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Two-step tilting for standardly stratified algebras

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Abstract. We study the class of standardly stratified algebras introduced by Cline, Parshall and Scott, and its subclass of the so-called weakly properly stratified algebras, which generalizes the class of properly stratified algebras introduced by Dlab. We characterize when the Ringel dual of a standardly stratified algebra is weakly properly stratified and show the existence of a two-step tilting module. This allows us to calculate the finitistic dimension of such algebras. Finally, we also give a construction showing that each finite partially pre-ordered set gives rise to a weakly properly stratified algebras with a simple preserving duality.

1. Introduction

The class of standardly stratified algebras, which appears in [CPS], generalizes the smaller class with the same name studied in [AHLU, ADL2], the later being closely related to the so-called properly stratified algebras which were introduced in [D]. The corresponding concept for standardly stratified algebras appeared in [F] under the name weakly properly stratified algebras. This is a natural subclass of the standardly stratified algebras, which has an additional advantage of possessing the the left-right symmetry.

The concept of tilting modules and the Ringel duality for the restricted case of stratified algebras has been studied in [AHLU] and extended to all stratified algebras in [F]. The results, in this paper, which correspond to results for SSS-algebras in [FM] are (mostly) stated without proofs. We give the proof only in the case when some steps differ from those used in [FM] (see Lemma 2, Lemma 5 and Lemma 6).

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In the present paper we extend to all stratified algebras the results of [FM] about the two-step tilting module and the corresponding Ringel duality. In more details, in Subsection 3.1 we characterize the situation, when the Ringel dual of a standardly stratified algebra is weakly properly stratified. An example of a standardly stratified algebra whose Ringel dual is not weakly properly stratified is given in Section 6. In Subsection 3.2 we present two classes of stratified algebras, which are closed under taking the Ringel dual.

For a stratified algebra A, whose Ringel dual R is weakly properly stratified, we can introduce the two-step tilting A-module H. Under some assumptions on the "local" behavior of A we compute the finitistic dimension of A and show that the category of A-modules of finite projective dimension is contravariantly finite. All this is done in Subsections 4.2 and 4.3. Under the same assumptions we explore the two-step duality for such algebras in Section 5.

In the last section we present examples illustrating calculations of the finitistic dimension of A in terms of p.d.(H) and the projective dimension of the characteristic tilting module. We also show that for every finite partially pre-ordered set there is a weakly properly stratified algebra having a simple preserving duality for which this set indexes the isoclasses of simple modules.

2. Stratified algebras

2.1. General definitions

Let A be a finite dimensional associative algebra with identity over an algebraically closed field $\Bbbk$. We denote by A-mod the category of all finite dimensional left A-modules. In the case when more then one algebra will be around, we will use the notation $M^{(B)}$ to indicate that M is a left module over the algebra B.

Denote by Λ a set indexing the isomorphism classes of simple A-modules, which we denote by $L(\lambda)$, $\lambda \in \Lambda$. Let $\preceq$ be a partial pre-order on Λ. For $\lambda, \mu \in \Lambda$ we will write $\lambda \prec \mu$ provided that $\lambda \preceq \mu$ and $\mu \not\preceq \lambda$; and $\lambda \sim \mu$ provided that $\lambda \preceq \mu$ and $\mu \preceq \lambda$. We define the set $\overline{\Lambda} = \{\overline{\lambda} | \lambda \in \Lambda\}$ as the collection of all equivalence classes under the equivalence relation $\sim$. By the definition $\lambda \in \overline{\lambda}$ for all $\lambda \in \Lambda$. The partial pre-order $\preceq$ on Λ induces in a natural way a partial order $\leq$ on $\overline{\Lambda}$. We write $P(\lambda)$ for the projective cover and $I(\lambda)$ for the injective hull of $L(\lambda)$.

The pair $(A, \preceq)$ is called a *standardly stratified algebra*, [CPS], if there exists a family, $\{\Delta(\lambda) | \lambda \in \Lambda\}$, of A-modules, such that the following two conditions are satisfied:

(SS1) there exists an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ whose kernel has a filtration with subquotients $\Delta(\mu)$, $\lambda \prec \mu$;

(SS2) there exists an epimorphism $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ whose kernel has a filtration with subquotients $L(\mu)$, $\mu \preceq \lambda$.

In several papers, see for example [AHLU, ADL2], a smaller class of algebras was called standardly stratified algebras. These were standardly stratified algebras in the sense of the above definition, for which $\preceq$ was a linear order. This smaller class of algebras is contained in the class of *strongly standardly stratified algebras*, or simply *SSS-algebras*, defined in [FM], where $\preceq$ is assumed to be a partial order. Actually, a given SSS-algebra $(A, \preceq)$ is also standardly stratified for some linear order with the same standard modules (see Lemma 8).

The class of standardly stratified algebras, in the sense of [AHLU, ADL2], contains a smaller class, namely the class of properly stratified algebras [D]. In [F] a natural generalization of this subclass is defined. The pair $(A, \preceq)$ is called a *weakly properly stratified algebra* if there exist two families $\{\Delta(\lambda)|\lambda \in \Lambda\}$ and $\{\overline{\Delta}(\lambda)|\lambda \in \Lambda\}$ of A-modules such that the following conditions are satisfied:

(wPS1) there exists an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$ whose kernel has a filtration with subquotients $\Delta(\mu)$, $\lambda \prec \mu$;

(wPS2) there exists an epimorphism $\Delta(\lambda) \twoheadrightarrow L(\lambda)$ whose kernel has a filtration with subquotients $L(\mu)$, $\mu \preceq \lambda$;

(wPS3) there is an epimorphism $\overline{\Delta}(\lambda) \twoheadrightarrow L(\lambda)$ whose kernel has a filtration with subquotients $L(\mu)$, $\mu \prec \lambda$;

(wPS4) $\Delta(\lambda)$ has a filtration with subquotients $\overline{\Delta}(\mu)$, $\mu \sim \lambda$, for all $\lambda \in \Lambda$.

The properly stratified algebras, in the sense of [D], are those weakly properly stratified algebras with $\preceq$ assumed to be a linear order. Abusing the language we will call weakly properly stratified algebras for which $\preceq$ is assumed to be a partial order simply *properly stratified* in this paper as well. Both these notions of properly stratified algebras and the notion of weakly properly stratified algebras are left-right symmetric.

If $(A, \preceq)$ is standardly (or weakly properly) stratified and $\lambda \in \Lambda$, then the module $\Delta(\lambda)$ is called *standard module* and the module $\overline{\Delta}(\lambda)$ is called *proper standard module.* The *costandard module* $\nabla(\lambda)$ and the *proper costandard module* $\overline{\nabla}(\lambda)$ are defined dually, see [F].

Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be two standardly stratified (or weakly properly stratified) algebras. An isomorphism of algebras $f : A_1 \to A_2$

induces a canonical equivalence of categories $F_f : A_1\text{-mod} \to A_2\text{-mod}$. If the bijection $\hat{f} : \Lambda_1 \to \Lambda_2$, defined by $F_f(L^{(A_1)}(\lambda)) \cong L^{(A_2)}(\hat{f}(\lambda))$, is order preserving we say that f is an *isomorphism of stratified algebras.*

2.2. Tilting modules and the Ringel dual

Let $(A, \preceq)$ be a standardly stratified (or weakly properly stratified) algebra. For a subclass $\mathcal{C}$ from A-mod define $\mathcal{F}(\mathcal{C})$ as the full subcategory of A-mod which consists of all modules M having a filtration, whose subquotients are isomorphic to modules from $\mathcal{C}$. Denote by $\mathcal{F}(\Delta)$ the category $\mathcal{F}(\mathcal{C})$, where $\mathcal{C} = \{\Delta(\lambda) | \lambda \in \Lambda\}$, and define $\mathcal{F}(\overline{\Delta})$, $\mathcal{F}(\nabla)$ and $\mathcal{F}(\overline{\nabla})$ similarly. Set $L = \oplus_{\lambda\in\Lambda} L(\lambda)$, and define similarly P, I, Δ, $\overline{\Delta}$, ∇ and $\overline{\nabla}$. Put $L(\overline{\lambda}) = \oplus_{\mu\in\overline{\lambda}} L(\mu)$ and similarly for $P(\overline{\lambda})$, $I(\overline{\lambda})$, $\Delta(\overline{\lambda})$, $\overline{\Delta}(\overline{\lambda})$, $\nabla(\overline{\lambda})$ and $\overline{\nabla}(\overline{\lambda})$.

