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RESEARCH ARTICLE

Twin signed domination numbers in directed graphs

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Abstract. Let $D = (V, A)$ be a finite simple directed graph (shortly digraph). A function $f : V \longrightarrow \{-1, 1\}$ is called a twin signed dominating function (TSDF) if $f(N^-[v]) \geqslant 1$ and $f(N^+[v]) \geqslant 1$ for each vertex $v \in V$. The twin signed domination number of D is $\gamma_s^*(D) = \min\{\omega(f) \mid f$ is a TSDF of $D\}$. In this paper, we initiate the study of twin signed domination in digraphs and we present sharp lower bounds for $\gamma_s^*(D)$ in terms of the order, size and maximum and minimum indegrees and outdegrees. Some of our results are extensions of well-known lower bounds of the classical signed domination numbers of graphs.

1. Introduction

Throughout this paper, D is a finite simple directed graph with vertex set $V(D)$ and arc set $A(D)$ (briefly V and A). A digraph without directed cycles of length 2 is an *oriented graph*. We write $d_D^+(v)$ for the outdegree of a vertex v and $d_D^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. A digraph D is called *regular* or r*-regular* if $\delta^-(D) = \delta^+(D) = \Delta^-(D) = \Delta^+(D) = r$. If uv is an arc of D, then we also write $u \to v$, and we say that v is an

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out-neighbor of u and u is an *in-neighbor* of v. For every vertex v, we denote the set of in-neighbors and out-neighbors of v by $N^-(v) = N_D^-(v)$ and $N^+(v) = N_D^+(v)$, respectively. Let $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$ and $N_D^+[v] = N^+[v] = N^+(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $A(X, v)$ is the set of arcs from X to v. We denote by $A(X, Y)$ the set of arcs from a subset X to a subset Y. We denote by D^{-1} the digraph obtained from D by reversing the arcs of D. For a real-valued function $f : V(D) \longrightarrow \mathbb{R}$ the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Consult [8] for the notation and terminology which are not defined here.

Let $D = (V, A)$ be a finite simple digraph. A *signed dominating function* (abbreviated SDF) of D is a function $f : V \to \{-1, 1\}$ such that $f(N^-[v])] \geqslant 1$ for every $v \in V$. The signed domination number for a directed graph (digraph) D is

$$\gamma_s(D) = \min\{\omega(f) \mid f \text{ is a SDF of } D\}.$$

A $\gamma_s(D)$-function is a SDF of D of weight $\gamma_s(D)$. The signed domination number of a digraph was introduced by by Zelinka in [9] and has been studied by several authors (see for example [2, 6]).

A signed dominating function of D is called a *twin signed dominating function* (briefly TSDF) if it also is a signed dominating function of D^{-1}, i.e., $f(N^+[v]) \geqslant 1$ for every $v \in V$. The twin signed domination number for a digraph D is $\gamma_s^*(D) = \min\{\omega(f) \mid f \text{ is a TSDF of } D\}$. This definition is analogously to the definition of twin domination number in digraphs which was introduced by Chartrand et al. [3] and has been studied by Arumugam et al. [1]. For any function $f : V \to \{-1, 1\}$, we define $P = P_f = \{v \in V \mid f(v) = 1\}$ and $M = M_f = \{v \in V \mid f(v) = -1\}$. Since every TSDF of D is a SDF on both D and D^{-1} and since the constant function 1 is a TSDF of D, we have

$$\max\{\gamma_s(D), \gamma_s(D^{-1})\} \leqslant \gamma_s^*(D) \leqslant n. \tag{1}$$

In this paper, we initiate the study of the twin signed domination number and establish some sharp lower bounds on twin signed domination number of digraphs.

We make use of the following results and observations in this paper.

Proposition A ([9]). *For any digraph D of order $n \geqslant 2$, $\gamma_s(D) \equiv n \pmod 2$.*

Observation 1. *For any digraph D of order $n \geqslant 2$, $\gamma_s^*(D) \equiv n \pmod 2$.*

Proof. Let f be a $\gamma_s^*(D)$-function . Since $n = |P|+|M|$ and $\gamma_s^*(D) = |P|-|M|$, we deduce that $n-\gamma_s^*(D) = 2|M|$ and hence $\gamma_s^*(D) \equiv n \pmod 2$. □

Corollary 2. *For any digraph D of order $n \geqslant 2$, $\gamma_s^*(D) \equiv \gamma_s(D) \pmod 2$.*

Observation 3. *Let D be a digraph of order n. Then $\gamma_s^*(D) = n$ if and only if every vertex has either an out-neighbor with indegree one or an in-neighbor with outdegree one.*

Proof. One side is clear. For the other side, assume that $\gamma_s^*(D) = n$. Suppose to the contrary that there exists a vertex $v \in V(D)$ such that $d^-(u) \geqslant 2$ for each $u \in N^+[v]$ and $d^+(w) \geqslant 2$ for each $w \in N^-[v]$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a twin signed dominating function of D of weight less than n, a contradiction. This completes the proof. □

Corollary 4. *If C_n and P_n are the directed cycle and path on n vertices, then $\gamma_s^*(C_n) = \gamma_s^*(P_n) = n$.*

Here we determine the exact value of the twin signed domination number for particular types of tournaments. Let $n = 2r + 1$ for some positive integer r. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$ and for each i, the arcs go from u_i to the vertices $u_{i+1}, \ldots, u_{i+r}$, where the indices are taken modulo n. The proof of the next result can be found in [2].

