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RESEARCH ARTICLE

On the spectrum and spectrum multiplicities of a sum of orthogonal projections

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Let H be a unitary (finite dimensional Hilbert) space.

It is known [1] that a self-adjoint operator on H, $A = A^* \geq 0$, is a sum of n orthogonal projections for some $n \in \mathbb{N}$ if and only if 1) $\operatorname{tr} A \in \mathbb{N} \cup \{0\}$, 2) $\operatorname{tr} A \geq \dim \operatorname{Im} A$ ($\operatorname{Im} A = AH$).

A known problem is the following: what are necessary and sufficient conditions on the spectrum and the spectrum multiplicities of the operator $A = A^*$ on a unitary space H, $\dim H = m < \infty$, so that A can be represented as a sum of n orthogonal projections (n is fixed), that is, what conditions should be imposed on the spectrum $\sigma(A) = \{0 \leq \lambda_1 < \ldots < \lambda_s \leq n\}$, $s \leq m$, and the collection of dimensions $\dim H_{\lambda_j} = m_j$ of the eigen spaces corresponding to the eigen values λ_s, $j = 1, \ldots, s$, $\sum_{j=1}^{s} m_j = m$, of a self-adjoint operator A so that there exist orthogonal projections $P_1, \ldots, P_n$ such that $A = \sum_{i=1}^{n} P_j$, $n \geq 3$. "The problem of the characterization of the sum of projections has been open for many years" (Pei Yuan Wu [2]).

However, using the fact that this problem is a particular case of the known problem to characterize the spectrum and the spectrum multiplicities of the sum of operators A_k, $k = 1, \ldots, n$, that have given spectra and spectrum multiplicities, and making use of the solution recently proposed in [3, 4] of Horn's problem (see the survey [5]), we can propose an algorithmic solution of the problem on the sum of orthogonal projections.

So, what can be said about precise statements of theorems? For $n = 2$, it immediately follows from the spectral theorem for a pair of self-adjoint idempotents, i.e., orthogonal projections on a Hilbert space (see for example [6] and bibl. there), that an operator $A = A^*$ is a sum of two orthogonal projections if and only if its spectrum is contained in the set $\{0,1,2\} \cup \bigcup_i \{1-\varepsilon_i, 1+\varepsilon_i\}$, $0 < \varepsilon_i < 1, i \geq 1$, and the multiplicities of the eigen values $1-\varepsilon_i, 1+\varepsilon_i$ coincide.

For $n \geq 3$, the corresponding spectral theorem for n orthogonal projections is not proved, this is a $*$-wild problem [6], so the spectral methods can be applied only with additional assumptions.

In Sections 1–3 we use methods of representation theory for obtaining a number of sufficient conditions that should be imposed on the spectrum and spectrum multiplicities of an operator $A = A^*$ so that it can be represented as a sum of n orthogonal projections ($n \geq 3$).

In Section 1, following [7, 8] we give a solution of the above-mentioned problem in the case where the spectrum of the operator A consists of a single point.

In [9], the authors give necessary and sufficient conditions for a self-adjoint operator A with a two-point spectrum to be a sum of three orthogonal projections (see Theorem 3 [9]), however, these conditions look rather cumbersome. In Section 2 of this paper, we use results from [7, 8] to solve this problem if the spectrum of the operator A consists of two points ε and $1+\varepsilon$. We formulate necessary and sufficient conditions on ε and multiplicities of the spectrum points ε and $1+\varepsilon$ so that A can be written as a sum of three orthogonal projections. These conditions are simpler for the above eigen values than those in the general case. In Section 3, we treat the case of n orthogonal projections, $n \geq 3$.