In [F] it has been shown that the category $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ is closed under taking direct summands. The modules in $\mathcal{F}(\Delta) \cap \mathcal{F}(\overline{\nabla})$ are called *tilting modules*. The indecomposable tilting modules are indexed by $\lambda \in \Lambda$ in a natural way. We denote by $T(\lambda)$ the indecomposable tilting module whose standard filtration starts with $\Delta(\lambda)$. The module $T = \oplus_{\lambda\in\Lambda} T(\lambda)$ is called the *characteristic tilting module* and is in fact a (generalized) tilting module.

If $(A, \preceq)$ is weakly properly stratified, we can also define the dual notion of *cotilting modules*, namely the objects in $\mathcal{F}(\overline{\Delta}) \cap \mathcal{F}(\nabla)$. We denote by $C(\lambda)$ the indecomposable cotilting module whose costandard filtration ends with $\nabla(\lambda)$. The module $C = \oplus_{\lambda\in\Lambda} C(\lambda)$ is called the *characteristic cotilting module.*

2.3. The Ringel dual

Let $(A, \preceq)$ be standardly stratified. The *Ringel dual* R is defined via $R = \mathrm{End}_A(T)$. The functor $F : A\text{-mod} \to R\text{-mod}$, defined via $F(\_) = \mathrm{Hom}_A(T, \_)$ is called the *Ringel duality functor*. The following statements are proved in [F].

Theorem 1. *Let $(A, \preceq)$ be standardly stratified, and $\preceq_R$ be the opposite order with respect to $\preceq$. Then*

(i) $F(\overline{\nabla}^{(A)}(\lambda)) = \overline{\Delta}^{(R)}(\lambda)$;

(ii) the functor F restricts to an exact equivalence between $\mathcal{F}(\overline{\nabla}^{(A)})$ and $\mathcal{F}(\overline{\Delta}^{(R)})$;

(iii) the opposite R^{opp} of the Ringel dual is standardly stratified with respect to $\preceq_R$;

(iv) *the Ringel dual of R^{opp} is Morita equivalent to A^{opp}.*

Proposition 1. *Let $(A, \preceq)$ be a standardly stratified algebra and $(R, \preceq_R)$ be the Ringel dual. Then the functor $J : A$-mod $\to R$-mod, defined by $J(\_) = D \circ Hom_R(\_, T^{(R)})$ has the following properties:*

(i) $J(\Delta^{(A)}(\lambda)) = \nabla^{(R)}(\lambda)$;

(ii) *the functor J restricts to an exact equivalence between $\mathcal{F}(\Delta^{(A)})$ and $\mathcal{F}(\nabla^{(R)})$.*

3. Weakly properly stratified Ringel duals

3.1. A criterion when the Ringel dual is weakly properly stratified

For a standardly stratified algebra $(A, \preceq)$ and $\lambda \in \Lambda$ set $T^{\prec\lambda} = \oplus_{\mu\prec\lambda} T(\mu)$, then $S(\lambda) = \mathrm{Tr}_{T^{\prec\lambda}}(T(\lambda))$ and $N(\lambda) = T(\lambda)/S(\lambda)$. Set further $N = \oplus_{\lambda\in\Lambda} N(\lambda)$ and $\mathcal{F}(N) = \mathcal{F}(\mathcal{C})$, where $\mathcal{C} = \{N(\lambda) | \lambda \in \Lambda\}$.

Using the same arguments as in [FM, Lemma 1] one obtains

Proposition 2. *Let $(A, \preceq)$ be standardly stratified. For every $\lambda \in \Lambda$ the module $S(\lambda)$ is the unique submodule M of $T(\lambda)$ which is characterized by the following properties:*

(a) $M \in \mathcal{F}(\{\overline{\nabla}(\mu) | \mu \prec \lambda\})$.

(b) $T(\lambda)/M \in \mathcal{F}(\{\overline{\nabla}(\mu) | \mu \sim \lambda\})$.

Similarly to [FM, Theorem 1] we obtain:

Theorem 2. *Let $(A, \preceq)$ be standardly stratified. Then the following assertions are equivalent:*

(I) *The Ringel dual $(R, \preceq_R)$ is weakly properly stratified.*

(II) *For each $\lambda \in \Lambda$ we have $S(\lambda) \in \mathcal{F}(N)$.*

3.2. Two classes of weakly properly stratified algebras which are closed under taking the Ringel dual

Denote by $\mathcal{C}_1$ the class of those weakly properly stratified algebras for which all tilting modules are also cotilting. Denote also by $\mathcal{C}_2$ the class of those weakly properly stratified algebras for which the endomorphism algebras of the characteristic tilting and the characteristic cotilting modules are isomorphic as stratified algebras. It is clear that $\mathcal{C}_1 \subset \mathcal{C}_2$. The class $\mathcal{C}_1$ contains, in particular, all quasi-hereditary algebras.

The following theorem is proved by the same arguments as in [FM, Theorem 2].

Theorem 3. *If* $(A, \preceq) \in \mathcal{C}_1$*, then* $(R, \preceq_R) \in \mathcal{C}_1$*.*

In fact we can prove that the class $\mathcal{C}_2$ is also closed under the Ringel dual.

Proposition 3. *If* $(A, \preceq) \in \mathcal{C}_2$ *is basic, then* $(R, \preceq_R) \in \mathcal{C}_2$*.*

Proof. From

$$\mathrm{End}_A(T^{(A)}) \cong \mathrm{End}_A(C^{(A)}).$$

and the usual duality we conclude that A^{opp} has the Ringel dual R^{opp}. Hence from Theorem 1 we conclude that both R and R^{opp} are standardly stratified. From [F] we conclude that R is weakly properly stratified. That $A \cong \mathrm{End}_R(T^{(R)})$ and $A \cong \mathrm{End}_R(C^{(R)})$ is obtained from the Ringel dualities of R^{opp} and R respectively. Thus $A \cong \mathrm{End}_R(T^{(R)}) \cong \mathrm{End}_R(C^{(R)})$ and the proposition is proved. □

Remark that Theorem 3 implies that the Ringel duality sends $\mathcal{C}_2 \setminus \mathcal{C}_1$ to itself. In Section 6 one can find an example of algebra from $\mathcal{C}_2 \setminus \mathcal{C}_1$.

We recall an algebra A has a *simple preserving duality* if there exists an exact contravariant and involutive equivalence $^\circ : A\text{-mod} \to A\text{-mod}$ that preserves the isomorphism classes of simple modules. For a weakly properly stratified algebra $(A, \preceq)$ with a simple preserving duality we have $\mathrm{End}_A(T) \cong \mathrm{End}_A(C)^{\mathrm{opp}}$ as stratified algebras. In fact, the last statement can be reversed.

Lemma 1. *Let* $(A, \preceq)$ *be a weakly properly stratified algebra and assume that* $\mathrm{End}_A(T) \cong \mathrm{End}_A(C)^{opp}$ *as stratified algebras. Then* A *has a simple preserving duality.*

Proof. From the asserted equality it follows that the Ringel duals of A and A^{opp} coincide. Thus, the two Ringel duality functors send $I^{(A)}$ and $I^{(A^{\mathrm{opp}})}$ to the dual of the characteristic tilting module $D(T^{(R^{\mathrm{opp}})})$. Hence, by the same argument as in the end of the proof of Theorem 1, we obtain $A \cong A^{\mathrm{opp}}$. This gives rise to a simple preserving duality. □

In Section 6 one can find an example of weakly properly stratified algebra with a simple preserving duality, whose Ringel dual does not have such a duality.

