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Identities related to integer partitions and complete Bell polynomials

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Abstract. Using the (universal) Theorem for the integer partitions and the q-binomial Theorem, we give arithmetical and combinatorial identities for the complete Bell polynomials as generating functions for the number of partitions of a given integer into k parts and the number of partitions of n into a given number of parts.

Introduction

The (exponential) partial Bell polynomials $B_{n,k}(x_1, x_2, \ldots)$ are defined by their generating function as follows

$$\sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \ldots)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k, \tag{1}$$

and the (exponential) complete Bell polynomials $A_n(x_1, x_2, \ldots)$ are defined by their generating function

$$1 + \sum_{n=1}^{\infty} A_n(x_1, x_2, \ldots)\frac{t^n}{n!} = \exp\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right) \text{ with } A_0(x_1, x_2, \ldots) = 1.$$

Comtet [13] gave an important impulsion for the development of Bell polynomials. The first author uses polynomials of binomial type, in [16,17],

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to give applications related to congruences and to determine inverse relations, see [15, 18]. For the exponential bipartitional polynomials and polynomial sequences of trinomial type we refer to [6, 19], and for The exponential multipartitional polynomials and polynomial sequences of multinomial type, we refer to [7, 20]. Belbachir et al. give connection of Bell polynomials with ordinary multinomials in [5], see also [4]. Collins [12] gives some applications in integration, Germano and Martinelli [14] in generalized Blissard problem and some others.

In this paper, using the (universal) Theorem for the integer partitions (see [1, 2]), q-binomial Theorem, we give properties and identities for the complete Bell polynomials as generating functions for the number of partitions of a given integer into k parts and for the number of partitions of n into a fixed given number of parts.

1. Complete Bell polynomials, integer partitions and q-binomial Theorem

1.1. Complete Bell polynomials and integer partitions. Let $A=(a_{i,j})$, $i=1,2,\ldots$, $j=0,1,2,\ldots$ be an infinite matrix with elements $a_{i,j}\in\{0,1\}$ and $Y_i=\{j : a_{i,j}=1\}$ for $i=1,2,\ldots$, we denote by $p(n,k;A)$ the number of partitions of n into k parts whose number y_i of parts that are equal to i belongs to the set Y_i.

A first use of partition's (universal) Theorem is given by the following.

Theorem 1. *Let*

$$\rho_n(q;A) := \sum_{i=1}^{\infty} b_n(i)q^{ni}, \quad |q|<1,$$

with $b_n(i) := \sum_{k=1}^{n}(-1)^{k-1}(k-1)!B_{n,k}(1!a_{i,1}, 2!a_{i,2}, \ldots, j!a_{i,j}, \ldots)$.

Then, for $a_{i,0}=1$ $(i\geqslant 1)$, *we have*

$$A_n(\rho_1(q;A),\ldots,\rho_n(q;A)) = n!\sum_{j=n}^{\infty} p(j,n;A)\,q^j, \quad |q|<1. \tag{2}$$

Proof. From [11, Thm. 10.3] (see also [2] and [13, Thm. B, p. 98]) we have

$$G(q,u;A) := \sum_{n=0}^{\infty}\sum_{k=0}^{n} p(n,k;A)\,u^k q^n = \prod_{i=1}^{\infty}\left(\sum_{j=0}^{\infty} a_{i,j}u^j q^{ij}\right). \tag{3}$$

For $a_{i,0} = 1$ $(i \geqslant 1)$, the last identity becomes

$$G(q,u;A) = \prod_{i=1}^{\infty}\left(1 + \sum_{j=1}^{\infty} a_{i,j} u^j q^{ij}\right), \quad |q| < 1, \quad |uq| < 1/2. \tag{4}$$