1. When the operator αI is a sum of n orthogonal projections

Let $\mathbb{H}$ be a finite or infinite dimensional Hilbert space and $\Sigma_n = \{\alpha |\ \exists P_i \in L(\mathbb{H}) : P_i^2 = P_i^* = P_i, \sum_{i=1}^n P_i = \alpha I\}$. For $n \geq 4$, introduce the following discrete sets:

$$\Lambda_n^{(1)} = \{0, 1+\frac{1}{n-1}, 1+\frac{1}{(n-2)-\frac{1}{n-1}}, \ldots, 1+\frac{1}{(n-2)-\frac{1}{(n-2)-\frac{1}{\ddots-\frac{1}{n-1}}}}, \ldots\},$$

$$\Lambda_n^{(2)} = \{1, 1+\frac{1}{n-2}, 1+\frac{1}{(n-2)-\frac{1}{n-2}}, \ldots, 1+\frac{1}{(n-2)-\frac{1}{(n-2)-\frac{1}{\ddots-\frac{1}{n-2}}}}, \ldots\}.$$

It is proved in [7] that

$$\Sigma_n = \{\Lambda_n^{(1)}, \Lambda_n^{(2)}, [\frac{n-\sqrt{n^2-4n}}{2}, \frac{n+\sqrt{n^2-4n}}{2}], n-\Lambda_n^{(1)}, n-\Lambda_n^{(2)}\}$$

for $n \geq 4$. The following result was proved in [8].

Proposition 1. *The operator αI on a Hilbert space $\mathbb{H}$, where $\alpha = \frac{p}{q}$ is an irreducible fraction, is a sum of n orthogonal projections if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$ and $\dim \mathbb{H} = qt$ $(t \geq 1)$.*

2. When an operator with the two-point spectrum $\varepsilon, 1+\varepsilon$ is a sum of three orthogonal projections

Consider, on a unitary space H, an operator $A = A^*$ that has the two-point spectrum $\sigma(A) = \{\varepsilon, 1+\varepsilon\}$, $\varepsilon > 0$, and that can be represented as a sum of n orthogonal projections $P_1 + \ldots + P_n = A$.

Lemma 2. *For an operator $A = A^*$ that has the spectrum $\{\varepsilon, 1+\varepsilon\}$ to be a sum of n projections, it is necessary that $(1+\varepsilon) \in \Sigma_{n+1}$. In particular, for the representation $A = P_1 + P_2 + P_3$ to take place, it is necessary that $(1+\varepsilon) \in \Sigma_4 = \{2, 2 \pm \frac{1}{k+1}, 2 \pm \frac{1}{k+\frac{1}{2}} : k \geq 0\}$.*

Proof. Let P be the projection onto the eigen space of the operator A corresponding to the eigen value ε. Then $P_1 + \ldots + P_n = \varepsilon P + (1+\varepsilon)(I-P)$ or $P_1 + \ldots + P_n + P = (1+\varepsilon)I$. This yields the claim. □

Theorem 3. *Let an operator $A = A^*$ have the two-point spectrum ε and $1+\varepsilon$, $\varepsilon > 0$, with multiplicities $r_1 \geq 1$ and $r_2 \geq 1$. Then it is a sum of three orthogonal projections if and only if the following holds.*

1. *for $\varepsilon < 1$,*
   - *(a) $\varepsilon = 1 - \frac{1}{k+\frac{1}{2}}$ has the multiplicity $r_1 = kt$ and $1+\varepsilon$ has the multiplicity $r_2 = (k+1)t$;*
   - *(b) $\varepsilon = 1 - \frac{1}{k+1}$ has the multiplicity $r_1 = r^{(1)}t_1 + r^{(2)}t_2$ and the multiplicity of $1+\varepsilon$ is $r_1 = r^{(2)}t_1 + r^{(1)}t_2$, where $r^{(1)} = \frac{2k+1+(-1)^{k-1}}{4}$, $r^{(2)} = \frac{2k+3+(-1)^k}{4}$;*
2. *for $\varepsilon = 1$, the multiplicities r_1, r_2 are arbitrary;*
3. *for $\varepsilon > 1$, a necessary and sufficient condition is that there exists a decomposition of an operator $B = B^*$ that has the two-point spectrum $2-\varepsilon$ and $3-\varepsilon$ with multiplicities $r_2 \geq 1$ and $r_1 \geq 1$ into a sum of three orthogonal projections.*

Proof. Let P be the projection onto the eigen space of the operator A corresponding to the eigen value ε. Then $P_1 + P_2 + P_3 + P = (1+\varepsilon)I$.