4. The module H and its properties

4.1. Definition of the module H

Let $(A, \preceq)$ be a standardly stratified algebra with a weakly properly stratified Ringel dual $(R, \preceq_R)$. We assume that A has these properties to the end of this section.

Using $T^{(R)}(\lambda) \in \mathcal{F}(\overline{\Delta}^{(R)})$ and Theorem 1 we define

$$H^{(A)}(\lambda) = F^{-1}\big(T^{(R)}(\lambda)\big)$$

for all $\lambda \in \Lambda$. Set $H^{(A)} = \oplus_{\lambda\in\Lambda} H^{(A)}(\lambda)$. One can easily see that $F(N^{(A)}(\lambda)) = \Delta^{(R)}(\lambda)$.

We recall that a module M over an associative algebra A is called a *(generalized) tilting module* if it has finite projective dimension, is ext-self-orthogonal and finitely coresolves ${}_AA$. By similar arguments as in [FM] we obtain:

Proposition 4. *The module $H^{(A)}$ is a (generalized) tilting module.*

4.2. Modules of finite projective dimension

For an algebra A and for a full subcategory $\mathcal{C}$ of A-mod, denote by $\check{\mathcal{C}}$ the full subcategory of A-mod, which contains all modules M for which there is a finite exact sequence

$$0 \to M \to C_0 \to C_1 \to \cdots \to C_k \to 0$$

with $C_i \in \mathcal{C}$. We also define the *(projective) finitistic dimension* of A to be

$$\text{fin.dim}(A) = \sup\{\text{p.d.}(M) | M \in A\text{-mod},\ \text{p.d.}(M) < \infty\}.$$

Recall that a full subcategory $\mathcal{C}$ of A-mod is called *contravariantly finite* provided that it is closed under direct summands and isomorphisms, and if for each A-module X there exists a homomorphism $f : C_X \to X$ where $C_X \in \mathcal{C}$, such that for any homomorphism $g : C \to X$ with $C \in \mathcal{C}$ there is a homomorphism $h : C \to C_X$ such that $f \circ h = g$.

Recall also that a subcategory $\mathcal{B}$ of A-mod is called *resolving* if it contains all projective modules and is closed under extensions and kernels of epimorphisms. Obviously, the category $\mathcal{P}(A)^{<\infty}$, defined as the subcategory of A-mod consisting of all modules of finite projective dimension, is a resolving category. However, $\mathcal{P}(A)^{<\infty}$ is not contravariantly finite in general, see [IST].

Let $A(\overline{\lambda}) = \mathrm{End}_A(\Delta^{(A)}(\overline{\lambda}))$ and assume until the end of the subsection that the algebra A also satisfies the assumption

$$\mathrm{fin.dim}\big(A(\overline{\lambda})\big) = 0 \tag{1}$$

for each $\overline{\lambda} \in \overline{\Lambda}$. Note that in the case when $\preceq$ is a partial order the condition (1) is trivially satisfied since $A(\overline{\lambda})$ is local.

Proposition 5. *Let $M \in A-\mathrm{mod}$ and p.d.$(M) < \infty$. Then there exists a finite coresolution*

$$0 \to M \to H_0 \to \cdots \to H_k \to 0,$$

where $H_i \in \mathrm{add}(H)$ and $k \geq 0$.

The proof of Proposition 5 is analogous to [FM, Proposition 3] using the following lemma:

Lemma 2. *Let $(A, \preceq)$ be weakly properly stratified satisfying* (1)*. Then*

(i) $\mathcal{F}(\Delta) = \{M \in \mathcal{F}(\overline{\Delta}) \,|\, p.d.(M) < \infty\}$

(ii) $\mathcal{F}(\nabla) = \{M \in \mathcal{F}(\overline{\nabla}) \,|\, i.d.(M) < \infty\}$.

Proof. We prove (2). The statement (2) is proved by dual arguments. The inclusion $\mathcal{F}(\Delta) \subset \{M \in \mathcal{F}(\overline{\Delta}) \,|\, \mathrm{p.d.}(M) < \infty\}$ follows easily from the definition of a weakly properly stratified algebra, see [F].

Let us prove the inverse inclusion. Let $M \in \mathcal{F}(\overline{\Delta})$ with p.d.$(M) < \infty$. We choose a minimal projective resolution of M

$$0 \to P_k \to \cdots \to P_1 \to P_0 \to M \to 0.$$

We will prove by induction in $\overline{\lambda}$ that $M \in \mathcal{F}(\Delta)$. Let $\overline{\lambda} \in \overline{\Lambda}$ be maximal. Applying [F] we obtain the exact sequence

$$0 \to \mathrm{Tr}_{P(\overline{\lambda})}(P_k) \to \cdots \to \mathrm{Tr}_{P(\overline{\lambda})}(P_1) \to \mathrm{Tr}_{P(\overline{\lambda})}(P_0) \to \mathrm{Tr}_{P(\overline{\lambda})}(M) \to 0,$$

moreover $\mathrm{Tr}_{P(\overline{\lambda})}(P_i) \in \mathrm{add}(P(\overline{\lambda}))$ for each i. Define $U : A\text{-mod} \to A(\overline{\lambda})\text{-mod}$ by $U(\_) = \mathrm{Hom}_A(\Delta(\overline{\lambda}), \_)$ (by maximality of $\overline{\lambda}$ we have $P(\overline{\lambda}) = \Delta(\overline{\lambda})$). Since U is exact we obtain a finite projective resolution of $U(\mathrm{Tr}_{P(\overline{\lambda})}(M))$. Thus $U(\mathrm{Tr}_{P(\overline{\lambda})}(M))$ has finite projective resolution and from (1) we conclude that $U(\mathrm{Tr}_{P(\overline{\lambda})}(M))$ is a projective module. Hence $\mathrm{Tr}_{P(\overline{\lambda})}(M) \in \mathrm{add}(P(\overline{\lambda}))$ and therefore $\mathrm{Tr}_{P(\overline{\lambda})}(M) \in \mathcal{F}(\Delta)$. From [F] we also have the exact sequence

$$0 \to \overline{P_k} \to \cdots \to \overline{P_1} \to \overline{P_0} \to \overline{M} \to 0,$$

where each $\overline{P_i}$ and $\overline{M}$ belong to $\mathcal{F}(\{\overline{\Delta}(\mu) | \mu \in \overline{\Lambda} \setminus \overline{\lambda}\})$. Thus we can proceed by induction and finally obtain $M \in \mathcal{F}(\Delta)$. □

Theorem 4. *The category $\mathcal{P}^{<\infty}$ is contravariantly finite. Moreover, fin.dim$(A) < \infty$.*

Proof. Since H is a (generalized) tilting module, the subcategory $\check{\mathrm{add}}(H)$ is contravariantly finite and resolving by [AR, Section 5]. From Proposition 5 we see that $\mathcal{P}^{<\infty} \subset \check{\mathrm{add}}(H)$. On the other hand, p.d.$(H) < \infty$ implies $\mathcal{P}^{<\infty} \supset \check{\mathrm{add}}(H)$. The second statement follows now from [AR]. □

We calculate the finitistic dimension of A in terms of p.d.(H). The proof uses Proposition 5 and follows [FM, Theorem 3].

Theorem 5.

$$\text{fin.dim}(A) = p.d.(H).$$

4.3. The case of algebras with duality

This section generalizes the main results in [MO] and [FM].

Theorem 6. *Let $(A, \preceq)$ be a standardly stratified algebra with a simple preserving duality, whose Ringel dual is weakly properly stratified. Assume that $\mathrm{soc}\left(P^{(A(\overline{\lambda}))}(\lambda)\right)$ contains a simple module $L^{(A(\overline{\lambda}))}(\lambda)$ for all for all $\lambda \in \Lambda$. Then fin.dim$(A) = 2p.d.(T^{(R)})$.*

To prove the statement we will need several lemmas.

