(H; \mu)$. We will prove in particular that $D(\mathcal{A})$ is a suitable subspace of $W^{2,2}(H; \mu)$ (see Sec. 3 below). Note that, in the particular case where $K = I$, a characterization of $D(\mathcal{A})$ was given in [3].

In Sec. 4, this result is applied to studying the Kolmogorov equation with continuous coefficients

$$\begin{cases} v_t = \frac{1}{2}\operatorname{Tr}\left[K(x)D^2 v\right] + \langle Ax + F(x), Dv\rangle, \\ v(0, x) = \varphi(x), \end{cases} \tag{3}$$

where $K(x)$ are linear positive operators depending continuously on x, and F is a nonlinear Borel mapping from H into H. Using the characterization of $D(\mathcal{A})$, we are able to solve problem (3), under suitable assumptions, by a perturbation argument. Moreover, arguing as in [4], we show that there is an invariant measure ν for this problem that is absolutely continuous with respect to μ.

2. Notation and setting of the problem. We are given a separable Hilbert space H (norm $|\cdot|$, inner product $\langle \cdot, \cdot \rangle$), and two linear operators $A: D(A) \subset H \to H$ and $K: H \to H$ satisfying the following conditions:

***Hypothesis* 1.** (i) *A is the infinitesimal generator of a strongly continuous semigroup e^{tA} in H. Moreover, there exist $M > 0$ and $\omega > 0$ such that*

$$\left\| e^{tA} \right\| \leq M e^{-\omega t}, \quad t \geq 0.$$

(ii) *For any $t > 0$, $e^{tA} \in \mathcal{L}_2(H)^*$ and*

$$\int_0^t \operatorname{Tr}\left[e^{sA} e^{sA^*}\right] ds < +\infty.$$

(iii) *$K \in \mathcal{L}(H)$ is self-adjoint and bounded. Moreover, there exists $\nu > 0$ such that*

$$\langle Kx, x \rangle \geq \nu |x|^2, \quad x \in H.$$

Under Hypothesis 1, the linear operator Q defined by the relation

$$Qx = \int_0^\infty e^{sA} K e^{sA^*} x \, ds, \quad x \in H,$$

is a well-defined trace-class** operator.

* $\mathcal{L}(H)$ is the Banach algebra of all linear bounded operators on H endowed with the sup norm $\|\cdot\|$. By $\mathcal{L}_1(H)$ (norm $\|\cdot\|_{\mathcal{L}_1(H)}$) we denote the Banach space of all trace-class operators on H, and by $\mathcal{L}_2(H)$ (norm $\|\cdot\|_{\mathcal{L}_2(H)}$) the Hilbert space of all Hilbert–Schmidt operators in H. If $T \in \mathcal{L}_1(H)$, the trace of T is denoted by $\operatorname{Tr} T$.

** We denote by A^* the adjoint of A.

Let us consider the Ornstein–Uhlenbeck semigroup R_t, $t \geq 0$, in $C_b(H)$ defined as

$$R_t\varphi(x) = \int_H \varphi(y)\mathcal{N}(e^{tA}x, Q_t)(dy), \quad t \geq 0, \quad x \in H, \quad \varphi \in C_b(H), \tag{4}$$

where $\mathcal{N}(e^{tA}x, Q_t)(dy)$ is the Gaussian measure with mean $e^{tA}x$ and covariance operator Q_t:

$$Q_t x = \int_0^t e^{sA} K e^{sA^*} x\, ds, \qquad x \in H.$$

Under Hypothesis 1, one can show (see, e.g., [5]) that $\mu = \mathcal{N}(0, Q)$ is the unique invariant measure for the semigroup R_t, $t \geq 0$. Consequently, for any $t > 0$, the operator R_t has a unique extension to a linear bounded operator in $L^2(H; \mu)$, still denoted by R_t. Moreover, R_t, $t \geq 0$, is a contraction semigroup on $L^2(H; \mu)$.

