Algebra and Discrete Mathematics
Number 2. **(2004).** pp. 84 – 91 RESEARCH ARTICLE
© Journal "Algebra and Discrete Mathematics"

Generalized equivalence of collections of matrices and common divisors of matrices

Vasyl' M. Petrychkovych

Communicated by M. Ya. Komarnytskyj

Abstract. The collections $(A_1, ..., A_k)$ and $(B_1, ..., B_k)$ of matrices over an adequate ring are called generalized equivalent if $A_i = UB_iV_i$ for some invertible matrices U and $V_i,\ i = 1, ..., k$. Some conditions are established under which the finite collection consisting of the matrix and its the divisors is generalized equivalent to the collection of the matrices of the triangular and diagonal forms. By using these forms the common divisors of matrices is described.

1. Introduction

Let R be a commutative ring with $1 \neq 0$. We denote by $M(m, n, R)$ and $M(n, R)$ the set of $m \times n$ matrices and the ring of $n \times n$ matrices over R respectively. The collections of matrices

$$(A_1, ..., A_k), (B_1, ..., B_k),\ A_i, B_i \in M(m, n, R),\ i = 1, \ldots, k$$

are called *equivalent* if

$$A_i = UB_iV,\ i = 1, ..., k$$

for some invertible matrices $U \in GL(m, R)$ and $V \in GL(n, R)$.

Definition 1. *The collections of matrices*

$$(A_1, ..., A_k), (B_1, ..., B_k),\ A_i, B_i \in M(m, n_i, R),\ i = 1, \ldots, k$$

2000 Mathematics Subject Classification: *15A33, 15A21, 15A23.*
Key words and phrases: *collection of matrices, generalized equivalence, canonical diagonal form, common divisors.*

Journal Algebra Discrete Math.

are called generalized equivalent if

$$A_i = UB_iV_i,\ i = 1, ..., k$$

for some matrices $U \in GL(m, R)$ *and* $V_i \in GL(n_i, R)$.

It is known that the problem of equivalence of the collections of matrices is solved at most for pairs of matrices over the fields. This problem is wild even for pairs of matrices over the rings [1]. The reducibility of the finite collections of matrices over polynomial rings and pairs of matrices over the principal ideal rings and other rings by means of generalized equivalent transformations to the triangular and diagonal forms and their applications is considered in [2-7]. V.Dlab and C.M.Ringel [8] have established the canonical form of the pairs of complex matrices (A_1, A_2) with respect to the transformation $(A_1, A_2)(Q, P_1, P_2) = (QA_1P_1^{-1}, QA_2P_2^{-1})$, where Q is a complex invertible matrix P_1 and P_2 are real invertible matrices. A standard form of pairs of matrices over a principal ideal ring and an adequate ring with respect to generalized equivalence is established in [9,10]. In this paper we give conditions under which a finite collection consisting of the matrix and its divisors over an adequate ring is generalized equivalent to the collection of the matrices of the triangular and diagonal forms. By using these forms we describe the common divisors of matrices.

2. Generalized equivalence of collections of matrices

From now on R will denote an adequate ring, i.e. R be domain of integrity in which every finitely generated ideal is principal and for every $a, b \in R$ with $a \neq 0$, a can be represented as $a = cd$, where $(c, b) = 1$ and $(d_i, b) \neq 1$ for any non-unit factor d_i of d [11]. Notice that more general concept of the adequate rings was introduced in [12,13]. We shall denote by D^A the canonical diagonal form (Smith normal form) of the matrix $A \in M(m, n, R)$, i.e

$$D^A = UAV = \operatorname{diag}(\mu_1, \ldots, \mu_r, 0, \ldots, 0),\ \ \mu_r \neq 0,\ \ \mu_1|\mu_2|\cdots|\mu_r$$

for some matrices $U \in GL(m, R)$ and $V \in GL(n, R)$.