Lemma 3. *Under assumptions of Theorem 6 we have*

$$p.d.(T^{(R)}) \leq p.d.(T^{(A)}).$$

The proof follows the arguments of [FM, Lemma 10] using the following lemmas:

Lemma 4. *Under assumptions of Theorem 6 let $\mathcal{C}^\bullet$ be a finite negative complex in the category Com$^b(\mathrm{add}(C))$, then there exists a negative (possibly infinite) complex $\mathcal{J}^\bullet$ in Com$(\mathrm{add}(T))$ such that $\mathcal{C}^\bullet$ is quasi-isomorphic to $\mathcal{J}^\bullet$. Moreover, $\mathcal{J}^0 = (\mathcal{C}^0)^\circ \oplus \hat{T}$, where $\hat{T}$ is some tilting module.*

Lemma 5. *Let $\lambda \in \Lambda$. Under assumptions of Theorem 6 there exists a minimal tilting resolution $\cdots \to T_1 \to T(\lambda) \oplus T_0 \to C(\lambda) \to 0$ of $C(\lambda)$.*

Proof. Since $C^{(A)}(\lambda) \in \mathcal{F}(\overline{\nabla}^{(A)})$, we can choose a minimal projective resolution

$$\cdots \to P_1^{(R)} \to P_0^{(R)} \to F(C^{(A)}(\lambda)) \to 0$$

of $F(C^{(A)}(\lambda))$. We have that $C^{(A)}(\lambda)$ surjects onto $\nabla^{(A)}(\lambda)$. By assumption, we can choose a composition series of $P^{(A(\overline{\lambda}))}(\lambda)$ which starts with the submodule $L^{(A(\overline{\lambda}))}(\lambda)$. Thus, applying [F], we obtain a proper standard filtration of $\Delta^{(A)}(\lambda)$, which starts with the submodule $\overline{\Delta}^{(A)}(\lambda)$. From the duality it follows that $\nabla^{(A)}(\lambda)$ has a proper costandard filtration which ends with the quotient $\overline{\nabla}^{(A)}(\lambda)$. We apply F and it follows that there is an epimorphism from $F(C^{(A)}(\lambda))$ to $\overline{\Delta}(\lambda)$. Hence we conclude that the head of $F(C^{(A)}(\lambda))$ contains $L^{(R)}(\lambda)$, and therefore $P_0^{(R)} = P^{(R)}(\lambda) \oplus \hat{P}^{(R)}$, where $\hat{P}^{(R)}$ is some projective module. The lemma now follows by applying F^{-1} to

$$\cdots \to P_1^{(R)} \to P^{(R)}(\lambda) \oplus \hat{P}^{(R)} \to F(C^{(A)}(\lambda)) \to 0.$$

□

Proof of Lemma 4. Using the tilting resolutions of the indecomposable cotilting modules, constructed in Lemma 5, the proof is similar to that of [FM, Lemma 6]. □

Lemma 6. *Let $k = p.d.(H^{(A)})$. Under assumptions of Theorem 6 we have*

(i) $\mathrm{Ext}_A^k\left(H^{(A)}, \left(H^{(A)}\right)^\circ\right) \neq 0$;

(ii) $\mathrm{Ext}_A^i\left(H^{(A)}, \left(H^{(A)}\right)^\circ\right) = 0$ *for all* $i > 2p.d.(T^{(R)})$.

Proof. Choose a minimal projective resolution

$$0 \to P_k^{(A)} \to \cdots \to P_1^{(A)} \to P_0^{(A)} \to H^{(A)} \to 0$$

of $H^{(A)}$ and let

$$\mathcal{P}_1^\bullet : \qquad \cdots \to 0 \to P_k^{(A)} \to \cdots \to P_1^{(A)} \to P_0^{(A)} \to 0 \to \dots$$

be the corresponding complex in $\mathrm{Com}^b(\mathrm{add}(P^{(A)}))$. Choose also a minimal (possibly infinite) projective resolution

$$\cdots \to P^{(A)}(1) \to P^{(A)}(0) \to \left(H^{(A)}\right)^\circ \to 0$$

of $\left(H^{(A)}\right)^\circ$, and construct the corresponding (possibly infinite) complex

$$\mathcal{P}_2^\bullet : \qquad \cdots \to P^{(A)}(1) \to P^{(A)}(0) \to 0 \to \dots .$$

The characteristic tilting module $T^{(R)}$ has a standard filtration which starts with $\Delta^{(R)}$. To proceed we show that for all $\overline{\lambda} \in \overline{\Lambda}$ we have $R(\overline{\lambda}) \cong$

$A(\overline{\lambda})$ as stratified algebras. Let $\overline{\lambda} \in \overline{\Lambda}$. Then, by Proposition 1, and the usual duality, we have

$$A(\overline{\lambda}) = \mathrm{End}_A(\Delta^{(A)}(\overline{\lambda})) \cong \mathrm{End}_R(\nabla^{(R)}(\overline{\lambda})) \cong \\ \cong \mathrm{End}_{R^{\mathrm{opp}}}(\Delta^{(R^{\mathrm{opp}})}(\overline{\lambda})) = R^{\mathrm{opp}}(\overline{\lambda}).$$

Moreover, from the usual duality and the simple preserving duality we get

$$A(\overline{\lambda}) = \mathrm{End}_A(\Delta^{(A)}(\overline{\lambda})) \cong \\ \cong \mathrm{End}_A(\nabla^{(A)}(\overline{\lambda}))^{\mathrm{opp}} \cong \mathrm{End}_{A^{\mathrm{opp}}}(\Delta^{(A^{\mathrm{opp}})}(\overline{\lambda})) = A^{\mathrm{opp}}(\overline{\lambda}). \quad (2)$$

Hence $R(\overline{\lambda}) \cong A(\overline{\lambda})$ and thus the socle of $P^{(R(\overline{\lambda}))}(\lambda)$ contains a simple module $L^{(R(\overline{\lambda}))}(\lambda)$ for all $\lambda \in \Lambda$. Again, by the same argument as in Lemma 5, it follows that $\Delta^{(R)}$ has a proper standard filtration, which starts with the submodule $\overline{\Delta}^{(R)}$. Thus, by using F^{-1}, we obtain the short exact sequence

$$0 \to \overline{\nabla}^{(A)} \to H^{(A)} \to \mathrm{Coker} \to 0.$$

Applying $^\circ$ gives the short exact sequence

$$0 \to (\mathrm{Coker})^\circ \to \left(H^{(A)}\right)^\circ \to \overline{\Delta}^{(A)} \to 0.$$

It follows that the head of $\left(H^{(A)}\right)^\circ$ contains the head of $\overline{\Delta}^{(A)}$, which is isomorphic to $L^{(A)}$. Hence $P^{(A)}(0) = P^{(A)} \oplus Q^{(A)}$, where $Q^{(A)}$ is some projective module. Using the same arguments as in the proof of Lemma 3 we obtain $\mathrm{Hom}_{D^-(A)}(\mathcal{P}_1^\bullet, \mathcal{P}_2^\bullet[k]) \neq 0$ and hence $\mathrm{Ext}_A^k\left(H^{(A)}, \left(H^{(A)}\right)^\circ\right) \neq 0$. This proves the first statement.