Proposition B. *Let $n \geqslant 5$ and $n = 2r + 1$, where r is a positive integer. Then*

$$\gamma_s(\mathrm{CT}(n)) = \begin{cases} 3 & \text{if } r \text{ is even} \\ 5 & \text{if } r \text{ is odd.} \end{cases}$$

The next Proposition shows that $\gamma_s^*(\mathrm{CT}(n)) = \gamma_s(\mathrm{CT}(n))$

Proposition 5. *Let $n \geqslant 5$ and $n = 2r + 1$, where r is a positive integer. Then $\gamma_s^*(\mathrm{CT}(n)) = \gamma_s(\mathrm{CT}(n))$.*

Proof. By (1) and Proposition B, we have

$$\gamma_s^*(\mathrm{CT}(n)) \geqslant \begin{cases} 3 & \text{if } r \text{ is even} \\ 5 & \text{if } r \text{ is odd.} \end{cases}$$

Assume that $s = \lfloor \frac{r-2}{2} \rfloor$, $V^- = \{u_0, u_1, \ldots, u_s, u_{r+1}, \ldots, u_{r+s}\}$ and $V^+ = V(\mathrm{CT}(n)) - V^-$. For any vertex $v \in V(\mathrm{CT}(n))$, we have $|N^-[v]| = r+1$, $|N^+[v]| = r+1$, $|N^+[v] \cap V^-| \leqslant s+1$ and $|N^-[v] \cap V^-| \leqslant s+1$. Define $f : V(\mathrm{CT}(n)) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in V^+$ and $f(v) = -1$ when $v \in V^-$. Clearly, $f(N^-[v]) \geqslant r - 2s - 1 \geqslant 1$ and $f(N^+[v]) \geqslant r - 2s - 1 \geqslant 1$ for each $v \in V$. Therefore f is a TSDF on $\mathrm{CT}(n)$ of weight 3 if r is even and 5 when r is odd. Thus

$$\gamma_s^*(\mathrm{CT}(n)) \leqslant \omega(f) = \begin{cases} 3 & \text{if } r \text{ is even} \\ 5 & \text{if } r \text{ is odd,} \end{cases}$$

and the proof is complete. □

As we observed in (1), $\gamma_s^*(D) \geqslant \max\{\gamma_s(D), \gamma_s(D^{-1})\}$. Now we show that the difference $\gamma_s^*(D) - \max\{\gamma_s(D), \gamma_s(D^{-1})\}$ can be arbitrarily large.

Theorem 6. *For every positive integer k, there exists a digraph D such that*

$$\gamma_s^*(D) - \max\{\gamma_s(D), \gamma_s(D^{-1})\} \geqslant 4k - 4.$$

Proof. Let $k \geqslant 1$ be an integer, and let D be a digraph with vertex set

$$V(D) = \{x, y, u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_{2k}\}$$

and edge set

$$E(D) = \{(x, u_i), (v_{k+i}, u_i), (v_{k+i}, y), (u_{k+i}, x), (u_{k+i}, v_i), (y, v_i) \mid 1 \leqslant i \leqslant k\}.$$

Obviously, $D \cong D^{-1}$ and so, $\gamma_s(D) = \gamma_s(D^{-1})$. It is easy to verify that the function $f : V(D) \to \{-1, 1\}$ defined by $f(u_i) = f(v_i) = -1$ for $1 \leqslant i \leqslant k$ and $f(u) = 1$ otherwise, is a SDF of D and so $\gamma_s(D) \leqslant 2$. Now let g be a $\gamma_S^*(D)$-function. Since $N^+[u] = \{u\}$ for each $u \in \{u_i, v_i \mid 1 \leqslant i \leqslant k\}$ and $N^-[u] = \{u\}$ for each $u \in \{u_{k+i}, v_{k+i} \mid 1 \leqslant i \leqslant k\}$, we must have $g(u) = 1$ for each $u \in V(D) - \{x, y\}$. It follows that $\gamma_s^*(D) \geqslant 4k - 2$. Thus $\gamma_s^*(D) - \max\{\gamma_s(D), \gamma_s(D^{-1})\} \geqslant 4k - 4$, and the proof is complete. □

2. Lower bounds on $\gamma_s^*(D)$

In this section we establish lower bounds for $\gamma_s^*(D)$ in terms of the order, size, the maximum and minimum indegree and outdegree of D. We start with the following lemma.

Lemma 7. *Let D be a digraph of order n and let f be a $\gamma_s^*(D)$-function. Then*

1) $(1+\lceil\frac{\delta^-}{2}\rceil)|M| \leqslant |A(P,M)| \leqslant \lfloor\frac{\Delta^+}{2}\rfloor|P|$.
2) $(1+\lceil\frac{\delta^+}{2}\rceil)|M| \leqslant |A(M,P)| \leqslant \lfloor\frac{\Delta^-}{2}\rfloor|P|$.
3) $|A(P,P)| \geqslant \max\{\lceil\frac{\delta^+}{2}\rceil|P|, \lceil\frac{\delta^-}{2}\rceil|P|\}$.

Proof. (1) Let $v \in M$. Since $f(N^-[v]) \geqslant 1$, we deduce that $|A(P,v)| \geqslant 1+\lceil\frac{d^-(v)}{2}\rceil \geqslant 1+\lceil\frac{\delta^-}{2}\rceil$. It follows that $|A(P,M)| \geqslant (1+\lceil\frac{\delta^-}{2}\rceil)|M|$. Assume now that $v \in P$. Since $f(N^+[v]) \geqslant 1$, we have $|A(v,M)| \leqslant \lfloor\frac{d^+(v)}{2}\rfloor \leqslant \lfloor\frac{\Delta^+}{2}\rfloor$ and so $|A(P,M)| \leqslant \lfloor\frac{\Delta^+}{2}\rfloor|P|$. Combining the inequalities, we obtain (1).

(2) The proof is similar to the proof of (1).