Let the matrix $C_i \in M(m, R)$ with the canonical diagonal form $D^{C_i} = \Phi$ is a left divisor of the matrix $A \in M(m, n, R), m \leq n$, i.e

$$A = C_iA_i,\ A_i \in M(m, n, R). \tag{1}$$

Journal Algebra Discrete Math.

Then to the factorization (1) of the matrix A there corresponds a factorization of its canonical diagonal form D^A

$$D^A = \Phi\Psi = \operatorname{diag}(\varphi_1, \dots, \varphi_s, 0, \dots, 0)\operatorname{diag}(\psi_1, \dots, \psi_t, 0, \dots, 0),$$
$$\varphi_1|\varphi_2|\cdots|\varphi_r, D^{C_i} = \Phi.$$

Theorem 1. *Let $A \in M(m,n,R)$, $m \le n$. Suppose that the matrices $C_i \in M(m,R)$, $i = 1,2,\dots$ with canonical diagonal forms $D^{C_i} = \Phi$ are the left divisors of the matrix A. Then every finite collection of matrices $(A, C_1, ..., C_k)$ is generalized equivalent to the collection of the diagonal matrices $(D^A, D^{C_1}, \dots, D^{C_k})$ if and only if the matrices A_i in (1) are equivalent to Ψ in (2) for every $i = 1, \dots, k$.*

Proof. Let the collection of matrices $(A, C_1, ..., C_k)$ is such that the matrices A_i in (1) are equivalent to Ψ in (2) for every $i = 1, \dots, k$. The proof is by induction with respect to k.

For $k = 1$ the proof follows from Theorem 2 in [7]. We assume that Theorem 1 is valid for $k-1$ and prove it for an arbitrary k.

According to the induction assumption the collection of the matrices $(A, C_1, ..., C_{k-1})$ is generalized equivalent to the collection of the diagonal matrices $(D^A, D^{C_1}, \dots, D^{C_{k-1}})$, i.e. $UAV = D^A$, $UC_iV_i = D^{C_i}$ for some matrices $U, V_i \in GL(m,R)$, $i = 1, \dots, k-1$ and $V \in GL(n,R)$. Then the collection of the matrices $(A, C_1, ..., C_{k-1}, C_k)$ is generalized equivalent to the collection of the matrices $(D^A, D^{C_1}, \dots, D^{C_{k-1}}, \tilde{C}_k)$, where $\tilde{C}_k = UC_k$. Since $A = C_kA_k$ then $D^A = \tilde{C}_k\tilde{A}_k$, where $D^A = UAV$. Therefore the pair of matrices $(D^A, \tilde{C}_k)$ is generalized equivalent to the pair (D^A, D^{C_k}), i.e. $\tilde{U}D^A\tilde{V} = D^A$, $\tilde{U}\tilde{C}_k\tilde{V}_k = D^{C_k}$, where $\tilde{U}, \tilde{V}_k \in GL(m,R)$, $\tilde{V} \in GL(n,R)$. The matrix $\tilde{U}$ has the form $\tilde{U} = \|u_{ij}\|_1^m$, where

$$u_{ij} = \begin{cases} \frac{\mu_i}{\mu_j}u'_{ij}, & \text{if } i,j = 1,\dots,r,\ i > j, \\ 0, & \text{if } i = r+1,\dots,m,\ j = 1,\dots,r. \end{cases}$$

Therefore $\tilde{U}\Phi = \Phi S$ for some matrix $S \in GL(m,R)$. From this we obtain that the collection of the matrices $(D^A, D^{C_1}, \dots, D^{C_{k-1}}, \tilde{C}_k)$ is generalized equivalent to the collection $(D^A, D^{C_1}, \dots, D^{C_{k-1}}, D^{C_k})$ of the diagonal matrices.