To prove the second statement, it is enough to take the tilting resolution of $H^{(A)}$, apply duality, and use Lemma 4 to change the last resolution to a tilting complex. The necessary extensions then will be annihilated as a homomorphism in the derived category between two tilting complexes, which do not share places with non-zero components. □

Proof of Theorem 6. Let $k = \mathrm{p.d.}(H^{(A)})$ and $b = \mathrm{p.d.}(T^{(R)})$. From Lemma 6, we obtain that $\mathrm{Ext}_A^k\left(H^{(A)}, \left(H^{(A)}\right)^\circ\right) \neq 0$. From our assumptions, (2) and [B, Theorem 6.3] (see also chapter 6 in [Z]) it follows that $\mathrm{fin.dim}\big(A(\overline{\lambda})\big) = 0$ for all $\overline{\lambda} \in \overline{\Lambda}$. Hence we can apply Theorem 5 and conclude that $k = \mathrm{fin.dim}(A)$. From Lemma 3 and Lemma 6 it also follows that $k = 2b$. This completes the proof. □

To relate the finitistic dimension of A to the projective dimension of the characteristic tilting A-module we follow [FM, Proposition 4] under a stronger assumption.

Proposition 6. *Let A be as in Theorem 6 and assume that R also have simple preserving duality. Then fin.dim$(A) = 2$p.d.$(T^{(A)})$.*

5. Two-step duality for standardly stratified algebras

5.1. General theory

Let $(A, \preceq)$ be standardly stratified and $(R, \preceq_R)$ be weakly properly stratified. Define the *two-step dual algebra* of A via $B(A) = \mathrm{End}_A(H^{(A)})$ and the *two-step duality functor* $G : A$-mod $\to B(A)$-mod via $G(\_) = D \circ \mathrm{Hom}_A(\_, H^{(A)})$.

Theorem 7. *[(i)]*

1. *The algebra $(B(A)^{\mathrm{opp}}, \preceq)$ is standardly stratified and is isomorphic to End$_R(T^{(R)})$.*

2. *$B(A)^{\mathrm{opp}}$ has the Ringel dual $(R^{\mathrm{opp}}, \preceq_R)$, which is weakly properly stratified, and the algebra $B(B(A)^{\mathrm{opp}})^{\mathrm{opp}}$ is Morita equivalent to A.*

Proof. The proof is similar to that of [FM, Theorem 6]. □

Proposition 7. *Assuming* (1)*, the functor G induces an exact equivalence between the categories $\mathcal{P}(A)^{<\infty}$ and $\mathcal{I}(B(A))^{<\infty}$.*

Proof. The proof is similar to that of [FM, Proposition 5]. □

Define $N^* = D(N^{(B(A)^{\mathrm{opp}})})$ and $H^* = D(H^{(B(A)^{\mathrm{opp}})})$.

Proposition 8. *For every $\lambda \in \Lambda$ we have*

$$\begin{array}{ll} G(H^{(A)}(\lambda)) = I^{(B(A))}(\lambda), & G(N^{(A)}(\lambda)) = \nabla^{(B(A))}(\lambda), \\ G(T^{(A)}(\lambda)) = C^{(B(A))}(\lambda), & G(\Delta^{(A)}(\lambda)) = (N^*)^{(B(A))}(\lambda), \\ G(P^{(A)}(\lambda)) = (H^*)^{(B(A))}(\lambda). & \end{array}$$

Proof. The proof is similar to that of [FM, Proposition 6]. □

5.2. Investigation of the class $\mathcal{C}_2$

Let $(A, \preceq)$ be basic and weakly properly stratified such that $\mathrm{End}_A(T^{(A)}) \cong \mathrm{End}_A(C^{(A)})$ as stratified algebras. We have already seen in Subsection 3.2 that $\mathrm{End}_R(T^{(R)}) \cong \mathrm{End}_R(C^{(R)})$. Let $R(A) = \mathrm{End}_A(T^{(A)})$ and $R(A^{\mathrm{opp}}) = \mathrm{End}_{A^{\mathrm{opp}}}(T^{(A^{\mathrm{opp}})})$. From the assumption we get $R(A^{\mathrm{opp}}) \cong R(A)^{\mathrm{opp}}$. It follows that the Ringel duality functor F' for A^{opp} given by $F' : A^{\mathrm{opp}}\text{-mod} \to R(A^{\mathrm{opp}})\text{-mod}$ restricts to an exact equivalence between the categories $\mathcal{F}(\overline{\nabla}^{(A^{\mathrm{opp}})})$ and $\mathcal{F}(\overline{\Delta}^{(R(A^{\mathrm{opp}}))})$. From Proposition 1 we also have the contravariant functor $D \circ J : A\text{-mod} \to R(A^{\mathrm{opp}})\text{-mod}$, which restricts to an exact contravariant functor between $\mathcal{F}(\Delta^{(A)})$ and $\mathcal{F}(\Delta^{(R(A^{\mathrm{opp}}))})$. Since the $R(A)$ is weakly properly stratified, it follows that the image of $D \circ J$ is contained in $\mathcal{F}(\overline{\Delta}^{(R(A^{\mathrm{opp}}))})$. Thus we can form the composition

$$K = F^{-1} \circ D \circ J : \mathcal{F}(\Delta^{(A)}) \to \mathcal{F}(\overline{\nabla}^{(A^{\mathrm{opp}})}).$$

Lemma 7. *Let $(A, \preceq) \in \mathcal{C}_2$. Then the functor K induces an exact contravariant equivalence $K : \mathcal{F}(\Delta^{(A)}) \to \mathcal{F}(N^{(A^{opp})})$. Moreover, for all $\lambda \in \Lambda$ we have $K(\Delta^{(A)}(\lambda)) = N^{(A^{opp})}(\lambda)$ and $K(P^{(A)}(\lambda)) = H^{(A^{opp})}(\lambda)$.*

Proof. The proof follows immediately from the definitions of N and H. □

Using Proposition 3 we get $B(A) \cong A$ and in this case Proposition 8 and Proposition 7 state the following:

Proposition 9. *Let $(A, \preceq) \in \mathcal{C}_2$ and assume that (1) holds. Then G induces an exact equivalence between $\mathcal{P}(A)^{<\infty}$ and $\mathcal{I}(A)^{<\infty}$, and for every $\lambda \in \Lambda$ we have*

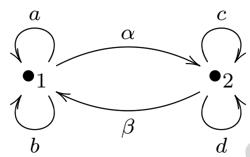
$$\begin{aligned} G(H^{(A)}(\lambda)) &= I^{(A)}(\lambda), \\ G(N^{(A)}(\lambda)) &= \nabla^{(A)}(\lambda), \\ G(T^{(A)}(\lambda)) &= C^{(A)}(\lambda), \\ G(\Delta^{(A)}(\lambda)) &= (N^*)^{(A)}(\lambda), \\ G(P^{(A)}(\lambda)) &= (H^*)^{(A)}(\lambda). \end{aligned}$$

Corollary 1. *Let $(A, \preceq)$ be as in Proposition 9. Then fin.dim(A) = fin.dim(A^{opp}).*

6. Examples

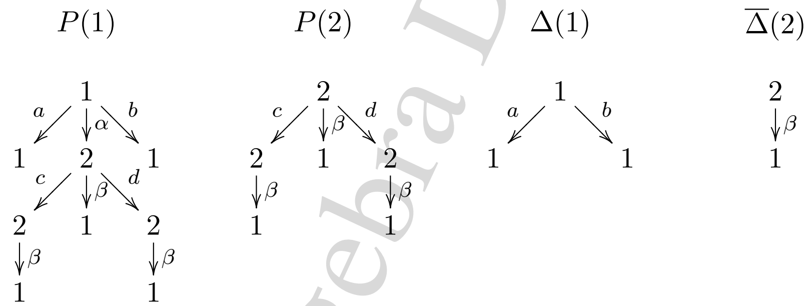
6.1. A properly stratified algebra with a simple preserving duality such that the Ringel dual is not properly stratified

Let $(A, \leq)$ be the quotient of the path algebra of the following quiver



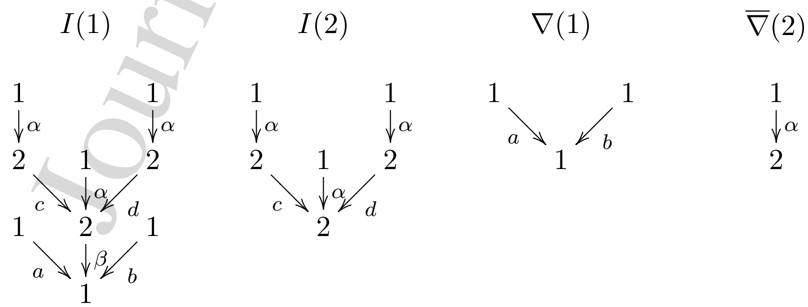
modulo relations $\alpha\beta = a\beta = b\beta = \alpha a = \alpha b = a^2 = b^2 = ab = ba = c^2 = d^2 = cd = dc = 0$.