Then from [11, Thm. 11.7] (see also [13]), we have

$$\ln\left(1 + \sum_{k=1}^{\infty} g_k \frac{q^k}{k!}\right) = \sum_{n=1}^{\infty} c_n \frac{q^n}{n!}, \tag{5}$$

with $c_n = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! B_{n,k}(g_1, g_2, \ldots)$. We have

$$\begin{aligned} G(q,u;A) &= \exp\left(\sum_{i=1}^{\infty} \ln\left(1 + \sum_{j=1}^{\infty} a_{i,j} u^j q^{ij}\right)\right) \\ &= \exp\left(\sum_{i=1}^{\infty} \ln\left(1 + \sum_{j=1}^{\infty} (j! a_{i,j} q^{ij}) \frac{u^j}{j!}\right)\right) = \exp\left(\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} q^{ki} b_k(i) \frac{u^k}{k!}\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{u^k}{k!} \sum_{i=1}^{\infty} b_k(i) q^{ki}\right) = \exp\left(\sum_{k=1}^{\infty} \rho_k(q;A) \frac{u^k}{k!}\right) \\ &= 1 + \sum_{k=1}^{\infty} A_k\left(\rho_1(q;A), \rho_2(q;A), \ldots, \rho_k(q;A)\right) \frac{u^k}{k!}, \end{aligned}$$

which implies (2). □

Corollary 1. *We have*

$$\begin{aligned} A_n\left(\frac{0!}{1-q}, \frac{1!}{1-q^2}, \ldots, \frac{(n-1)!}{1-q^n}\right) &= \frac{n!}{\prod_{i=1}^{n}(1-q^i)}, \; |q| < 1, \\ A_n\left(-\frac{0!}{1-q}, -\frac{1!}{1-q^2}, \ldots, -\frac{(n-1)!}{1-q^n}\right) &= \frac{n!\,(-1)^n q^{n(n-1)/2}}{\prod_{i=1}^{n}(1-q^i)}, \; |q| > 1. \end{aligned} \tag{6}$$

Proof. Let $a_{i,j} = 1$ for every $i = 1, 2, \ldots$ and $j = 0, 1, \ldots$ in Theorem 1. We have $p(n;A) = \sum_{i=0}^{n} p(n,k;A)$ represents the number of all partitions of n, and so $p(n,k;A) = p(n,k)$ the number of all partitions of n into k parts. Then, using the well known identity,

$$B_{n,k}(1!, 2!, 3!, \ldots) = \frac{(n-1)!}{(k-1)!}\binom{n}{k}, \tag{7}$$

we get $b_n(i) = (n-1)!$ and $\rho_n(q;A) = (n-1)! \frac{q^n}{1-q^n}$.

Then, using (2) and from the well-known identity (see [11, Thm. 10.2])

$$\sum_{j=n}^{\infty} p\,(j,n)\,q^j = q^n \prod_{i=1}^{n}\left(1-q^i\right)^{-1}, \tag{8}$$

we get

$$A_n\left(\frac{0!q}{1-q},\frac{1!q^2}{1-q^2},\ldots,\frac{(n-1)!q^n}{1-q^n}\right) = n!q^n\prod_{i=1}^{n}\left(1-q^i\right)^{-1},\quad |q|<1,$$

and from the property of complete Bell polynomials

$$A_n\left(ax_1,a^2x_2,\ldots,a^nx_n\right) = a^nA_n\left(x_1,x_2,\ldots,x_n\right) \tag{9}$$

we obtain the first identity of (6).

Now, for $|q|>1$ the first identity (6) becomes

$$A_n\left(\frac{0!}{1-q^{-1}},\frac{1!}{1-q^{-2}},\ldots,\frac{(n-1)!}{1-q^{-n}}\right) = n!\prod_{i=1}^{n}\left(1-q^{-i}\right)^{-1},\ |q|>1,$$

and this gives, in virtue of (9), the second identity of (6). □

Corollary 2. *We have*

$$\begin{aligned} A_n\left(-\frac{0!}{1-q},-\frac{1!}{1-q^2},\ldots,-\frac{(n-1)!}{1-q^n}\right) &= \frac{n!\,(-1)^n\,q^{n(n-1)/2}}{\prod_{i=1}^{n}(1-q^i)},\ |q|<1,\\ A_n\left(\frac{0!}{1-q},\frac{1!}{1-q^2},\ldots,\frac{(n-1)!}{1-q^n}\right) &= \frac{n!}{\prod_{i=1}^{n}(1-q^i)},\ |q|>1. \end{aligned} \tag{10}$$

Proof. For $i=1,2,\ldots$, let $a_{i,j}=1$ for $j=0,1$ and $a_{i,j}=0$ for $j=2,3,\ldots$ in Theorem 1. We have, $p\,(n;A)=\sum_{k=0}^{n}q(n,k)$ represents the number of partitions of n into unequal parts, and so, $q(n,k)$ represents the number of partitions of n into k unequal parts. Then, using the identity

$$B_{n,n}\,(1!,0,0,\ldots)=1 \ \text{ and } \ B_{n,k}\,(1!,0,0,\ldots)=0 \ \text{ if } \ k\neq n, \tag{11}$$

we get $b_n(i)=(-1)^{n-1}\,(n-1)!$ and $\rho_n(q;A)=(-1)^{n-1}\,(n-1)!\frac{q^n}{1-q^n}$.