We shall denote by $\{e_k\}$ a complete orthonormal system of eigenvectors of Q and by $\{\lambda_k\}$ a corresponding sequence of eigenvalues:

$$Q e_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, we denote by $D_k\varphi$ the derivative of φ in the direction of e_k, and we set $x_k = \langle x, e_k \rangle$, $x \in H$. It is well known that D_k is closable. We shall still denote by D_k its closure.

We recall now the definition of Sobolev spaces. We denote by $W^{1,2}(H;\mu)$ the linear space of all functions $\varphi \in L^2(H;\mu)$ such that $D_k\varphi \in L^2(H;\mu)$ for all $k \in \mathbb{N}$ and

$$\int_H |D\varphi(x)|^2 \mu(dx) = \sum_{k=1}^{\infty} \int_H |D\varphi(x)|^2 \mu(dx) < +\infty.$$

The space $W^{1,2}(H;\mu)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_1 = \int_H \varphi(x)\psi(x)\mu(dx) + \int_H \langle D\varphi(x), D\psi(x) \rangle \mu(dx)$$

is a Hilbert space.

In a similar way, we can define the Sobolev space $W^{2,2}(H;\mu)$ consisting of all functions $\varphi \in W^{1,2}(H;\mu)$ such that $D_h D_k \varphi \in L^2(H;\mu)$ for all $h, k \in \mathbb{N}$ and

$$\int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \mu(dx) = \sum_{h,k=1}^{\infty} \int_H |D_h D_k \varphi(x)|^2 \mu(dx) < +\infty.$$

The space $W^{2,2}(H;\mu)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_2 = \langle \varphi, \psi \rangle_1 + \sum_{h,k=1}^{\infty} \int_H D_h D_k \varphi(x) D_h D_k \psi(x) \mu(dx) =$$

$$= \langle \varphi, \psi \rangle_1 + \int_H \left\langle D^2\varphi(x), D^2\psi(x) \right\rangle_{\mathcal{L}_2(H)} \mu(dx)$$

is a Hilbert space.

We shall also need some weighted Sobolev spaces. Let $B: D(B) \subset H \to H$ be a self-adjoint operator such that

$$\langle Bx, x \rangle \geq \beta |x|^2$$

for some $\beta > 0$. Then we consider the linear operator D_B in $L^2(H; \mu)$,

$$D_B \varphi(x) = \sqrt{B}\, D\varphi(x), \quad x \in H,$$

defined on all $\varphi \in W^{1,2}(H; \mu)$ such that $D\varphi(x) \in D(\sqrt{B})$ μ-a.e. and $\sqrt{B}\, D\varphi \in L^2(H; \mu)$. It is easy to see that D_B is closable; we still denote by D_B its closure. We define $W_B^{1,2}(H; \mu)$ as the domain of the closure of $D(B)$. The space $W_B^{1,2}(H; \mu)$ endowed with the norm

$$\| \varphi \|^2_{W_B^{1,2}(H;\mu)} = \int_H \left(|\varphi(x)|^2 + \left| \sqrt{B}\, D\varphi(x) \right|^2 \right) \mu(dx)$$

is a Banach space.

We can now return to the semigroup R_t, $t \geq 0$. We denote by $\mathcal{A}$ its infinitesimal generator. As is well known, $\mathcal{A}$ is m-dissipative on $L^2(H; \mu)$. Moreover, one can show [5] that a core for $\mathcal{A}$ is given by the space $\mathcal{E}$ of all finite linear combinations of functions φ of the form

$$\varphi(x) = e^{i\langle h, x \rangle}, \quad x \in H, \quad h \in D(A^*).$$

For any $\varphi \in \mathcal{E}$, we have, as can easily be checked,

$$\mathcal{A}\varphi = \frac{1}{2} \operatorname{Tr}\left[K D^2 \varphi(x) \right] + \langle Ax, D\varphi(x) \rangle, \quad \varphi \in \mathcal{E}.$$

We end this section by recalling some formulas of integration by parts, which will be used later.

Propositions 1 and 2 below are well known (see, e.g., [6] and [7]).