We set $\{1 < 2\}$. The radical filtrations of $P(\lambda)$, $\Delta(\lambda)$, and $\overline{\Delta}(\lambda)$, $\lambda = 1, 2$, look as follows:



and $\Delta(2) = P(2)$, $\overline{\Delta}(1) = L(1)$. It follows that A is properly stratified. The algebra A has a simple preserving duality, associated with the anti-isomorphism, defined via $\alpha \mapsto \beta$, $\beta \mapsto \alpha$, $a \mapsto b$, $b \mapsto a$, $c \mapsto d$ and $d \mapsto c$.

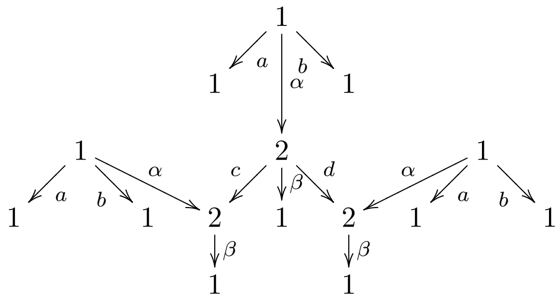
The modules $I(\lambda)$, $\nabla(\lambda)$, and $\overline{\nabla}(\lambda)$, $\lambda = 1, 2$, have the following socle filtrations:



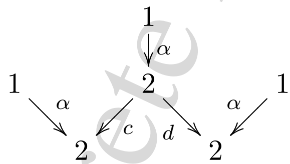
and $\nabla(2) = I(2)$, $\overline{\nabla}(1) = L(1)$.

The modules $T(\lambda)$, $S(\lambda)$ and $N(\lambda)$ have the following radical filtration:

$T(2)$



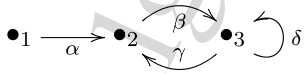
$N(2)$



and $T(1) = N(1) = \Delta(1)$, $S(1) = 0$, $S(1) = L(1)^9$. It follows immediately from [FM, Theorem 1] that the Ringel dual of A is not properly stratified. Thus, in particular, the Ringel dual does not have a simple preserving duality.

6.2. Computation of the finitistic dimension with Theorem 5

Let $(A, \preceq)$ be the quotient of the path algebra of the following quiver

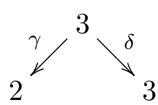


modulo relations $\delta\beta = \delta^2 = \gamma\delta = \beta\gamma = 0$. We set $\Lambda = \{1 \prec 2 \sim 3\}$. The radical filtrations of $P(\lambda)$, $\Delta(\lambda)$, and $\overline{\Delta}(\lambda)$, $\lambda = 1, 2, 3$, look as follows:

$P(1)$

$P(2)$

$P(3)$

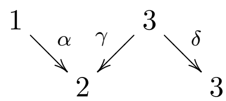


and $\Delta(1) = L(1)$, $\Delta(2) = P(2)$, $\Delta(3) = P(3)$. The proper standard modules are all simple. It follows that A is weakly properly stratified.

Note that there is no partial order on Λ such that A becomes an SSS-algebra.

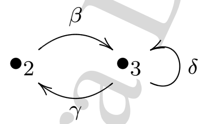
The modules $I(\lambda)$, $\nabla(\lambda)$, and $\overline{\nabla}(\lambda)$, $\lambda = 1, 2, 3$, have the socle filtrations, which are dual to the corresponding radical filtrations above. The modules $T(\lambda)$, $C(\lambda)$, $S(\lambda)$, $N(\lambda)$ and $H(\lambda)$ have the following radical filtration:

$T(3)$



and $T(1) = L(1)$, $T(2) = I(2)$. We also have $C(\lambda) = I(\lambda)$, $S(\lambda) = 0$ and $N(\lambda) = T(\lambda)$ for $\lambda = 1, 2, 3$. Thus the Ringel dual is properly stratified and we can also conclude that $H(\lambda) = T(\lambda)$ for $\lambda = 1, 2, 3$.

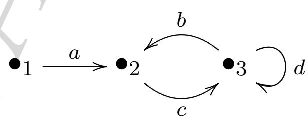
By a simple calculation we see that $\text{p.d.}(T) = 1$. Moreover, the algebra $A(\bar{1})$ is simple and the algebra $A(\bar{2})$ is given by



modulo relations $\delta\beta = \delta^2 = \gamma\delta = 0$. We have immediately that $\text{fin.dim}(A(\bar{1})) = 0$ and from [Z, Observation 8] it follows that $\text{fin.dim}\big(A(\bar{2})\big) = 0$. Thus we can apply Theorem 5 and obtain

$$\text{fin.dim(A)=p.d.(H)=1.}$$

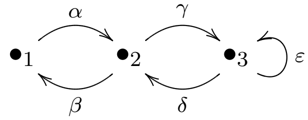
By a simple calculation we also observe that $\text{End}_A(T) \cong \text{End}_A(C)$ and the later is the quotients of the path algebra of the following quiver



modulo relations $cb = bd = dc = d^2 = 0$. Note also that $T \neq C$ and hence the class $\mathcal{C}_2 \setminus \mathcal{C}_1$ is non-empty. From Corollary 1 we can also conclude that $\text{fin.dim}(A^{\text{opp}}) = 1$.

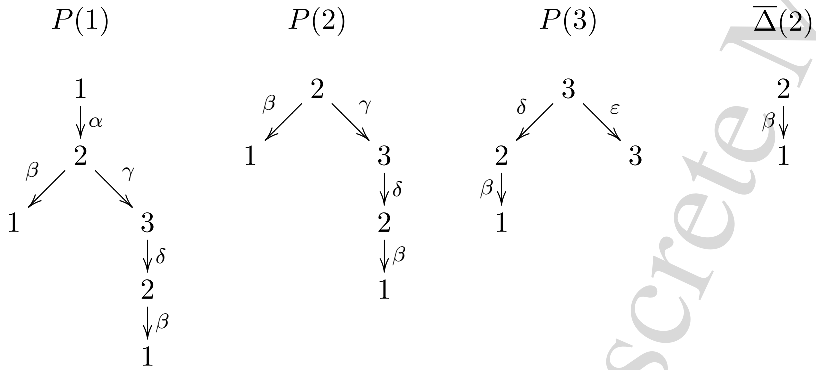
6.3. Computation of the finitistic dimension with Proposition 6

Let $(A, \preceq)$ be the quotient of the path algebra of the following quiver



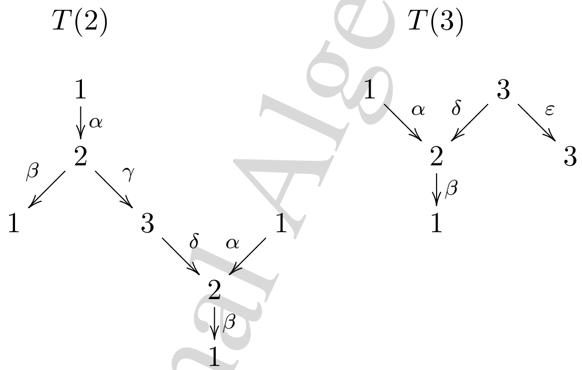
modulo relations $\alpha\beta = \varepsilon^2 = \varepsilon\gamma = \delta\varepsilon = \gamma\delta = 0$. We set $\Lambda = \{1 \prec 2 \sim 3\}$.