The first identity of (10) results from (2) and from the well-known identity (see [11, Exp. 10.2]),

$$\sum_{j=n}^{\infty} q\,(j,n)\,q^j = q^{n(n+1)/2}\prod_{i=1}^{n}\left(1-q^i\right)^{-1}. \tag{12}$$

Now, for $|q| > 1$ the first identity of (10) becomes

$$A_n\left(-\frac{0!}{1-q^{-1}},-\frac{1!}{1-q^{-2}},\ldots,-\frac{(n-1)!}{1-q^{-n}}\right)=\frac{n!\,(-1)^n\,q^{-n(n-1)/2}}{\prod\limits_{i=1}^{n}\left(1-q^{-i}\right)},$$

and this gives, in virtue of (9), the second identity of (10). □

Remark 1. From Corollaries 1 and 2, we deduce the following

$$A_n\left(\varepsilon\frac{0!}{1-q},\varepsilon\frac{1!}{1-q^2},\ldots,\varepsilon\frac{(n-1)!}{1-q^n}\right)=\frac{n!q^{\left(\frac{\varepsilon-|\varepsilon|}{2}\right)n}}{\prod\limits_{i=1}^{n}\left(1-q^{\varepsilon i}\right)},\ |q|\neq 1,\ \varepsilon=\pm 1.$$

Corollary 3. *Let $p_{r,s}(n,k)$ be the number of partitions of n into k parts in form $(s \mod r)$, $1 \leqslant s \leqslant r-1$. Then, for $|q| < 1$, we have*

$$A_n\left(0!\frac{q^s}{1-q^r},1!\frac{q^{2s}}{1-q^{2r}},\ldots,(n-1)!\frac{q^{ns}}{1-q^{nr}}\right)=n!\sum_{j=n}^{\infty}p_{r,s}\left(j,n\right)q^j. \quad (13)$$

Proof. $a_{i,0}=1$, for $i=1,2,\ldots$ let $a_{ir+s,j}=1$ and $a_{ir+s',j}=0$, $s'\neq s$, for $j=1,2,3,\ldots$ in Theorem 1, we get $p\left(n,k;A\right)=p_{r,s}(n,k)$. Then, using the identity (7) and the identity $B_{n,k}\left(0,0,\ldots 0\right)=0$, we get $b_n\left(ir+s\right)=(n-1)!$, $b_n\left(ir+s'\right)=0$, $\left(s'\neq s\right)$, and $\rho_n(q;A)=(n-1!)\,\frac{q^{ns}}{1-q^{nr}}$. Then (13) follows from identity (2). □

1.2. Complete Bell polynomials and q-binomial Theorem.

We give the link between the q-binomial Theorem and the complete Bell polynomials.

Theorem 2. *We have*

$$\begin{aligned} A_n\left(0!\frac{1-a}{1-q},\ldots,(n-1)!\frac{1-a^n}{1-q^n}\right)&=n!\frac{(a;q)_n}{(q;q)_n},\ |q|<1,\\ A_n\left(-0!\frac{1-a}{1-q},\ldots,-(n-1)!\frac{1-a^n}{1-q^n}\right)&=n!\prod_{j=1}^{n}\frac{a-q^{j-1}}{1-q^j},\ |q|>1, \end{aligned} \quad (14)$$

where $(a;q)_n=\prod\limits_{j=0}^{n-1}\left(1-aq^j\right)$ *for* $n\geqslant 1$ *and* $(a;q)_0=1$.