Now suppose that the collection of matrices $(A, C_1, ..., C_k)$ is generalized equivalent to the collection $(D^A, D^{C_1}, \dots, D^{C_k})$ of the diagonal matrices. Then it is easy verified that the matrices A_i in (1) are equivalent to Ψ in (2) for every $i = 1, \dots, k$. This completes the proof. $\square$

Corollary 1. *Suppose that the canonical diagonal form D^A of the matrix $A \in M(m,n,R)$, $m \le n$, can be represented as*

$$D^A = \Phi\Psi = \operatorname{diag}(\varphi_1, \dots, \varphi_m)\operatorname{diag}(\psi_1, \dots, \psi_m),\ \varphi_1|\varphi_2|\cdots|\varphi_m,$$

Journal Algebra Discrete Math.

$\Phi \in M(m,R), \Psi \in M(m,n,R)$. *If*

$$\left(\frac{\varphi_i}{\varphi_j}, (\psi_i, \psi_j)\right) = 1,\ i,j = 1,\dots,m,\ i > j,$$

then every finite collection $(A, C_1, ..., C_k)$ *consisting of the matrix* A *and its the divisors* C_i *with canonical diagonal forms* $D^{C_i} = \Phi,\ i = 1,\dots,k$ *is generalized equivalent to the collection* $(D^A, D^{C_1},\dots,D^{C_k})$ *of the diagonal matrices.*

Theorem 2. *Let* $A \in M(m,n_1,R),\ B \in M(m,n_2,R),\ m \le n_1,n_2,$ *and* $C \in M(m,R),\ D^C = \Phi = \operatorname{diag}(\varphi_1,\dots,\varphi_s,0,\dots,0)$. *Suppose that the matrix* C *is a common left divisor of the matrices* A *and* B, *i.e.*

$$A = CA_1,\ B = CB_1, \tag{2}$$

and thus

$$D^A = \Phi\Psi,\ D^B = \Phi\Lambda, \tag{3}$$

where

$$\Psi = \operatorname{diag}(\psi_1,\dots,\psi_t,0,\dots,0),\ \Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_q,0,\dots,0).$$

If the matrix A_1 *is equivalent to* Ψ *then the three of matrices* (A,B,C) *is generalized equivalent to the three* $(D^A, \Phi T, D^C)$, *where* T *is lower triangular matrix.*

Proof. By Theorem 1 the pair of matrices (A,C) is generalized equivalent to the pair of the diagonal matrices (D^A, Φ), i.e. $UAV = D^A$, $UCW = D^C = \Phi$ for some matrices $U, W \in GL(m,R)$ and $V \in GL(n,R)$. Then the three of matrices (A,B,C) is generalized equivalent to the three matrix $(D^A, \tilde{B} = \Phi\tilde{B}_1, \Phi)$, where $\tilde{B} = UB$, $\tilde{B}_1 = W^{-1}B_1$. By Lemma 1 in [9,10] there exist an upper unitriangular matrix $U_1 \in GL(m,R)$ and an invertible matrix $S \in GL(n,R)$ such that $U_1\tilde{B}S = T^B$, where T^B is the lower triangular matrix with the principal diagonal D^B. Since $U_1\Phi = \Phi\tilde{U_1}$ for some matrix $\tilde{U_1} \in GL(m,R)$ then $T^B = \Phi T$, where T is the lower triangular matrix. Then $U_1 D^A V_1 = D^A$, $U_1\Phi W_1 = \Phi$ for some matrices $U_1, W_1 \in GL(m,R)$ and $V_1 \in GL(n_1,R)$. This proves the theorem. $\square$

Remark 1. If the matrix B_1 in (3) is equivalent to Λ in (4) then the three of matrices (A,B,C) is generalized equivalent to the three $(\Phi\tilde{T}, D^B, D^C)$, where $\tilde{T}$ is lower triangular matrix.

Journal Algebra Discrete Math.

3. Common divisors of matrices

Let $A \in M(m, n_1, R),\ B \in M(m, n_2, R),\ m \leq n_1, n_2$. Then the pair of matrices (A, B) is generalized equivalent to the pair $(D^A, T^B = TD^B)$, i.e.

$$UAV_1 = D^A,\ UBV_2 = T^B, \tag{4}$$

where $U \in GL(m, R),\ V_1 \in GL(n_1, R),\ V_2 \in GL(n_2, R)$, and T is the lower triangular matrix [10].