***Proposition* 1.** *Let $\psi_1, \psi_2 \in W^{1,2}(H; \mu)$ and $\alpha \in H$. Then we have*

$$\int_H \langle D\psi_1(x), Q\alpha \rangle \psi_2(x) \mu(dx) + \int_H \langle D\psi_2(x), Q\alpha \rangle \psi_1(x) \mu(dx) = \\ = \int_H \psi_1(x) \psi_2(x) \langle \alpha, x \rangle \mu(dx). \tag{5}$$

***Proposition* 2.** *Let $\varphi \in W^{1,2}(H; \mu)$ and $\alpha \in H$. Then the function*

$$H \to \mathbb{R}, \quad x \mapsto \langle x, \alpha \rangle \varphi(x),$$

belongs to $L^2(H; \mu)$ and we have

$$\int_H |\langle \alpha, x \rangle|^2 \varphi^2(x) \mu(dx) \leq 2 \left| Q^{1/2} \alpha \right|^2 \int_H \varphi^2(x) \mu(dx) + 16 |Q\alpha|^2 \int_H |D\varphi(x)|^2 \mu(dx). \tag{6}$$

By Proposition 2, we easily get the following result:

***Corollary* 1.** *Let $\varphi \in W^{1,2}(H; \mu)$. Then the function*

$$H \to \mathbb{R}, \quad x \mapsto |x| \varphi(x),$$

belongs to $L^2(H; \mu)$. Moreover,

$$\int_H |x|^2 \varphi^2(x)\mu(dx) \leq 2\operatorname{Tr} Q \int_H \varphi^2(x)\mu(dx) + 16\operatorname{Tr}[Q]^2 \int_H |D\varphi(x)|^2 \mu(dx). \quad (7)$$

***Corollary* 2.** *Let $\varphi \in W^{2,2}(H;\mu)$ and $L \in \mathcal{L}(H)$. Then we have*

$$\int_H |\langle Lx, D\varphi(x)\rangle|^2 \mu(dx) \leq 2\|L\|^2 \operatorname{Tr} Q \int_H |D\varphi(x)|^2 \mu(dx) +$$
$$+ 16\|L\|^2 \operatorname{Tr}[Q^2] \int_H \|D^2\varphi(x)\|^2_{\mathcal{L}_2(H)} \mu(dx). \quad (8)$$

Proof. Taking (7) into account, we have

$$\int_H |\langle Lx, D\varphi(x)\rangle|^2 \mu(dx) \leq \|L\|^2 \sum_{i=1}^{\infty} \int_H |x|^2 |D_i\varphi(x)|^2 \mu(dx) \leq$$
$$\leq \|L\|^2 \sum_{i=1}^{\infty} \left[2\operatorname{Tr} Q \int_H |D_i\varphi(x)|^2 \mu(dx) + 16\operatorname{Tr}[Q]^2 \int_H |DD_i\varphi(x)|^2 \mu(dx) \right],$$

which yields the conclusion.

We end this section recalling the following property of $\mathcal{A}$ proved in [6] (see also [7]):

***Proposition* 3.** *Assume that Hypotheses 1 holds. Then for any $\varphi \in D(\mathcal{A})$, one has*

$$\int_H (\mathcal{A}\varphi)(x)\varphi(x)\mu(dx) = -\frac{1}{2}\int_H \langle KD\varphi(x), D\varphi(x)\rangle \mu(dx). \quad (9)$$

***Remark* 1.** It follows from Proposition 3 and Hypothesis 1 (iii) that $D(\mathcal{A}) \subset \subset W^{1,2}(H;\mu)$.

3. Characterization of $D(\mathcal{A})$. This section is devoted to the characterization of $D(\mathcal{A})$. We shall assume, besides Hypotheses 1, the following:

***Hypothesis* 2.** (i) *$D(A) \cap Q(H)$ is dense in H. Moreover, the linear operator*

$$Lx := Ax + \frac{1}{2}KQ^{-1}x, \quad x \in D(A) \cap Q(H), \quad (10)$$

has a bounded extension (still denoted by L) to H.