Proof. From the q-binomial Theorem ([9, Ch. 16]),

$$\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n=\frac{(ax;q)_\infty}{(x;q)_\infty},\ |q|<1,\ |x|<1,$$

where $(a;q)_\infty=\prod_{j=0}^{\infty}\left(1-aq^j\right)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n&=\prod_{j=0}^{\infty}\left(1-axq^j\right)\left(1-xq^j\right)^{-1}\\&=\exp\left(\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}\frac{x^i}{i}\left(-a^iq^{ij}+q^{ij}\right)\right)\\&=\exp\left(\sum_{i=1}^{\infty}\frac{x^i}{i}\left(\frac{1-a^i}{1-q^i}\right)\right)\\&=1+\sum_{n=1}^{\infty}A_n\left(0!\frac{1-a}{1-q},\ldots,(n-1)!\frac{1-a^n}{1-q^n}\right)\frac{x^n}{n!}.\end{aligned}$$

Then we obtain the first identity of (14) when $|q|<1$.

Now, for $|q|>1$, the first identity of (14) becomes

$$A_n\left(0!\frac{1-a}{1-q^{-1}},1!\frac{1-a^2}{1-q^{-2}},\ldots,(n-1)!\frac{1-a^n}{1-q^{-n}}\right)=n!\prod_{j=1}^{n}\frac{1-aq^{-(j-1)}}{1-q^{-j}},$$

and using (9), this identity can be written as the second identity of (14). □

1.3. Complete Bell polynomials and multivariate Lagrange polynomials. The multivariate Lagrange polynomials are introduced and investigated systematically by Chan et al. [10]. In [3], Altin et al. suggest a multivariate q-Lagrange polynomials as follows

$$\prod_{i=1}^{r}\frac{1}{(x_it;q)_{\alpha_i}}=\sum_{n=0}^{\infty}g_{n,q}^{(\alpha_1,\ldots,\alpha_r)}\left(x_1,\ldots,x_r\right)t^n, \tag{15}$$

where $|t|<\min\left\{|x_1|^{-1},\ldots,|x_r|^{-1}\right\}$, $0<|q|<1$, $(\lambda;q)_\mu:=\frac{(\lambda;q)_\infty}{(\lambda q^\mu;q)_\infty}$ with $(\lambda;q)_\infty:=\prod_{k=0}^{\infty}\left(1-\lambda q^k\right)$.

This yields the following explicit representation

$$g_{n,q}^{(\alpha_1,\ldots,\alpha_r)}\left(x_1,\ldots,x_r\right)=\sum_{k_1+\cdots+k_r=n}\prod_{i=1}^{k}(q^{\alpha_i};q)_{k_i}\frac{x_1^{k_i}}{(q;q)_{k_i}}.$$

The next theorem gives an another expression for $g_{n,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r)$ in terms of complete Bell polynomials.

Theorem 3. *Link with multivariate q-Lagrange polynomials. We have*

$$n!g_{n,q}^{(\alpha_1,\ldots,\alpha_r)}(x_1,\ldots,x_r) = A_n\left(\frac{0!}{1-q}\sum_{i=1}^{r}(1-q^{\alpha_i})\,x_i,\ldots,\frac{(n-1)!}{1-q^n}\sum_{i=1}^{r}(1-q^{n\alpha_i})\,x_i^n\right). \tag{16}$$

Proof. For $y_n(i) := (n-1)!\frac{1-q^{n\alpha_i}}{1-q^n}x_i^n,\ i=1,2,\ldots,r$, the identity (16) follows from (15) and from the expansion of

$$\prod_{i=1}^{r}\frac{1}{(x_it;q)_{\alpha_i}} = \exp\left(\sum_{j=1}^{\infty}(y_j(1)+\cdots+y_j(r))\frac{t^j}{j!}\right) = \sum_{n=0}^{\infty}A_n\left(\sum_{i=1}^{r}y_1(i),\ldots,\sum_{i=1}^{r}y_n(i)\right)\frac{t^n}{n!}.$$

□

2. Complete Bell polynomials and integer partitions

As a second use of partition's (universal) Theorem, we obtain

Theorem 4. *Let*

$$\sigma_n(u;A) := n!\sum_{k|n}b_k\binom{n}{k}\frac{u^k}{k!}.$$

Then for $a_{i,0}=1$ $(i\geqslant 1)$, we have

$$A_n(\sigma_1(u;A),\sigma_2(u;A),\ldots,\sigma_n(u;A)) = n!\sum_{j=0}^{n}p(n,j;A)\,u^j. \tag{17}$$

Proof. From [11, Thm. 11.17] (see also [13]), we have

$$\ln\left(1+\sum_{k=1}^{\infty}g_k\frac{t^k}{k!}\right) = \sum_{n=1}^{\infty}c_n\frac{t^n}{n!},$$

with $c_n := \sum_{k=1}^{n}(-1)^{k-1}(k-1)!B_{n,k}(g_1,g_2,\ldots)$.