(3) Let $v \in P$. Then $|A(v,P)| \geqslant \lceil\frac{d^+(v)}{2}\rceil \geqslant \lceil\frac{\delta^+}{2}\rceil$ and $|A(P,v)| \geqslant \lceil\frac{d^-(v)}{2}\rceil \geqslant \lceil\frac{\delta^-}{2}\rceil$ because $f(N^+[v])\geqslant 1$ and $f(N^-[v]) \geqslant 1$. Thus $|A(P,P)| \geqslant \max\{\lceil\frac{\delta^+}{2}\rceil|P|, \lceil\frac{\delta^-}{2}\rceil|P|\}$, and the proof is complete. □

Theorem 8. *Let D be a digraph of order n, minimum indegree δ^-, minimum outdegree δ^+, maximum indegree Δ^- and maximum outdegree Δ^+. Then*

$$\gamma_s^*(D) \geqslant \max\left\{\frac{1+\lceil\frac{\delta^-}{2}\rceil-\lfloor\frac{\Delta^+}{2}\rfloor}{1+\lceil\frac{\delta^-}{2}\rceil+\lfloor\frac{\Delta^+}{2}\rfloor}n, \frac{1+\lceil\frac{\delta^+}{2}\rceil-\lfloor\frac{\Delta^-}{2}\rfloor}{1+\lceil\frac{\delta^+}{2}\rceil+\lfloor\frac{\Delta^-}{2}\rfloor}n\right\}.$$

Furthermore, this bound is sharp for directed cycles and paths.

Proof. Let f be a minimum TSDF of D. Using Lemma 7 and replacing $|M|$ and $|P|$ by $\frac{n-\gamma_s^*(D)}{2}$ and $\frac{n+\gamma_s^*(D)}{2}$ in (1) and (2), the desired inequality follows. □

Theorem 8 implies the next result immediately.

Corollary 9. *If D is an r-regular digraph with $r \geqslant 1$, then $\gamma_s^*(D) \geqslant n/(r+1)$ when r is even and $\gamma_s^*(D) \geqslant 2n/(r+1)$ when r is odd.*

Example 1. If K_n^* is the complete digraph of order n, then $\gamma_s^*(K_n^*) = 1$ when n is odd and $\gamma_s^*(K_n^*) = 2$ when n is even.

Proof. According to Corollary 9, we have $\gamma_s^*(K_n^*) \geqslant 2$ when n is even and $\gamma_s^*(K_n^*) \geqslant 1$ when n is odd. On the other hand if $n = 2p$ is even, then assign to $p+1$ vertices the value 1 and to $p-1$ vertices the value -1. Then $f(N^-[v]) = f(N^+[v]) = 2$ for each vertex v. Thus f is a TSDF of K_n^* of weight 2 and so $\gamma_s^*(K_n^*) = 2$ when n is even. If $n = 2p+1$ is odd, then assign to $p+1$ vertices the value 1 and to p vertices the value -1. Then $f(N^-[v]) = f(N^+[v]) = 1$ for each vertex v. Thus f is a TSDF of K_n^* of weight 1 and so $\gamma_s^*(K_n^*) = 1$ when n is odd. $\square$

This is another example that shows the sharpness of Theorem 8. If G is a graph, then a *signed dominating function* is defined in [4] as a function $f : V(G) \longrightarrow \{-1, 1\}$ such that $f(N[v]) \geqslant 1$ for all $v \in V(G)$. The *signed domination number* $\gamma_s(G)$ of G is the minimum weight of a signed dominating function on G. The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. Since $N^-_{D(G)}[v] = N^+_{D(G)}[v] = N_G[v]$ for each $v \in V(G) = V(D(G))$, the following useful observation is valid.

Observation 10. *If $D(G)$ is the associated digraph of a graph G, then $\gamma_s^*(D(G)) = \gamma_s(G)$.*

Theorem 8 and Observation 10 lead immediately to an lower bound for the signed domination number of graphs.

Corollary 11. *Let G be a graph of order n, minimum degree δ and maximum degree Δ. Then*

$$\gamma_s(G) \geqslant \frac{1 + \lceil \frac{\delta}{2} \rceil - \lfloor \frac{\Delta}{2} \rfloor}{1 + \lceil \frac{\delta}{2} \rceil + \lfloor \frac{\Delta}{2} \rfloor} n.$$

Since

$$\frac{1 + \lceil \frac{\delta}{2} \rceil - \lfloor \frac{\Delta}{2} \rfloor}{1 + \lceil \frac{\delta}{2} \rceil + \lfloor \frac{\Delta}{2} \rfloor} n \geqslant \frac{\delta + 2 - \Delta}{\delta + 2 + \Delta} n,$$

Corollary 11 implies the following known bound.

Corollary 12 ([10])**.** *If G is a graph of order n, minimum degree δ and maximum degree Δ, then*

$$\gamma_s(G) \geqslant \frac{\delta + 2 - \Delta}{\delta + 2 + \Delta} n,$$

Using Corollary 9 and Observation 10, we obtain the next result.

Corollary 13. *If G is an r-regular graph with $r \geqslant 1$, then $\gamma_s(G) \geqslant n/(r+1)$ when r is even and $\gamma_s(G) \geqslant 2n/(r+1)$ when r is odd.*

Corollary 13 can be found in [4] and [5].

Theorem 14. *If D is a digraph of order n and maximum indegree Δ^-, then*

$$\gamma_s^*(D) \geqslant 2\left\lceil \frac{\Delta^-}{2} \right\rceil + 2 - n.$$

Proof. Let $w \in V(D)$ be a vertex of maximum indegree $d^-(w) = \Delta^-$, and let f be a $\gamma_s^*(D)$-function. Assume first that $w \in M$. Since $f(N^-[w]) \geqslant 1$, we deduce that $|A(P,w)| \geqslant 1 + \lceil \frac{\Delta^-}{2} \rceil$. It follows that

$$\frac{n + \gamma_s^*(D)}{2} = |P| \geqslant |A(P,w)| \geqslant 1 + \left\lceil \frac{\Delta^-}{2} \right\rceil,$$

and this leads to the desired inequality. If $w \in P$, then $f(N^-[w]) \geqslant 1$ implies that $|A(P,w)| \geqslant \lceil \frac{\Delta^-}{2} \rceil$. We conclude that

$$\frac{n + \gamma_s^*(D)}{2} = |P| \geqslant |A(P,w)| + 1 \geqslant 1 + \left\lceil \frac{\Delta^-}{2} \right\rceil,$$

and the proof is complete. □

The condition $f(N^+[v]) \geqslant 1$ for each vertex v, yields analogously the next result.