(ii) *Either*

(a) *KA^* is self-adjoint and negative,*

or

(b) *$A + A^*$ is self-adjoint and $K = 1 + S$ with $S \geq 0$ and SA^* bounded.*

We start with some identities which will play a key role in what follows.

***Proposition* 4.** *Assume that Hypothesis 1 holds, and let $\varphi \in \mathcal{E}$ and $f = \mathcal{A}\varphi$. Then we have*

$$\frac{1}{2}\int_H \operatorname{Tr}\left[(KD^2\varphi(x))^2\right]\mu(dx) - \int_H \left\langle KA^* D\varphi(x), D\varphi(x)\right\rangle \mu(dx) =$$
$$= -\int_H \langle Df(x), KD\varphi(x)\rangle \mu(dx). \quad (11)$$

Proof. Let $\varphi \in \mathscr{E}$ and $f = \mathscr{A}\varphi$. For any $\alpha \in H$, we set $\psi_\alpha(x) = \langle D\varphi(x), \alpha \rangle$. Then we have

$$\mathscr{A}\psi_\alpha + \langle A\alpha, D\varphi \rangle = \langle Df, \alpha \rangle.$$

Multiplying both sides of this identity by ψ_α and integrating in H with respect to μ, we obtain

$$\frac{1}{2}\int_H \langle KD\psi_\alpha(x), D\psi_\alpha(x) \rangle \mu(dx) - \int_H \left\langle \alpha, KA^* D\varphi(x) \right\rangle \langle D\varphi(x), \alpha \rangle \mu(dx) =$$

$$= -\int_H \langle Df(x), \alpha \rangle \langle D\varphi(x), \alpha \rangle \mu(dx). \tag{12}$$

We will prove now identity (11) under the additional assumption that K is a diagonal operator. Note that this assumption can easily be removed by approximating K with the finite-dimensional operators

$$K_n x = \sum_{j,h=1}^{n} \langle Ke_h, e_j \rangle \langle x, e_h \rangle e_j, \quad X \in H.$$

Thus, we assume that there exists a complete orthonormal set $\{f_h\}$ in H, and positive numbers $\{k_h\}$ such that $Kf_h = k_h f_h$, $h \in \mathbb{N}$. Then, setting in (12) $\alpha = K^{1/2} f_h$ and summing up over h, we arrive at (11).

Proposition 5. *Assume that Hypotheses* 1 *and* 2 (i) *hold. Let* $\varphi \in \mathscr{E}$ *and* $f = \mathscr{A}\varphi$. *Then we have*

$$\frac{1}{2}\int_H \mathrm{Tr}\left[\left(KD^2\varphi(x)\right)^2 \right] \mu(dx) - \int_H \left\langle KA^* D\varphi(x), D\varphi(x) \right\rangle \mu(dx) =$$

$$= 2\int_H |f(x)|^2 \mu(dx) - 2\int_H f(x) \langle Lx, D\varphi(x) \rangle \mu(dx), \tag{13}$$

where L *is defined by* (10).

Proof. Using Proposition 2 and setting $K_{h,k} = \langle Ke_k, e_h \rangle$, we get

$$\int_H \langle Df(x), KD\varphi(x) \rangle \mu(dx) = \sum_{h,k=1}^{\infty} \int_H K_{h,k} D_h f(x) D_k \varphi(x) \mu(dx) =$$

$$= -\sum_{h,k=1}^{\infty} \int_H f(x) K_{h,k} D_h D_k \varphi(x) \mu(dx) + \sum_{h,k=1}^{\infty} \int_H f(x) K_{h,k} \frac{x_h}{\lambda_h} D_k \varphi(x) \mu(dx),$$

which yields

$$\int_H \langle Df(x), KD\varphi(x) \rangle \mu(dx) = -\int_H f(x) \mathrm{Tr}[KD^2\varphi(x)] \mu(dx) +$$

$$+ \int_H f(x) \left\langle KQ^{-1}x, D\varphi(x) \right\rangle \mu(dx).$$

Now the conclusion follows in view of Proposition 2.