Then, by (3) and (4) we obtain for $|t| < 1$ and $|ut| < 1/2$

$$\begin{aligned} G(t,u;A) &= \prod_{i=1}^{\infty}\left(1+\sum_{j=1}^{\infty}a_{i,j}u^j t^{ij}\right) \\ &= \exp\left(\sum_{i=1}^{\infty}\ln\left(1+\sum_{j=1}^{\infty}(j!a_{i,j})\frac{(ut^i)^j}{j!}\right)\right) = \exp\left(\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}b_k(i)u^k\frac{t^{ik}}{k!}\right) \\ &= \exp\left(\sum_{n=1}^{\infty}t^n\sum_{k|n}b_k\left(\frac{n}{k}\right)\frac{u^k}{k!}\right) = \exp\left(\sum_{n=1}^{\infty}\sigma_n(u,A)\frac{t^n}{n!}\right) \\ &= 1+\sum_{n=1}^{\infty}A_n(\sigma_1(u;A),\sigma_2(u;A),\ldots,\sigma_n(u;A))\frac{t^n}{n!}, \end{aligned}$$

which implies (17). □

Corollary 4. *Let*

$$\sigma_n(u) := \sum_{d|n} d u^{n/d}, \quad \sigma_n := \sigma_n(1), \quad \sigma_{n,k} := \sum_{d|n,\ d\leqslant k} d.$$

Then

$$A_n(0!\sigma_1(u), 0!\sigma_2(u), \ldots, (n-1)!\sigma_n(u)) = n!\sum_{k=0}^{n}p(n,k)u^k. \tag{18}$$

In particular, for $u = 1$, we have

$$A_n(0!\sigma_1, 1!\sigma_2, \ldots, (n-1)!\sigma_n) = n!p(n). \tag{19}$$

We also have

$$A_n(0!\sigma_{1,k}, 1!\sigma_{2,k}, \ldots, (n-1)!\sigma_{n,k}) = n!p(n+k,k). \tag{20}$$

Proof. For $a_{i,j} = 1$ for $i = 1, 2, \ldots$ and $j = 0, 1, \ldots$ in Theorem 4, we get $p(n,k;A) = p(n,k)$, and from the well known identity (7) we get $b_n(i) = (n-1)!$ and $\sigma_n(u;A) = (n-1)!\sum_{d|n} du^{n/d}$. Then (18) follows from (17). The identity (20) results from [11, Thm. 10.2] and gives

$$\sum_{n=0}^{\infty}p(n+k,k)t^n = \prod_{i=1}^{k}\left(1-t^i\right)^{-1} = \exp\left(\sum_{i=1}^{\infty}\sigma_{n,k}\frac{t^n}{n}\right). \qquad \square$$

Corollary 5. *Let $q(n,k)$ be the number of partitions of n into k unequal (different) parts and*

$$\sigma_n(u) := \sum_{d|n} d u^{n/d}.$$

Then

$$A_n\left(-0!\sigma_1(-u), -1!\sigma_2(-u), \ldots, -(n-1)!\sigma_n(-u)\right) = n! \sum_{k=0}^{n} q(n,k) u^k. \tag{21}$$

Proof. For $i = 1, 2, \ldots$ set $a_{i,j} = 1$ for $j = 0, 1$ and $a_{i,j} = 0$ for $j = 2, 3, \ldots$ in Theorem 4, we then get $p(n,k;A) = q(n,k)$. Using (11), we obtain

$$b_n(i) = (-1)^{n-1}(n-1)! \text{ and } \sigma_n(u;A) = -(n-1)! \sum_{d|n} d(-u)^{n/d}.$$

Then (21) follows from the identity (17). □

Corollary 6. *For r, s nonnegative integers, with $r \geqslant 1$ and $0 \leqslant s \leqslant r-1$, let $p_{r,s}(n,k)$ be the number of partitions of n into k parts in form (s mod r) and*

$$\sigma_n^{r,s}(u) := (n-1)! \sum_{d \in \{j:\ j|n,\ r|(j-s)\}} d u^{n/d}.$$

Then

$$A_n\left(\sigma_1^{r,s}(u), \sigma_2^{r,s}(u), \ldots, \sigma_n^{r,s}(u)\right) = n! \sum_{j=0}^{n} p_{r,s}(n,j) u^j. \tag{22}$$