Theorem 15. *If D is a digraph of order n and maximum outdegree Δ^+, then $\gamma_s^*(D) \geqslant 2\lceil \frac{\Delta^+}{2} \rceil + 2 - n$.*

Example 1 demonstrates that Theorems 14 and 15 are sharp.

Theorem 16. *For any digraph D of order n, size m, minimum indegree δ^- and minimum outdegree δ^+,*

$$\gamma_s^*(D) \geqslant \frac{n(2 + 2\lceil \frac{\delta^+}{2} \rceil + \lceil \frac{\delta^-}{2} \rceil) - 2m}{2 + \lceil \frac{\delta^-}{2} \rceil}.$$

This bound is sharp for directed cycles.

Proof. Let f be a $\gamma_s^*(D)$-function. By Lemma 7, we have

$$\begin{aligned} m &\geqslant |A(M,P)| + |A(P,M)| + |A(P,P)| \\ &\geqslant (1 + \lceil \frac{\delta^-}{2} \rceil)|M| + (1 + \lceil \frac{\delta^+}{2} \rceil)|M| + \lceil \frac{\delta^+}{2} \rceil |P| \\ &= \lceil \frac{\delta^+}{2} \rceil n + (2 + \lceil \frac{\delta^-}{2} \rceil)(\frac{n - \gamma_s^*(D)}{2}). \end{aligned}$$

This leads to the desired inequality. □

Theorem 17. *Let D be a digraph of order n, maximum indegree Δ^- and maximum outdegree Δ^+. Then $\gamma_s^*(D) \geqslant \frac{4-\lfloor\frac{\Delta^-}{2}\rfloor-\lfloor\frac{\Delta^+}{2}\rfloor}{4+\lfloor\frac{\Delta^-}{2}\rfloor+\lfloor\frac{\Delta^+}{2}\rfloor}n$. This bound is sharp for directed cycles and paths.*

Proof. Let f be a $\gamma_s^*(D)$-function and let $v \in M$. Since $f(N^+[v]) \geqslant 1$ and $f(N^-[v]) \geqslant 1$, we conclude that $|A(v,P)| \geqslant 2$ and $|A(P,v)| \geqslant 2$ and thus $|A(M,P)| + |A(P,M)| \geqslant 4|M|$. Using Lemma 7 (Parts 1, 2), it follows that

$$|P|(\lfloor\frac{\Delta^-}{2}\rfloor + \lfloor\frac{\Delta^+}{2}\rfloor) \geqslant 4|M|. \tag{2}$$

Replacing $|M|$ and $|P|$ by $\frac{n-\gamma_s^*(D)}{2}$ and $\frac{n+\gamma_s^*(D)}{2}$ in (2), we obtain the desired bound. □

Theorem 18. *For any digraph D of order n and size m,*

$$\gamma_s^*(D) \geqslant n - \frac{m}{3}.$$

Furthermore, the bound is sharp.

Proof. Let f be a $\gamma_s^*(D)$-function. In view of the proof of Theorem 17, $|A(P,M)| \geqslant 2|M|$ and $|A(M,P)| \geqslant 2|M|$. If $x \in P$, then it follows from $f(N^+[x]) \geqslant 1$ that $|A(x,P)| \geqslant |A(x,M)|$. This implies that $|A(P,P)| \geqslant |A(P,M)| \geqslant 2|M|$. Hence $m \geqslant |A(M,P)|+|A(P,M)|+|A(P,P)| \geqslant 6|M|$. Since $n = |P| + |M|$, we deduce that $\gamma_s^*(D) = |P| - |M| = n - 2|M| \geqslant n - \frac{m}{3}$.

To prove the sharpness, let G be a graph of order $n(G)$ and size $m(G)$. Assume that D is a digraph obtained from G by replacing every edge $xy \in E(G)$, by three new vertices z_{xy}, u_{xy}, v_{xy} and arcs (x, z_{xy}), (y, z_{xy}), (x, u_{xy}), (y, v_{xy}), (z_{xy}, u_{xy}) and (z_{xy}, v_{xy}). Then $n(D) = n(G)+3m(G)$ and $m(D) = 6m(G)$. It is easy to see that the function $f : V(D) \to \{-1, 1\}$ that assigns -1 to z_{xy} for each edge $xy \in E(G)$ and $+1$ otherwise, is a TSDF of D with $\omega(f) = n(D) - \frac{m(D)}{3}$ as desired. □

Using an idea in [10], we prove the next sharp lower bound.

Theorem 19. *Let D be a digraph of order n. Then*

$$\gamma_s^*(D) \geqslant 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil - n,$$

and this bound is sharp.

Proof. Let f be a $\gamma_s^*(D)$-function. In view of the proof of Theorem 17, $|A(M,P)| \geqslant 2|M|$. Hence ther exists a vertex $v \in P$ such that $|A(M,v)| \geqslant 2|M|/|P|$. Since $f(N^-[v]) \geqslant 1$, we have $|A(P,v)| \geqslant |A(M,v)|$. Therefore it follows that

$$|P| \geqslant |A(P,v)| + 1 \geqslant |A(M,v)| + 1 \geqslant \frac{2|M|}{|P|} + 1$$

and so $|P|^2 + |P| - 2n \geqslant 0$. This implies that

$$|P| \geqslant \frac{-1+\sqrt{8n+1}}{2},$$

and thus we obtain

$$\gamma_s^*(D) = 2|P| - n \geqslant 2\left\lceil \frac{-1+\sqrt{8n+1}}{2} \right\rceil - n.$$