We can now prove the main result of this section.

Theorem 1. *Assume that Hypotheses* 1 *and* 2 *hold and let* $\mathscr{A}$ *be the infinitesimal generator of the semigroup* R_t, $t \geq 0$, *defined in* (4). *Then we have*

$$D(\mathcal{A}) = W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu). \tag{14}$$

Proof. We present the proof assuming that condition (b) of Hypothesis 2 holds. When condition (a) holds, the proof is completely similar. We first prove that $D(\mathcal{A}) \subset \subset W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu)$. Let $\varphi \in D(\mathcal{A})$. Since $\mathcal{E}$ is a core for $\mathcal{A}$, there exists a sequence $\{\varphi_n\} \subset D(\mathcal{A})$ such that

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{and} \quad \lim_{n\to\infty} \mathcal{A}\varphi_n = \mathcal{A}\varphi \quad \text{in} \quad L^2(H;\mu).$$

We set $f_n = \mathcal{A}\varphi_n$. Since, obviously,

$$\mathrm{Tr}\Big[\big(KD^2\varphi_n(x)\big)^2\Big] \ge \nu^2\,\mathrm{Tr}\Big[\big(D^2\varphi_n(x)\big)^2\Big],$$

taking (13) and Corollary 2 into account, we find that, for any $\varepsilon > 0$, we have

$$\frac{\nu^2}{2}\int_H \|D^2\varphi_n(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) - \int_H \Big\langle KA^*D\varphi_n(x), D\varphi_n(x)\Big\rangle\mu(dx) \le$$

$$\le 2\Big(1+\frac{1}{\varepsilon}\Big)\int_H |f_n(x)|^2\mu(dx) + 2\varepsilon\int_H |\langle Lx, D\varphi_n(x)\rangle|^2\mu(dx) +$$

$$+ \|SA^*\|\int_H |D\varphi_n(x)|^2\mu(dx) \le 2\Big(1+\frac{1}{\varepsilon}\Big)\int_H |f_n(x)|^2\mu(dx) +$$

$$+ 4\varepsilon\|L\|^2\,\mathrm{Tr}\,Q\int_H |D\varphi_n(x)|^2\mu(dx) +$$

$$+32\varepsilon\|L\|^2\,\mathrm{Tr}\big[Q^2\big]\int_H \|D^2\varphi_n(x)\|^2_{\mathcal{L}_2(H)}\mu(dx) + \|SA^*\|\int_H |D\varphi_n(x)|^2\mu(dx).$$

Choosing ε sufficiently small, we see that there exists $N > 0$ such that

$$\|\varphi_n\|^2_{W^{2,2}(H;\mu)} + \|\varphi_n\|^2_{W^{1,2}_{A+A^*}(H;\mu)} \le N\int_H |f_n(x)|^2\mu(dx).$$

By a classical argument, this implies $\varphi \in W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu)$.

We finally prove that, conversely, $W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu) \subset D(\mathcal{A})$. For any $\varphi \in W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu)$, we set

$$\gamma(\varphi) = \|\varphi\|^2_{W^{2,2}(H;\mu)} - \int_H \Big\langle A^*D\varphi(x), D\varphi(x)\Big\rangle\mu(dx).$$

Let $\varphi \in W^{2,2}(H;\mu) \cap W^{1,2}_{A+A^*}(H;\mu)$ and let $\{\varphi_n\} \subset \mathcal{E}$ such that

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{and} \quad \lim_{n\to\infty} \gamma(\varphi_n) = \gamma(\varphi) \quad \text{in} \quad L^2(H;\mu).$$