The radical filtrations of $P(\lambda)$, $\Delta(\lambda)$, and $\overline{\Delta}(\lambda)$, $\lambda = 1, 2, 3$, look as follows:

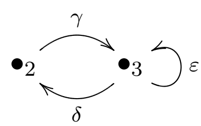


and $\Delta(1) = \overline{\Delta}(1) = L(1)$, $\Delta(2) = P(2)$, $\Delta(3) = P(3)$, $\overline{\Delta}(3) = L(3)$. It follows that A is weakly properly stratified. Note that there is no partial order on Λ such that A becomes an SSS-algebra. The algebra A has a simple preserving duality, associated with the anti-isomorphism defined via $\alpha \mapsto \beta$, $\beta \mapsto \alpha$, $\gamma \mapsto \delta$, $\delta \mapsto \gamma$ and $\varepsilon \mapsto \varepsilon$.

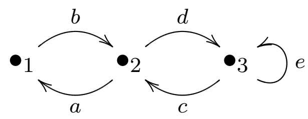
The modules $I(\lambda)$, $\nabla(\lambda)$, and $\overline{\nabla}(\lambda)$, $\lambda = 1, 2, 3$, have the following socle filtrations which are dual to the corresponding radical filtrations above. The modules $T(\lambda)$ have the following radical filtration:



and $T(1) = L(1)$. We see that the algebra $A(\bar{1})$ is simple and the algebra $A(\bar{2})$ is the quotient of the path algebra of the following quiver



modulo relations $\varepsilon^2 = \varepsilon\gamma = \delta\varepsilon = \gamma\delta = 0$. We have immediately that $\text{fin.dim}(A(\bar{1})) = 0$ and from [Z, Observation 8.] it follows that fin.dim$(A(\bar{2})) = 0$. By a direct computation we also obtain the Ringel dual R is the quotient of the path algebra of the following quiver



modulo relations $e^2 = ce = ed = dc = ab = acdb = 0$. Then $(R, \preceq_R)$ is weakly properly stratified and has a simple preserving duality. Since p.d.$(T^{(A)}) = 1$, from Proposition 6 we obtain fin.dim$(A) = 2$p.d.$(T^{(A)}) = 2$.

6.4. For each partial pre-order there exists a weakly properly stratified algebras with a simple preserving duality

Every algebra (in the sense of Section 2) is standardly stratified if we choose the full relation to be the partial pre-order. In fact, it is even weakly properly stratified. A natural question is if the opposite is true, that is, if we are given a finite partially pre-ordered set X, does there exist a weakly properly stratified algebra with indexing set X? This question has a positive answer for the less general case when X is partially ordered, see [DX1, DX2]. We will show that the question has a positive answer even in the general situation.

Let X be a finite partially pre-ordered set. For $x, y \in X$ we set

$$x \prec_0 y \text{ if and only if } x \prec y \text{ and there is no } z \in X \text{ such that } x \prec z \prec y.$$

Define the graph $H = (H_0, H_1)$ with

$$H_0 = X \text{ and } H_1 = \{x \leftarrow y \mid x \prec_0 y \text{ for } x, y \in X\}.$$

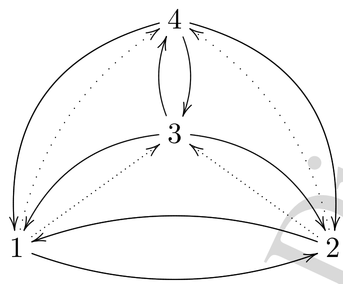
Note that H is the Hasse diagram in the case when $\preceq$ is a partial order. Define the quiver $Q = (Q_0, Q_1)$ as follows: $Q_0 = H_0 = X$, and $Q_1 = H_1 \cup \{a^{\mathrm{op}} : x \to y \mid a : x \leftarrow y \in H_1\} \cup K$, where $K = \{x \leftarrow y \mid x \neq y \text{ and } x \sim y \text{ and } x, y \in X\}$. Analogously to [DX1, DX2] we define the *dual extension algebra* $A(X)$ as $A(X) = \Bbbk Q/I$, where I is the ideal in $\Bbbk Q$ generated by $\{b^{\mathrm{op}}a \mid a, b \in H_1\} \cup \{ab \mid a, b \in K\} \cup \{c^{\mathrm{op}}ba \mid a, c \in H_1 \text{ and } b \in K\}$.

Theorem 8. *The algebra $A(X)$ is weakly properly stratified and has a simple preserving duality.*

Proof. The algebra $A(X)$ has a simple preserving duality which arises from the anti-isomorphism defined via $a \leftrightarrow a^{\mathrm{op}}$, $a \in H_1 \cup K$.

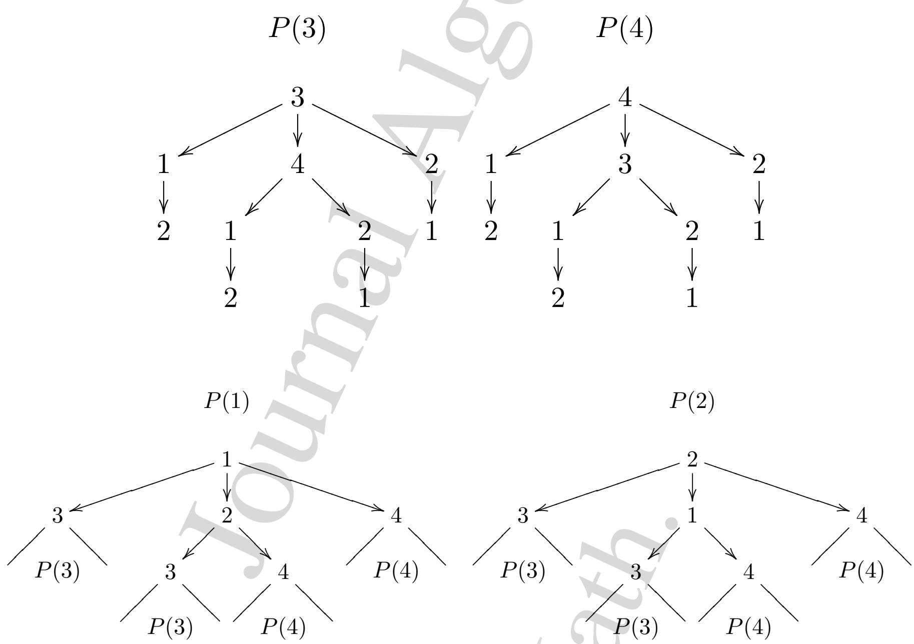
Now we show that $A(X)$ is weakly properly stratified. Let $\lambda \in X$. We define $\Delta(\lambda) = P(\lambda)/\operatorname{Tr}_{P^{\succ\lambda}}(P(\lambda))$ and $\overline{\Delta}(\lambda) = P(\lambda)/\operatorname{Tr}_{P^{\succeq\lambda}}(\operatorname{rad}(P(\lambda)))$, where $P^{\succ\lambda} = \oplus_{\mu\succ\lambda}P(\mu)$ and $P^{\succeq\lambda} = \oplus_{\mu\succeq\lambda}P(\mu)$. Following the paths in Q it is easy to check that all the conditions (wPS1), (wPS2), (wPS3) and (wPS4) are satisfied. This completes the proof. □

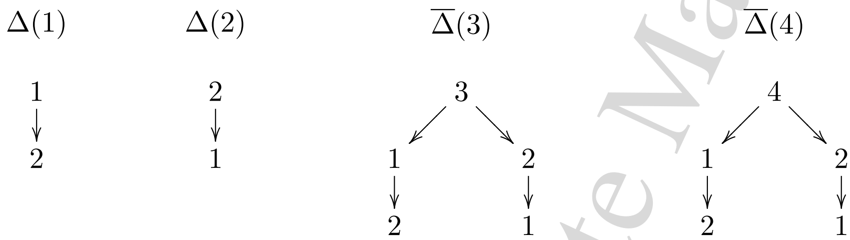
Let us give an example. We take $X = \{1 \sim 2 \prec 3 \sim 4\}$. Then the dual extension algebra $A(X)$ is the path algebra of the quiver Q:



modulo the relations above. Here the dotted arrows denote a^{op} for $a \in H_1$.