It was shown in [7] that irreducible quadruples of orthogonal projections p_1, p_2, p_3, p_4 such that $p_1 + p_2 + p_3 + p_4 = (1+\varepsilon)I$, where $\varepsilon < 1$, exist only in the following cases. 1) $\varepsilon = 1 - \frac{1}{k+\frac{1}{2}}$ and the space has dimension $2k+1$; here $\dim \operatorname{Im} p_4 = k$. So orthogonal projections P_1, P_2, P_3, P such that $P_1 + P_2 + P_3 + P = (1+\varepsilon)I$ exist only on a space of dimension $(2k+1)t$, $t \geq 1$ and $\dim \operatorname{Im} P = kt$, whence (1.a) of the theorem follows. 2) $\varepsilon = 1 - \frac{1}{k+1}$ and the dimension of the space is $k+1$; here $\dim \operatorname{Im} p_4 = r^{(1)} = \frac{2k+1+(-1)^{k-1}}{4}$ or $\dim \operatorname{Im} p_4 = r^{(2)} = \frac{2k+3+(-1)^k}{4}$. So orthogonal projections P_1, P_2, P_3, P such that $P_1 + P_2 + P_3 + P = (1+\varepsilon)I$ exist only on the space of dimension $(k+1)t$, $t \geq 1$ and $\dim \operatorname{Im} P = r^{(1)}t_1 + r^{(2)}t_2$, here $t_1 + t_2 = t$, $t_i \geq 0$, which gives (1.b).

For $\varepsilon = 1$ the claim is obvious.

If $\varepsilon > 1$ and $A = A^*$, with the spectrum containing the two points ε and $1+\varepsilon$ of multiplicities $r_1 \geq 1$ and $r_2 \geq 1$, is a sum of three orthogonal projections $A = P_1 + P_2 + P_3$, then the operator $B = P_1^{\perp} + P_2^{\perp} + P_3^{\perp} = (I - P_1) + (I - P_2) + (I - P_3) = 3I - A$, where $P_i^{\perp} = I - P_i$, is a sum of three orthogonal projections and has spectrum consisting of the two points $2 - \varepsilon$ and $3 - \varepsilon$ of multiplicities $r_2 \geq 1$ and $r_1 \geq 1$, which gives (3). $\square$

3. When an operator with the two-point spectrum ε and $1+\varepsilon$ is a sum of n orthogonal projection

Let $\Phi^{+}(\alpha) = 1 + \frac{1}{n-1-\alpha}$ and

$$\Phi^{+(k)}(\alpha) = \Phi(\Phi^{+(k-1)}(\alpha)),\ k \geq 1,\ \Phi^{+(0)}(\alpha) = \alpha,$$

then $\Lambda_n^{(1)} = \bigcup\limits_{k\geq 0} \Phi^{+(k)}(0)$, $\Lambda_n^{(2)} = \bigcup\limits_{k\geq 0} \Phi^{+(k)}(1)$. Let $1+\varepsilon = \frac{p}{q}$ be an irreducible fraction.