We set $f_n = \mathcal{A}\varphi_n$. It follows from (13) that

$$\int_H |\mathcal{A}\varphi_n|^2\mu(dx) \le \gamma(\varphi_n) + 2\int_H |f_n(x)||\langle Lx, D\varphi_n(x)\rangle|\mu(dx) +$$

$$+ \left\|SA^*\right\| \int_H |D\varphi_n(x)|^2 \mu(dx) \le \gamma(\varphi_n) + \frac{1}{2}\int_H |f_n(x)|^2 \mu(dx) +$$

$$+ 2\int_H |\langle Lx, D\varphi_n(x)\rangle|^2 \mu(dx) + \left\|SA^*\right\| \int_H |D\varphi_n(x)|^2 \mu(dx),$$

which implies

$$\int_H |\mathcal{A}\varphi_n|^2 \mu(dx) \le 2\gamma(\varphi_n) + 4\int_H |\langle Lx, D\varphi_n\rangle|^2 \mu(dx) +$$

$$+ \left\|SA^*\right\| \int_H |D\varphi_n(x)|^2 \mu(dx).$$

Using again Corollary 2, we get

$$\int_H |\mathcal{A}\varphi_n|^2 \mu(dx) \le 2\gamma(\varphi_n) + 8\|L\|^2 \operatorname{Tr} Q \int_H |D\varphi_n(x)|^2 \mu(dx) +$$

$$+ 64\|L\|^2 \operatorname{Tr}[Q]^2 \int_H \left\|D^2\varphi_n(x)\right\|^2_{\mathcal{L}_2(H)} \mu(dx) +$$

$$+ \left\|SA^*\right\| \int_H |D\varphi_n(x)|^2 \mu(dx) \le$$

$$\le 2\left(1 + 36\|L\|^2 \operatorname{Tr} Q + \left\|SA^*\right\|\right)\gamma(\varphi_n),$$

and the conclusion follows.

***Remark* 2** (Finite-dimensional case). Assume that H is a finite-dimensional space and let $A, K \in \mathcal{L}(H)$. Assume that all eigenvalues of A have the negative real part. Then Hypotheses 1 and 2 clearly hold. Thus, by Theorem 1, it follows that

$$D(\mathcal{A}) = W^{2,2}(H;\mu). \tag{15}$$

This result was obtained by A. Lunardi [8] by different methods based on interpolation.

***Remark* 3** (Commutative case). Assume that

(i) A is self-adjoint and there exists $\omega > 0$ such that

$$\langle Ax, x\rangle \le -\omega|x|^2, \quad x \in D(A).$$

Moreover, $A^{-1} \in \mathcal{L}_1(H)$.

(ii) K is self-adjoint, strictly positive, and such that $Ke^{tA} = e^{tA}K$ for all $t > 0$. Then KA is self-adjoint and

$$\langle KAx, x\rangle \le \omega\nu|x|^2, \quad x \in H.$$

Moreover, $L = 0$. Thus, Hypotheses 1 and 2 are fulfilled and, by Theorem 1, it follows that

$$D(\mathcal{A}) = W^{2,2}(H;\mu) \cap W^{1,2}_{KA}(H;\mu). \tag{16}$$

For $K = I$, this results was obtained in [3].

We end this section by giving an example where A and Q do not commute, but Theorem 1 still applies.

***Example* 1**. We assume here that condition (i) of Remark 3 is fulfilled and, moreover, that K is of the form

$$K = 1 + A^{-1}SA^{-1}$$

for some $S \in \mathcal{L}(H)$, symmetric nonnegative. Then Hypotheses 1 and 2 clearly hold. Let us check Hypothesis 2. First, we write Q as

$$Q := Q_0^{1/2}(1+S_1)Q_0^{1/2}, \tag{17}$$

where $Q_0 = A^{-1}/2$ and

$$S_1 x = 2\int_0^\infty (-A)^{-1/2} e^{tA} S(-A)^{-1/2} e^{tA} x\, ds, \quad x \in H.$$

It follows from (17) that

$$Q^{-1} = Q_0^{-1/2}(1+S_1)^{-1} Q_0^{-1/2} = Q_0^{-1/2}(1-T_1)^{-1} Q_0^{-1/2},$$

where $T_1 = 1-(1+S_1)^{-1}$. Then

$$A + \frac{1}{2}Q^{-1}K = -\frac{1}{2}Q_0^{-1/2} T_1 Q_0^{-1/2} K = S_2(1+Q_0 S_2),$$

where $S_2 = Q_0^{-1/2} S_1 Q_0^{-1/2}$. It is easy to see that S_2 is bounded and, hence, $A + Q^{-1}K/2$ is bounded too, as required.