Theorem 3. *Let $A \in M(m, n_1, R)$, $B \in M(m, n_2, R)$, $m \leq n_1, n_2$, and $D^A = \operatorname{diag}(\mu_1, \ldots, \mu_r, 0, \ldots, 0)$. Suppose that d-matrix*

$$\Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_s, 0, \ldots, 0),$$

$\varphi_1 | \varphi_2 | \cdots | \varphi_s$, $\Phi \in M(m, R)$ is common divisor of the canonical diagonal forms D^A and D^B of matrices A and B, i.e.

$$D^A = \Phi\Psi,\ D^B = \Phi\Lambda. \tag{5}$$

Then there exists a common left divisor C with the canonical diagonal form $D^C = \Phi$ of matrices A and B, i.e.

$$A = CA_1,\ B = CB_1, \tag{6}$$

where the matrix A_1 is equivalent to Ψ if and only if Φ is the common left divisor of matrices D^A and T^B. All common left divisors C_i with the canonical diagonal form $D^{C_i} = \Phi$ of matrices A and B up to associates have the form $C_i = U_i^{-1}\Phi$, where matrices U_i are given by (5).

Proof. Let a matrix $C \in M(m, R)$ is the common left divisor of the matrices A and B, i.e. the relation (7) holds. By Theorem 2 the pair of matrices (A, B) is generalized equivalent to the pair $(D^A, \tilde{T}^B = \Phi\tilde{T})$, i.e.

$$\tilde{U}A\tilde{V}_1 = D^A,\ \tilde{U}B\tilde{V}_2 = \tilde{T}^B \tag{7}$$

for some matrices $\tilde{U} \in GL(m, R),\ \tilde{V}_1 \in GL(n_1, R),\ \tilde{V}_2 \in GL(n_2, R)$. We now must show that the matrix Φ is the left divisor of the matrix $T^B = UBV_2$ in (5). For the matrices U in (5) and $\tilde{U}$ in (8) we have $U = H\tilde{U}_1$, where matrix H has the form

$$H = \|h_{ij}\|_1^m,\ h_{ij} = \begin{cases} \frac{\mu_i}{\mu_j} h'_{ij}, & \text{if } i, j = 1, \ldots, r,\ i > j, \\ 0, & \text{if } i = r+1, \ldots, m,\ j = 1, \ldots, r. \end{cases}$$

Then $T^B = H\tilde{T}^B V_3$, where $V_3 = \tilde{V}_2^{-1} V_2$. Since $D^A = \Phi\Psi$, then $H\Phi = \Phi\tilde{H}$ for some matrix $\tilde{H} \in GL(m, R)$. Thus $T^B = \Phi T$.

Journal Algebra Discrete Math.

Journal Algebra Discrete Math.

Now let Φ is the common left divisor of the matrices D^A and T^B in (5). Then every matrix $U_i^{-1}\Phi = C_i$, where U_i is given by (5), is the common left divisor of the matrices A and B.

If the matrix $C \in M(m, R)$ is the common left divisor of the matrices A and B, i.e. the relation (7) holds, then by Theorem 2 there exist the matrices $U, V_3 \in GL(m, R)$, and $V_1 \in GL(n_1, R)$, $V_2 \in GL(n_2, R)$ such that $UAV_1 = D^A$, $UBV_2 = T^B$, $UCV_3 = \Phi$. Therefore $C = U^{-1}\Phi$. This completes the proof of the theorem. □

Remark 2. If in (7) the matrix B_1 is equivalent to Λ in (6) then there exists a common left divisor C with the canonical diagonal form $D^C = \Phi$ of matrices A and B if and only if Φ is the common left divisor of matrices T^A and D^B, where $T^A = UAV_1$, $D^B = UBV_2$ for some matrices $U \in GL(m, R)$, $V_1 \in GL(n_1, R)$ and $V_2 \in GL(n_2, R)$.