Theorem 4. *An operator $A = A^*$ with the two-point spectrum ε and $1+\varepsilon$, where $1+\varepsilon \in (0, \frac{(n+1)-\sqrt{(n+1)^2-4(n+1)}}{2}) \cup (\frac{(n+1)+\sqrt{(n+1)^2-4(n+1)}}{2}, n+1)$, of multiplicities $r_1 \geq 1$ and $r_2 \geq 1$ is a sum of n orthogonal projections if and only if the following holds.*

1. *$(1+\varepsilon) \in \Lambda_{n+1}^1 \setminus \{0\}$ and has multiplicity $r_2 = (q - \frac{p}{n+1})t$, and ε has multiplicity $r_1 = \frac{p}{n+1}t$;*
2. *$(1+\varepsilon) \in \Lambda_{n+1}^2$ and has multiplicity $r_2 = (q - r^{(1)})t_1 + (q - r^{(2)})t_2$, and ε has multiplicity $r_1 = r^{(1)}t_1 + r^{(2)}t_2$, where $r^{(1)} = \frac{p+(-1)^{k-1}}{n+1}$, $r^{(2)} = r^{(1)} + (-1)^k$ and $k : \Phi^{+(k)}(1) = 1+\varepsilon$, $t_i \geq 0$;*
3. *for $1+\varepsilon \in (\frac{(n+1)+\sqrt{(n+1)^2-4(n+1)}}{2}, n+1)$, a necessary and sufficient condition for existence of the operator A is that there exists an*

operator $B = B^*$, having the two-point spectrum $n-\varepsilon-1, n-\varepsilon$ of multiplicities $r_2 \geq 1$ and $r_1 \geq 1$, that can be represented as a sum of n orthogonal projections.

Proof. Let P be the projection onto the eigen space of the operator A corresponding to the eigen value ε. Then $P_1 + \ldots + P_n + P = (1+\varepsilon)I$.

It was shown in [7] that irreducible orthogonal projections such that $p_1 + p_2 + \ldots + p_n + p_{n+1} = (1+\varepsilon)I$, where $1+\varepsilon < \frac{(n+1)-\sqrt{(n+1)^2-4(n+1)}}{2}$, exist only in the following cases.

1) $1+\varepsilon = \frac{p}{q} \in \Lambda^1_{n+1}$ and the space has dimension q and $\dim \operatorname{Im} p_{n+1} = \frac{p}{n+1}$. So orthogonal projections $P_1, \ldots, P_n, P$ such that $P_1 + \ldots + P_n + P = (1+\varepsilon)I$ exist only on the space of dimension qt, $t \geq 1$ and $\dim \operatorname{Im} P = \frac{p}{n+1}t$, whence (1) of the theorem follows.

2) $1+\varepsilon = \frac{p}{q} = \Phi^{+(k)}(1) \in \Lambda^2_{n+1}$ and the dimension of the space is q; here $\dim \operatorname{Im} p_{n+1} = r^{(1)} = \frac{p+(-1)^{k-1}}{n+1}$ or $\dim \operatorname{Im} p_{n+1} = r^{(2)} = r^{(1)} + (-1)^k$. So orthogonal projections $P_1, \ldots, P_n, P$ such that $P_1 + \ldots + P_n + P = (1+\varepsilon)I$ exist only on the space of dimension qt, $t \geq 1$ and $\dim \operatorname{Im} P = r^{(1)}t_1 + r^{(2)}t_2$, here $t_1 + t_2 = t$, $t_i \geq 0$, which gives (2).

If $1+\varepsilon > \frac{(n+1)+\sqrt{(n+1)^2-4(n+1)}}{2}$ and $A = A^*$, with the spectrum containing the two points ε and $1+\varepsilon$ of multiplicities $r_1 \geq 1$ and $r_2 \geq 1$, is a sum of n orthogonal projections $A = P_1 + \ldots + P_n$, then the operator $B = P_1^\perp + \ldots + P_n^\perp = (I-P_1) + \ldots + (I-P_n) = nI - A$, where $P_i^\perp = I - P_i$, is a sum of n orthogonal projections and has spectrum consisting of the two points $n-1-\varepsilon$ and $n-\varepsilon$ of multiplicities $r_2 \geq 1$ and $r_1 \geq 1$, which gives (3). $\square$

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