We now show that for any positive integer n, there exists a digraph H of order n with equality in the bound above. For $n = 1,2$ let $H = K_n^*$. Assume next that $n \geqslant 3$, and let $t = \lceil(-1+\sqrt{8n+1})/2\rceil$. Let H be the associated digraph of the following graph G. Let G be the graph obtained from $K_t \cup \overline{K_{n-t}}$ by joining each vertex of $\overline{K_{n-t}}$ to a pair of vertices of K_t such that each pair of vertices in K_t is joined to at most on vertex in $\overline{K_{n-t}}$. Note that this is possible since $n - t \leqslant \binom{t}{2}$. Define the function $f : V(H) \longrightarrow \{-1,1\}$ by $f(v) = 1$ for $v \in K_t^*$ and $f(v) = -1$ otherwise. Then f is a TSDF of H, and thus $\gamma_s^*(H) \leqslant 2t - n$. Applying the bound above, we obtain $\gamma_s^*(H) = 2t - n$. $\square$

Following a procedure in [7], we improve Theorem 19 for bipartite digraphs.

Theorem 20. *Let D be a bipartite digraph of order n. Then*

$$\gamma_s^*(D) \geqslant 4\sqrt{n+1} - n - 4,$$

and this bound is sharp.

Proof. Let f be a $\gamma_s^*(D)$-function, and let X and Y be the partite sets of D. In addition, let X^+ and X^- (Y^+ and Y^-) be the sets of vertices in X (in Y) assigning the values 1 and -1, respectively.

Assume first that $Y^- \neq \varnothing$. Since $f(N^+[y]) \geqslant 1$, we observe that $|A(y,X^+)| \geqslant 2$ for each vertex $y \in Y^-$ and thus $|A(Y^-,X^+)| \geqslant 2|Y^-|$.

Hence there exists a vertex $v \in X^+$ such that $|A(Y^-, v)| \geqslant 2|Y^-|/|X^+|$. It follows that

$$1 \leqslant f(N^-[v]) \leqslant 1 + |Y^+| - |A(Y^-, v)| \leqslant 1 + |Y^+| - \frac{2|Y^-|}{|X^+|}$$

and therefore $|Y^+||X^+| \geqslant 2|Y^-|$. Note that this inequality remains valid when $Y^- = \varnothing$. Analogously, one can show $|Y^+||X^+| \geqslant 2|X^-|$. Adding the last two inequalities, we obtain

$$2|X^+||Y^+| \geqslant 2|X^-| + 2|Y^-| = 2n - 2|X^+| - 2|Y^+|.$$

Using this inequality and the fact that $4|X^+||Y^+| \leqslant (|X^+| + |Y^+|)^2$, we deduce that

$$(|X^+| + |Y^+| + 2)^2 \geqslant 4n + 4$$

and so

$$|X^+| + |Y^+| \geqslant 2\sqrt{n+1} - 2.$$

This implies

$$\gamma_s^*(D) = 2(|X^+| + |Y^+|) - n \geqslant 4\sqrt{n+1} - n - 4.$$

To prove the sharpness, let $t \geqslant 1$ be an integer, and let H be the associated digraph of the following bipartite graph G. Let G be the disjoint union of $K_{2t,2t}$ with the partite sets X and Y, and the vertex sets A and B with $|A| = |B| = 2t^2$ by adding edges between A and X and B and Y such that each vertex in A is joined to exactly 2 vertices in X, each vertex in X is joined to exactly $2t$ vertices in A, and each vertex in B is joined to exactly 2 vertices in Y, and each vertex in Y is joined to exactly $2t$ vertices in B. Then H has order $n(H) = 4t^2 + 4t$. Define the function $f : V(H) \longrightarrow \{-1, 1\}$ by $f(v) = 1$ for $v \in K^*_{2t,2t}$ and $f(v) = -1$ otherwise. Then f is a TSDF of H and thus $\gamma_s^*(H) \leqslant 4t - 4t^2$. Applying the bound above, we obtain $\gamma_s^*(H) = 4t - 4t^2 = 4\sqrt{n(H)+1} - n(H) - 4$. □

Theorem 21. *Let D be a digraph of order n and let $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n$ be the degree sequence of the underling graph G of D. If s is the smallest positive integer for which $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geqslant 4(n - s)$, then*

$$\gamma_s^*(D) \geqslant 2s - n.$$

Furthermore, this bound is sharp.

Proof. Let f be a $\gamma_s^*(D)$-function and $p = |P|$. Since $f([N_D^+[v]) \geqslant 1$ and $f(N_D^-[v]) \geqslant 1$ for each $v \in V(D)$, we have

$$\begin{aligned} n &\leqslant \sum_{v \in V} f(N_D^+[v]) \\ &= \sum_{v \in V} (d_D^+(v) + 1) f(v) = |P| - |M| + \sum_{v \in P} d_D^+(v) - \sum_{v \in M} d_D^+(v) \end{aligned}$$

and

$$\begin{aligned} n &\leqslant \sum_{v \in V} f(N_D^-[v]) \\ &= \sum_{v \in V} (d_D^-(v) + 1) f(v) = |P| - |M| + \sum_{v \in P} d_D^-(v) - \sum_{v \in M} d_D^-(v). \end{aligned}$$

Summing the above inequalities, we deduce that

$$\begin{aligned} 2n &\leqslant 2(|P| - |M|) + \sum_{v \in P} (d_D^+(v) + d_D^-(v)) - \sum_{v \in M} (d_D^+(v) + d_D^-(v)) \\ &= 2(2p - n) + \sum_{v \in P} \deg_G(v) - \sum_{v \in M} \deg_G(v) \\ &\leqslant 4p - 2n + \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i. \end{aligned}$$

Thus $4(n-p) \leqslant \sum_{i=1}^{p} d_i - \sum_{i=p+1}^{n} d_i$. By the assumption on s, we must have $p \geqslant s$. This implies that $\gamma_s^*(D) = 2p - n \geqslant 2s - n$.