Corollary 2. *Suppose that the canonical diagonal forms D^A and D^B of the matrices $A \in M(m, n_1, R)$ and $B \in M(m, n_2, R)$, $m \leq n_1, n_2$, can be represented as*

$$D^A = \operatorname{diag}(\varphi_1, \ldots, \varphi_m)\operatorname{diag}(\psi_1, \ldots, \psi_m),$$

$$D^B = \operatorname{diag}(\varphi_1, \ldots, \varphi_m)\operatorname{diag}(\lambda_1, \ldots, \lambda_m), \ \varphi_1 | \varphi_2 | \cdots | \varphi_m.$$

If

$$\left(\frac{\varphi_i}{\varphi_j}, (\psi_i, \psi_j)\right) = 1, \ i, j = 1, \ldots, m, \ i > j,$$

or

$$\left(\frac{\varphi_i}{\varphi_j}, (\lambda_i, \lambda_j)\right) = 1, \ i, j = 1, \ldots, m, \ i > j$$

then there exists common left divisor C with the canonical diagonal form $D^C = \Phi = \operatorname{diag}(\varphi_1, \ldots, \varphi_m)$ of matrices A and B if and only if Φ is common left divisor of matrices D^A and T^B or T^A and D^B.

Corollary 3. *Let a pair (A, B) of matrices $A \in M(m, R)$, $\det A \neq 0$ and $B \in M(m, n, R)$ is generalized equivalent to the pair (D^A, D^B) of the diagonal matrices, i.e. the matrix $(\operatorname{adj} A)B$ is equivalent to the matrix $(\operatorname{adj} D^A)D^B$ [10]. Then there exists common left divisor C with the canonical diagonal form $D^C = \Phi$ of matrices A and B if and only if Φ is common divisor the canonical diagonal forms D^A and D^B of matrices A and B.*

Let $A, B \in M(m, R)$. The matrices A and B are called *left relatively prime* if every factorization

$$A = C_iA_i, \ B = C_iB_i, \ C_i \in M(m, R)$$

of the matrices A and B yields $C_i \in GL(m, R)$.

Theorem 4. *Let $A, B \in M(m, R)$ and $D^A = \operatorname{diag}(1, \dots, 1, \varphi_m)$, $D^B = \operatorname{diag}(\psi_1, \dots, \psi_m)$. Then the matrices A and B are left relatively prime if and only if the pair of matrices (A, B) is generalized equivalent to the pair (D^A, T^B), where*

$$T^B = \left\| \begin{array}{ccccc} \psi_1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ t\psi_1 & 0 & \dots & 0 & \psi_m \end{array} \right\| \qquad (8)$$

and

$$t = \begin{cases} 0, & \text{if } (\varphi_m, \psi_m) = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. Suppose that the matrices A and B are left relatively prime. By Theorem 1 in [10] the pair of matrices (A, B) is generalized equivalent to the pair (D^A, T_1^B), where

$$T_1^B = \left\| \begin{array}{ccccc} \psi_1 & 0 & \cdots & 0 & 0 \\ 0 & \psi_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \psi_{m-1} & 0 \\ t_1\psi_1 & t_2\psi_2 & \dots & t_{m-1}\psi_{m-1} & \psi_m \end{array} \right\|.$$

Then the matrices D^A and T_1^B are left relatively prime. Therefore

$$(\varphi_m, \ (t_1\psi_1, t_2\psi_2, \dots, t_{m-1}\psi_{m-1}, \psi_m)) = 1.$$

By Theorem 3 in [10] the pair of matrices (D^A, T_1^B) is generalized equivalent to the pair (D^A, T^B), where the matrix T^B has the form (9).