Proof. For $i = 1, 2, \ldots$ set $a_{i,0} = 1$, $a_{ir+s,j} = 1$ and $a_{ir+s',j} = 0$ for $s' \neq s$ and $0 \leqslant s' \leqslant r-1$ (if $s \neq 0$ resp. $s' \neq 0$ we consider the case $i = 0$ too) $j = 1, 2, 3, \ldots$ in Theorem 4, we get $p(n,k;A) = p_{r,s}(n,k)$. Using (7) and the fact that $B_{n,k}(0, 0, \ldots, 0) = 0$, we get $b_n(ir+s) = (n-1)!$, $b_n(ir+s') = 0$, $(s' \neq s)$, and $\sigma_n(u;A) = (n-1)! \sum_{d \in \{j:\ j|n,\ r|(j-s)\}} d u^{n/d}$. Then (22) follows from (17). □

Corollary 7. *Let $R_r(n,k)$ be the number of partitions of n into k parts with no part greater than r; and*

$$\sigma_n^r(u) := (n-1)! \sum_{d|n,\ d \leqslant r} d u^{n/d}.$$

Then

$$A_n\left(\sigma_1^r(u), \sigma_2^r(u), \ldots, \sigma_n^r(u)\right) = n! \sum_{j=0}^{n} R_r(n,j) u^j. \tag{23}$$

Proof. Let $a_{i,0} = 1$ for $i = 1, 2, \ldots,$ $a_{i,j} = 1$ for $i = 1, 2, \ldots, r$ and $a_{i,j} = 0$ for $i = r+1, r+2, \ldots,$ we have $p(n,k,A) = R_r(n,k)$. Then from (7) we get $b_n(i) = (n-1)!$, $b_n(i+r) = 0$, $i = 1, \ldots, r$, and $\sigma_n(u;A) = (n-1)! \sum_{d|n,\ d\leqslant r} du^{n/d}$, and (23) follows from (17). □

Corollary 8. *Let $q_0(n,k)$ be the number of partitions of n into k even unequal parts, $q_1(n,k)$ be the number of partitions of n into k odd unequal parts and*

$$\sigma_{n,0}(u) := -(n-1)! \sum_{d|n,\ d\ even} (-1)^{n/d} du^{n/d},$$

$$\sigma_{n,1}(u) := -(n-1)! \sum_{d|n,\ d\ odd} (-1)^{n/d} du^{n/d}.$$

Then

$$\begin{aligned} A_n\left(\sigma_{1,0}(u;A), \sigma_{2,0}(u;A), \ldots, \sigma_{n,0}(u;A)\right) &= n!\sum_{j=0}^{n} q_0(n,j)u^j, \\ A_n\left(\sigma_{1,1}(u;A), \sigma_{2,1}(u;A), \ldots, \sigma_{n,1}(u;A)\right) &= n!\sum_{j=0}^{n} q_1(n,j)u^j. \end{aligned} \tag{24}$$

Proof. For $p(n,k,A) = q_0(n,k)$ and $q_0(n) = \sum_k q_0(n,k)$, we have from (3), $a_{i,j} = 1$ if, and only if (i is even and $j = 0$ or 1) or (i is odd and $j = 0$). Using (11), we get $b_n(2i) = (-1)^{n-1}(n-1)!$, $b_n(2i-1) = 0$ and $\sigma_n(u;A) = -(n-1)! \sum_{d|n,d \text{ even}} (-1)^{n/d} du^{n/d}$. The first identity of (24) follows from (17). For $p(n,k,A) = q_1(n,k)$ and $q_1(n) = \sum_k q_1(n,k)$, we have from (3), $a_{i,j} = 1$ if, and only if (i is odd and $j = 0$ or 1) or (i is even and $j = 0$). We get $b_n(2i-1) = (-1)^{n-1}(n-1)!$, $b_n(2i) = 0$ and $\sigma_n(u;A) = -(n-1)! \sum_{d|n,\ d \text{ odd}} (-1)^{n/d} du^{n/d}$. The second identity of (24) follows from (17). □

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