To prove the sharpness, let D be the digraph obtained from two disjoint directed cycles $(u_1, u_2, \ldots, u_k)$ and $(v_1, v_2, \ldots, v_k)$ by adding k new vertices $w_1, w_2, \ldots, w_k$ and adding arcs $u_i w_i$, $v_i w_i$, $w_i u_{i+1}, w_i v_{i+1}$ for each i, where $n + 1 = 1$. Obviously, the order of D is $3k$ and the underlying graph of D is 4-regular. Therefore, the smallest positive integer s satisfying $\sum_{i=1}^{s} d_i - \sum_{i=s+1}^{n} d_i \geqslant 4(n-s)$ is $s = 2k$. Thus $\gamma_s^*(D) \geqslant k$. Now define $f : V(D) \to \{-1, 1\}$ by $f(w_i) = -1, f(u_i) = f(v_i) = 1$ for each i. Obviously f is a TSDF of D with $\omega(f) = k$. This completes the proof. □

3. Twin Signed Domination in Oriented Graphs

Let G be the complete bipartite graph $K_{4,4}$ with bipartite sets $V_1 = \{v_1, \ldots, v_4\}$ and $V_2 = \{u_1, \ldots, u_4\}$. Let D_1 be an orientation of G such that all arcs go from V_1 in to V_2 and D_2 be an orientation of G such that

$A(D_2) = \{(v_i, u_j), (u_j, v_r) \mid i = 1, 2, r = 3, 4 \text{ and } 1 \leqslant j \leqslant 4\}$. It is easy to see that $\gamma_s^*(D_1) = 8$ and $\gamma_s^*(D_2) = 4$. Thus two distinct orientations of a graph can have distinct twin domination numbers. Motivated by this observation, we define the lower orientable twin signed domination number $\mathrm{dom}_s^*(G)$ and the upper orientable twin signed domination number $\mathrm{Dom}_s^*(G)$ of a graph G as follows:

$$\mathrm{dom}_s^*(G) = \min\{\gamma_s^*(D) \mid \text{D is an orientation of } G\},$$

and

$$\mathrm{Dom}_s^*(G) = \max\{\gamma_s^*(D) \mid \text{D is an orientation of } G\}.$$

Proposition 22. *For any graph G of order n, $\gamma_s(G) \leqslant \mathrm{dom}_s^*(G)$.*

Proof. Let D be an orientation of G such that $\gamma_s^*(D) = \mathrm{dom}_s^*(G)$, and let f be a $\gamma_s^*(D)$-function. Then $f(N_G[v]) = f(N_D^+[v]) + f(N_D^-[v]) - f(v)$ for each $v \in V$. Since $f(N_D^+[v]) \geqslant 1$ and $f(N_D^-[v]) \geqslant 1$, we have $f(N_G[v]) \geqslant 1$ for each $v \in V$, and so f is a SDF of G. Therefore $\gamma_s(G) \leqslant \omega(f) = \mathrm{dom}_s^*(G)$ as desired. □

Lemma 23. *Let G be a graph of order n and $v \in V(G)$. Let D be an orientation of G and f be a $\gamma_s^*(D)$-function. If v is a support vertex or $\deg(v) \leqslant 3$, then $f(v) = 1$.*

Proof. If v is a support vertex and u is a leaf adjacent to v, then it follows from $f(N^+[u]) \geqslant 1$ and $f(N^-[u]) \geqslant 1$ that $f(v) = 1$. Assume that $\deg(v) \leqslant 3$. Then $\deg_D^+(v) \leqslant 1$ or $\deg_D^-(v) \leqslant 1$. Since $f(N^+[v]) \geqslant 1$ and $f(N^-[v]) \geqslant 1$, we deduce that $f(v) = 1$. □

Proposition 24. *Let G be a graph of order n. Then $\mathrm{dom}_s^*(G) = n$ if and only if every vertex of G either is a support vertex or has degree at most 3.*

Proof. The sufficiency follows from Lemma 23. To prove the necessity, assume that $\mathrm{dom}_s^*(G) = n$ and assume, to the contrary, that there exists a vertex $v \in V(G)$ such that v is not a support vertex and $\deg(v) \geqslant 4$. Let t be the maximum number of disjoint pairs u_i, w_i in $N(v)$ such that the subgraph induced by $\{v, u_i, w_i\}$ is a triangle. Assume that $r = \deg(v) - 2t$ and $N(v) \setminus \{u_i, w_i : \ 1 \leqslant i \leqslant t\} = \{z_1, z_2, \ldots, z_r\}$ if $r > 0$. Since v is not a support vertex, $N(z_i) \setminus \{v\} \neq \varnothing$ for each $1 \leqslant i \leqslant r$. Suppose that $x_i \in N(z_i) \setminus \{v\}$ for $1 \leqslant i \leqslant r$. If $t = 0$, then let D be an orientation of G such that

$$\{(v, z_i), (z_j, v), (x_i, z_i), (z_j, x_j) \mid 1 \leqslant i \leqslant 2 \text{ and } 3 \leqslant j \leqslant r\} \subseteq A(D).$$

If $t = 1$, then let D be an orientation of G such that

$$\{(v,u_1),(w_1,v),(w_1,u_1),(v,z_1),(z_j,v),(x_1,z_1),(z_j,x_j) : 2 \leqslant j \leqslant r\} \subseteq A(D).$$

Finally, if $t \geqslant 2$, then let D be an orientation of G such that

$$\{(v,u_i),(w_i,v),(w_i,u_i),(v,z_j),(x_j,z_j) \mid 1 \leqslant i \leqslant t \text{ and } 1 \leqslant j \leqslant r\} \subseteq A(D).$$

It is easy to verify that the function $f : V(D) \to \{-1,1\}$ that assigns -1 to v and $+1$ to the remaining vertices, is a TSDF of D of weight $n-2$ in all cases, and so $\mathrm{dom}_s^*(G) \leqslant n-2$ which is a contradiction. This completes the proof. □

An immediate consequence of Proposition 24 now follows.