Now let the pair of matrices (A, B) is generalized equivalent to the pair (D^A, T^B), where the matrix T^B has the form (9). Then [14] the common left divisors C_i, $i = 1, 2 \dots$ of matrices D^A and T^B up to right associates have the form $C_i = \operatorname{diag}(1, \dots, 1, \varphi_i)$, where φ_i are the divisors of φ_m. Taking into account the form of the matrices D^A and T^B we have that $\varphi_i = 1$. Therefore the matrices D^A and T^B are left relatively prime and thus the matrices A and B are left relatively prime. This proves the theorem. ☐

References

[1] P. M. Gudyvok, *On equivalence of matrices over a commutative rings*, Besk. gruppy i primyk. algebr. struktury, Kyiv, 1993, pp. 431–437 (in Russian).

[2] P.S. Kazimirs'kij, V.M.Petrychkovych, *Equivalence of polynomial matrices*, Teor. ta prykl. pytannia algebry i dyf. rivnian. Naukova Dumka, Kyiv, 1977, pp. 61-66 (in Ukrainian).

Journal Algebra Discrete Math.

[3] V. M. Petrychkovych, *Semiscalar equivalence and the Smith normal form of polynomial matrices*, Mat. Metody Fiz.-Mekh. Polya, N.**26**, 1987, pp. 13-16 (in Ukrainian).

[4] V. M.Petrychkovych, *Semiscalar equivalence and the factorization of polynomial matrices*, Ukr. Math. Zh., **42**, N. 5, 1990, pp. 644-649 (in Ukrainian).

[5] B. V.Zabavs'kij, P. S. Kazimirs'kij, *Reduction of a pair of matrices over an adequate ring to special triangular form by application of identical unilateral transformations*, Ukr. Math. Zh., **36**, N. 2, 1984, pp. 256–258 (in Ukrainian).

[6] V. M.Petrychkovych, *A criterion of diagonalizability of a pair of matrices over a ring of principal ideals by common row and separate column transformations*, Ukr. Math. Zh. **49**, N. 6, 1997, pp. 860-862 (in Ukrainian).

[7] V. M.Petrychkovych, *Reducibility of pairs of matrices by generalized equivalent transformations to triangular and diagonal forms and their applications*, Mat. Metody Fiz.-Mekh. Polya, **43**, N. 2, 2000, pp. 15-22 (in Ukrainian).

[8] V.Dlab, C. M.F.Ringel, *Canonical forms of pairs of complex matrices*, Linear Algebra Appl., **147**, 1991, pp. 387-410.

[9] V. Petrychkovych, *Generalized equivalence of pairs of matrices*, Linear and Multilinear Algebra, **48**, 2000, pp. 179-188.

[10] V. M. Petrychkovych, *Standard forms of pair of matrices with respect to generalized equivalence*, Visnyk Lviv. Univ., **61**, 2003, pp. 153-160.

[11] O.Helmer, *The elementary divisor theorem for certain rings without chain condition*, Bull. Amer. Math. Soc., **49**, 1943, pp. 225-236.

[12] N.Ya.Komarnitskij, *Commutative adequate Bezout domain and elementary divisor ring*, Algebr. Issled., Collection of papers, Institute of mathematics NAN of Ukraine, Kiev, **24**, 1996, pp. 97–113 (in Russian).

[13] M.Ya.Komarnyts'kij, V.M.Petrychkovych, *Structure-theoretic properties of matrices over finitely generated principal ideal rings*, Mat. Metody Fiz.-Mekh. Polya, **46**, N. 2, 2003, pp. 7-21 (in Ukrainian).

[14] V. M.Petrychkovych, *On parallel factorizations of matrices over principal ideal rings*, Mat. Metody Fiz.-Mekh. Polya, **40**, N. 4, 1997, pp. 96-100 (in Ukrainian).

Contact information

Vasyl' M. Petrychkovych

Department of Algebra, Pidstryhach Institute for Applied Problems of Mechanics and the Mathematics National Academy of Sciences of Ukraine, 3B Naukova Str., Lviv, 79053, Ukraine
E-Mail: `vas_petrych@yahoo.com`, `vpetrych@iapmm.lviv.ua`

Received by the editors: 21.04.2004
and final form in 25.05.2004.

Journal Algebra Discrete Math.