4. Perturbation results. Here, we assume that Hypotheses 1 and 2 hold. For the sake of simplicity, we also assume that A is self-adjoint. We denote by μ the Gaussian measure defined in Sec. 2.

We consider, besides the operator $\mathcal{A}$ defined by

$$\mathcal{A}\varphi = \frac{1}{2}\operatorname{Tr}\left[KD^2\varphi(x)\right] + \langle Ax, D\varphi(x)\rangle, \qquad \varphi \in W^{2,2}(H;\mu) \cap W_A^{1,2}(H;\mu), \tag{18}$$

the following one:

$$\mathcal{A}_1\varphi = \frac{1}{2}\operatorname{Tr}\left[K(x)D^2\varphi(x)\right] + \langle Ax + F(x), D\varphi(x)\rangle, \qquad \varphi \in W^{2,2}(H;\mu) \cap W_A^{1,2}(H;\mu), \tag{19}$$

where K and F satisfy the following assumption:

***Hypothesis* 3.** (i) *$K: H \to \mathcal{L}(H)$ is a Borel function. Moreover, $K(0) = K$ and $K(x) - K \in \mathcal{L}_2(H)$.*

(ii) *$F: H \to H$ is a Borel function. Moreover, $(-A)^{-1/2}F$ is bounded.*

Theorem 2. *Assume, besides Hypothesis 3, that*

$$\sup_{x \in H}\left(\operatorname{Tr}(K(x)-1)^2 + \left|(-A)^{1/2}F(x)\right|^2\right) < 1. \tag{20}$$

Then $\mathcal{A}_1$ generates a strongly continuous semigroup P_t, $t \ge 0$, on $L^2(H;\mu)$. Moreover, there exists an invariant measure for P_t, $t \ge 0$, which is, in addition, absolutely continuous with respect to μ.

Proof. We set

$$\mathcal{A}_1 = \mathcal{A} + \mathcal{B},$$

where

$$\mathcal{B}\varphi = \frac{1}{2}\operatorname{Tr}\big[(K(x)-K)D^2\varphi(x)\big] + \langle F(x), D\varphi(x)\rangle,$$

$$\mathcal{B}\varphi \in W^{2,2}(H;\mu) \cap W_A^{1,2}(H;\mu).$$

We are going to show that $\mathcal{B}$ is relatively bounded with respect to $\mathcal{A}$.

We have, in fact, for any $\varphi \in \mathcal{A}$,

$$\int_H \left|\operatorname{Tr}\big[(K(x)-K)D^2\varphi(x)\big]\right|^2 \mu(dx) \le$$

$$\le \int_H \left|\operatorname{Tr}\big[(K(x)-K)\big]\right|^2 \left|\operatorname{Tr}\big[D^2\varphi(x)\big]\right|^2 \mu(dx) \le \sup_{x\in H} \operatorname{Tr}(K(x)-1)^2 \|\mathcal{A}\varphi\|^2. \quad (21)$$

Moreover,

$$\int_H |\langle F(x), D\varphi(x)\rangle|^2 \mu(dx) = \int_H \left|\left\langle (-A)^{-1/2}F(x), (-A)^{1/2}D\varphi(x)\right\rangle\right|^2 \mu(dx) \le$$

$$\le \sup_{x\in H} \left|(-A)^{1/2}F(x)\right|^2 \|\mathcal{A}\varphi\|^2. \quad (22)$$

Now it follows from (21) and (22) that $\mathcal{B}$ is relatively bounded with respect to $\mathcal{A}$, as required. Now, by a well-known perturbation result (see, e.g., A. Pazy [9]), it follows that $\mathcal{A}_1$ generates a semigroup C_0 in $L^2(H;\mu)$.

Finally, the last statement follows by analogy with [3].

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