The radical filtrations of $P(\lambda)$, $\Delta(\lambda)$, and $\overline{\Delta}(\lambda)$, $\lambda = 1, 2, 3, 4$, look as follows:





and $\Delta(4) = P(4)$, $\Delta(3) = P(3)$, $\overline{\Delta}(1) = L(1)$ and $\overline{\Delta}(2) = L(2)$. We obtain that $A(X)$ is weakly properly stratified and has a simple preserving duality.

We remark that for a given standardly (weakly properly) stratified algebra $(A, \preceq)$ we can always assume that the induced partial order on $\overline{\Lambda}$ is linear. More precisely we have:

Lemma 8. *Let $(A, \preceq)$ be a standardly (weakly properly) stratified algebra and denote the induced partial order on $\overline{\Lambda}$ by $\leq$. Let $\leq_0$ be a linear order on $\overline{\Lambda}$ extending $\leq$ (i.e. $\overline{\lambda} \leq \overline{\mu} \Longrightarrow \overline{\lambda} \leq_0 \overline{\mu}$) and define the partial pre-order $\preceq_0$ on Λ by*

$$\lambda \preceq_0 \mu \Longleftrightarrow \overline{\lambda} \leq_0 \overline{\mu}.$$

Then $(A, \preceq_0)$ is a standardly (weakly properly) stratified algebra.

Proof. Let $\preceq_0$ be defined as above. Note that for all $\lambda, \mu \in \Lambda$ we have: $\lambda \preceq \mu \Longrightarrow \lambda \preceq_0 \mu$ and $\lambda \prec \mu \Longrightarrow \lambda \prec_0 \mu$. Let $\lambda \in \Lambda$ be arbitrary. Then, from the definition of $\Delta(\lambda)$ and $\Delta^0(\lambda)$, we obtain the short exact sequence

$$0 \to \mathrm{Tr}_{P^{\succ\lambda}}(P(\lambda)) \to \mathrm{Tr}_{P^{\succ_0\lambda}}(P(\lambda)) \to K \to 0, \tag{3}$$

where $K = \mathrm{Tr}_{P^{\succ_0\lambda}}(P(\lambda))/\,\mathrm{Tr}_{P^{\succ\lambda}}(P(\lambda)) \subset \Delta(\lambda)$. Let $\mu \succ_0 \lambda$ and apply $\mathrm{Hom}_A(P(\mu), {}_-)$ to (3). Since $\mathrm{Hom}_A(P(\mu), K) = 0$ (otherwise $\mu \preceq_0 \lambda$, which is a contradiction) we conclude that $K = 0$ and $\Delta(\lambda) = \Delta^0(\lambda)$. It follows that the conditions (SS1) and (SS2) are satisfied and thus $(A, \preceq_0)$ is standardly stratified. In the case when $(A, \preceq)$ is a weakly properly stratified algebra we conclude, using similar arguments, that $(A, \preceq_0)$ is weakly properly strafied and $\overline{\Delta}(\lambda) = \overline{\Delta}^0(\lambda)$. This proves the lemma. □

6.5. Tensor product of stratified algebras

Here we generalize the construction from [FM, Section 9.5] and show that the tensor product of two stratified algebras carries a natural structure of a stratified algebra.

Lemma 9. *Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be standardly stratified (resp. SSS-algebras). Set $A = A_1 \otimes_{\Bbbk} A_2$ and define the partial pre-order (resp. order) $\preceq$ on $\Lambda = \Lambda_1 \times \Lambda_2$ via*

$$(\lambda_1, \lambda_2) \preceq (\mu_1, \mu_2) \text{ if and only if } \lambda_1 \preceq_1 \mu_1 \text{ and } \lambda_2 \preceq_2 \mu_2.$$

Then $(A, \preceq)$ is standardly stratified (resp. SSS-algebra). Moreover, in the case when $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ are weakly properly stratified (resp. properly stratified), the algebra $(A, \preceq)$ is also weakly properly stratified (resp. properly stratified).

Proof. Let $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ be standardly stratified and choose $(\lambda_1, \lambda_2) \in \Lambda$. Then $P^{(A)}(\lambda_1, \lambda_2) = P^{(A_1)}(\lambda_1) \otimes_{\Bbbk} P^{(A_2)}(\lambda_2)$ and $L^{(A)}(\lambda_1, \lambda_2) = L^{(A_1)}(\lambda_1) \otimes_{\Bbbk} L^{(A_2)}(\lambda_2)$. We set $\Delta^{(A)}(\lambda_1, \lambda_2) = \Delta^{(A_1)}(\lambda_1) \otimes_{\Bbbk} \Delta^{(A_2)}(\lambda_2)$ and $\overline{\Delta}^{(A)}(\lambda_1, \lambda_2) = \overline{\Delta}^{(A_1)}(\lambda_1) \otimes_{\Bbbk} \overline{\Delta}^{(A_2)}(\lambda_2)$.

Since the tensor product is over a field the functors $M \otimes_{\Bbbk} {}_- : A_2\text{-mod} \to A\text{-mod}$ and ${}_- \otimes_{\Bbbk} N : A_1\text{-mod} \to A\text{-mod}$, with $M \in A_1$-mod and $N \in A_2$-mod, are always exact. Hence, using ${}_- \otimes_{\Bbbk} P^{(A_2)}(\nu_2)$ and $\Delta^{(A_1)}(\nu_1) \otimes_{\Bbbk} {}_-$, we obtain the short exact sequence

$$0 \to X \to P^{(A)}(\lambda_1, \lambda_2) \to \Delta^{(A)}(\lambda_1, \lambda_2) \to 0,$$

where X has a filtration with subquotients $\Delta^{(A)}(\mu_1, \mu_2)$, $(\mu_1, \mu_2) \succ (\lambda_1, \lambda_2)$. Moreover, using $\Delta^{(A_1)}(\nu_1) \otimes_{\Bbbk} {}_-$ and ${}_- \otimes_{\Bbbk} L^{(A_2)}(\nu_2)$, we get the short exact sequence

$$0 \to Y \to \Delta^{(A)}(\lambda_1, \lambda_2) \to L^{(A)}(\lambda_1, \lambda_2) \to 0,$$

where Y has a filtration with subquotients $L^{(A)}(\mu_1, \mu_2)$, $(\mu_1, \mu_2) \preceq (\lambda_1, \lambda_2)$. Then both conditions (SS1) and (SS2) are satisfied and so $(A, \preceq)$ is standardly stratified. In the case when $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ are weakly properly stratified $\Delta^{(A)}(\lambda_1, \lambda_2)$ has a filtration with modules $\overline{\Delta}^{(A)}(\lambda_1, \lambda_2)$, $(\mu_1, \mu_2) \sim (\lambda_1, \lambda_2)$, and we obtain a short exact sequence

$$0 \to Z \to \overline{\Delta}^{(A)}(\lambda_1, \lambda_2) \to L^{(A)}(\lambda_1, \lambda_2) \to 0,$$

where Z has a filtration with subquotients $L^{(A)}(\mu_1, \mu_2)$, $(\mu_1, \mu_2) \prec (\lambda_1, \lambda_2)$. Hence the conditions (wPS1), (wPS2),(wPS3) and (wPS4) are all satisfied and so $(A, \preceq)$ is weakly properly stratified. Finally, if both $(A_1, \preceq_1)$ and $(A_2, \preceq_2)$ are SSS-algebras (resp. properly stratified), then, by the same arguments as above, we conclude that $(A, \preceq)$ is also an SSS-algebra (resp. properly stratified). □

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