Corollary 25. *For $n \geqslant 3$, $\mathrm{dom}_s^*(P_n) = \mathrm{dom}_s^*(C_n) = n$.*

The wheel, W_n, is a graph with vertex set $\{v_0, v_1, \ldots, v_n\}$ and edge set $\{v_0v_i \mid 1 \leqslant i \leqslant n\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Next we determine the lower orientable twin signed domination number of wheels.

Proposition 26. *For $n \geqslant 4$, $\mathrm{dom}_s^*(W_n) = n-1$.*

Proof. Let D be an arbitrary orientation of W_n and f be a $\gamma_s^*(D)$-function. It follows from Lemma 23 that $f(v_i) = +1$ for each $i \geqslant 1$. This implies that $\mathrm{dom}_s^*(W_n) = \omega(f) \geqslant n-1$. Now, let D be an orientation of W_n in which $(v_1v_2\ldots v_n)$ is a directed cycle and $\{(v_0,v_1),(v_0,v_2),(v_j,v_0) : 3 \leqslant j \leqslant n\} \subseteq A(D)$. Then the function $f : V(D) \to \{-1,1\}$ that assigns -1 to v_0 and $+1$ to the remaining vertices, is a TSDF of D of weight $n-1$ implying that $\mathrm{dom}_s^*(G) \leqslant n-1$. Thus $\mathrm{dom}_s^*(G) = n-1$ as desired. □

We now proceed to determine the lower orientable twin domination numbers of several classes of graphs include complete graphs and complete bipartite graphs.

Lemma 27. *For $n \geqslant 3$,*

$$\mathrm{dom}_s^*(K_n) \geqslant \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Proof. If $n = 3, 4$, then the result follows from Lemma 23. Let $n \geqslant 5$. By Proposition 24 and Observation 1, we have $\mathrm{dom}_s^*(K_n) \leqslant n-2$. Let D be an orientation of K_n such that $\gamma_s^*(D) = \mathrm{dom}_s^*(K_n)$ and let f be a $\gamma_s^*(D)$-function. Assume that $v \in M$. We consider two cases.

Case 1: n is odd. Since $f(N^+[v]) \geqslant 1$ and $f(N^-[v]) \geqslant 1$ and since $N^+(v) \cup N^-(v)$ is a partition of $V(K_n) \setminus \{v\}$, we deduce that $\mathrm{dom}_s^*(K_n) = \omega(f) = f(N^+[v]) + f(N^-[v]) - f(v) \geqslant 3$.

Case 2: n is even. Since $n-1$ is odd and since $N^+(v) \cup N^-(v)$ is a partition of $V(K_n) \setminus \{v\}$, one of the $d^+(v)$ or $d^-(v)$ must be odd. Assume, without loss of generality, that $d^+(v)$ is odd. Then we must have $f(N^+[v]) \geqslant 2$ and $f(N^-[v]) \geqslant 1$. Proceeding as above, we obtain $\mathrm{dom}_s^*(K_n) \geqslant 4$. □

Theorem 28. *For $n \geqslant 3$,*

$$\mathrm{dom}_s^*(K_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Proof. The result is trivial for $n = 3, 4$, so assume $n \geqslant 5$. Let

$$V(K_n) = \Big\{u_i, v_i, w_j \mid 1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil - 2 \ \text{ and } \ 2\lceil \frac{n}{2} \rceil - 3 \leqslant j \leqslant n\Big\}$$

and let D be an orientation of K_n such that

$$\begin{aligned} A(D) = \Big\{&(u_k, u_l), (u_k, v_l), (v_k, v_l), (v_r, u_s) \mid 1 \leqslant k < l \leqslant \lceil \frac{n}{2} \rceil - 2 \\ &\text{and } 1 \leqslant r \leqslant s \leqslant \lceil \frac{n}{2} \rceil - 2\Big\} \\ \cup \Big\{&(u_i, w_j), (v_i, w_j), (w_{2\lceil \frac{n}{2} \rceil - 3}, u_i), (w_{2\lceil \frac{n}{2} \rceil - 3}, v_i) \mid 1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil - 2 \\ &\text{and } 2\lceil \frac{n}{2} \rceil - 2 \leqslant j \leqslant n\Big\} \\ \cup \Big\{&(w_k, w_l) \mid 2\lceil \frac{n}{2} \rceil - 3 \leqslant k < l \leqslant n\Big\}. \end{aligned}$$

It is easy to see that the function $f : V(D) \to \{-1, +1\}$ defined by $f(u_i) = -1$ for $1 \leqslant i \leqslant \lceil \frac{n}{2} \rceil - 2$ and $f(x) = +1$ otherwise, is a TSDF of D of weight 3 when n is odd and weight 4 when n is even. This implies that

$$\mathrm{dom}_s^*(K_n) \leqslant \omega(f) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even.} \end{cases}$$

Now the result follows from Lemma 27. □

Let $m \leqslant n$ and $K_{m,n}$ be the bipartite graph with bipartite sets V_1, V_2 such that $|V_1| = m$ and $|V_2| = n$.

Lemma 29. *Let D be an orientation of $K_{m,n}$ with $n \geqslant m \geqslant 4$. If f is a TSDF of D such that $V_i \cap M_f \neq \varnothing$ for $i = 1, 2$, then*

$$\omega(f) \geqslant \begin{cases} 8 & \text{if } n \text{ and } m \text{ are both even} \\ 9 & \text{if } n \text{ and } m \text{ have different parity} \\ 10 & \text{if } n \text{ and } m \text{ are both odd .} \end{cases}$$

Proof. Let $u \in V_1 \cap M_f$ and $v \in V_2 \cap M_f$. We consider three cases.

Case 1: m and n are both even. Since $f(N^+[u]) \geqslant 1$ and $f(N^-[u]) \geqslant 1$, we must have

$$|N^+(u) \cap P_f \cap V_2| \geqslant |N^+(u) \cap M_f \cap V_2| + 2$$

and

$$|N^-(u) \cap P_f \cap V_2| \geqslant |N^-(u) \cap M_f \cap V_2| + 2.$$

Since $V_2 = N^+(u) \cup N^-(u)$, we deduce that

$$|V_2 \cap P_f| \geqslant |V_2 \cap M_f| + 4. \tag{3}$$

Similarly, we have

$$|V_1 \cap P_f| \geqslant |V_1 \cap M_f| + 4. \tag{4}$$

Adding (3) and (4), we obtain $|P| \geqslant |M|+8$ and so $\omega(f) = |P_f|-|M_f| \geqslant 8$ as desired.

Case 2: m and n have different parity. Assume, without loss of generality, that m is even and n is odd. Since $d^+(u) + d^-(u) = n$ is odd, we may assume that $d^+(u)$ is odd. It follows that $f(N^+[u]) \geqslant 2$ and hence

$$|N^+(u) \cap P_f \cap V_2| \geqslant |N^+(u) \cap M_f \cap V_2| + 3.$$

Using an argument similar to that described in Case 1, we obtain $\omega(f) = |P_f| - |M_f| \geqslant 9$.

Case 3: m and n are both odd. Since $d^+(u) + d^-(u) = n$ and $d^+(v) + d^-(v) = m$ are both odd, we may assume, without loss of generality, that $d^+(u)$ and $d^+(v)$ are both odd. As Cases 1, 2, we have

$$\begin{aligned} |N^+(u) \cap P_f \cap V_2| &\geqslant |N^+(u) \cap M_f \cap V_2| + 3 \\ |N^-(u) \cap P_f \cap V_2| &\geqslant |N^-(u) \cap M_f \cap V_2| + 2 \\ |N^+(v) \cap P_f \cap V_1| &\geqslant |N^+(v) \cap M_f \cap V_1| + 3 \\ |N^-(v) \cap P_f \cap V_1| &\geqslant |N^-(v) \cap M_f \cap V_1| + 2. \end{aligned}$$

Summing the above inequalities, we deduce that $|P| \geqslant |M| + 10$ and so $\omega(f) \geqslant 10$ as desired. □

Lemma 30. *Let D be an orientation of $K_{m,n}$ and f be a TSDF of D. If $V_1 \cap M_f = \varnothing$, then*

$$\omega(f) \geqslant \begin{cases} m & \text{if } n \text{ is even} \\ m+1 & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $u \in V_1$. If n is even, then it follows from $f(N^+[u]) \geqslant 1$ and $f(N^-[u]) \geqslant 1$ that $|N^+(u) \cap P_f \cap V_2| \geqslant |N^+(u) \cap M_f|$ and $|N^-(u) \cap P_f| \geqslant |N^-(u) \cap M_f \cap V_2|$. This implies that $|V_2 \cap P_f| \geqslant |V_2 \cap M_f|$ and hence $\omega(f) = |P_f| - |M_f| = |V_1| + |V_2 \cap P_f| - |V_2 \cap M_f| \geqslant |V_1| = m$.

Assume that n is odd. Since $d^+(u) + d^-(u) = n$ is odd, we may assume, without loss of generality, that $d^+(u)$ is odd. This implies that $f(N^+[u]) \geqslant 2$. As above we have $|N^+(u) \cap P_f \cap V_2| \geqslant |N^+(u) \cap M_f \cap V_2| + 1$ and $|N^-(u) \cap P_f| \geqslant |N^-(u) \cap M_f \cap V_2|$ implying that $|V_2 \cap P_f| \geqslant |V_2 \cap M_f| + 1$. It follows that $\omega(f) = |P_f| - |M_f| = |V_1| + |V_2 \cap P| - |V_2 \cap M_f| \geqslant m + 1$ as desired. □

The next result is an immediate consequence of Lemmas 29 and 30.

Corollary 31. *For $4 \leqslant m \leqslant n$,*

$$\mathrm{dom}_s^*(K_{m,n}) \geqslant \begin{cases} \min\{m, 8\} & \text{if } n \text{ and } m \text{ are both even} \\ \min\{m+1, 9\} & \text{if } n \text{ is odd and } m \text{ is even} \\ \min\{m, 9\} & \text{if } m \text{ is odd and } n \text{ is even} \\ \min\{m+1, 10\} & \text{if } n \text{ and } m \text{ are both odd .} \end{cases}$$

Theorem 32. *For $4 \leqslant m \leqslant n$,*

$$\mathrm{dom}_s^*(K_{m,n}) = \begin{cases} \min\{m, 8\} & \text{if } n \text{ and } m \text{ are both even} \\ \min\{m+1, 9\} & \text{if } n \text{ is odd and } m \text{ is even} \\ \min\{m, 9\} & \text{if } m \text{ is odd and } n \text{ is even} \\ \min\{m+1, 10\} & \text{if } n \text{ and } m \text{ are both odd .} \end{cases}$$

Proof. Let $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$ be the partite sets of $K_{m,n}$. We consider four cases.

Case 1: assume that m and n are both even. First let $m \leqslant 6$. Let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (v_j, u_s) \mid 1 \leqslant i \leqslant \frac{m}{2},\ 1 \leqslant j \leqslant n \ \text{ and } \ \frac{m}{2} + 1 \leqslant s \leqslant m\}.$$

Define $f : V(G) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in V_1 \cup \{v_1, \ldots, v_{\frac{n}{2}}\}$ and $f(x) = -1$ otherwise. It is easy to see that f is an TSDF of D of weight m and so $\mathrm{dom}^*_s(K_{m,n}) \leqslant m$. Let now $m \geqslant 8$ and let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (u_r, v_s), (v_l, u_t) \mid i, j, t \notin \{1, 2\}, \\ 1 \leqslant s \leqslant n \text{ and } r, l \in \{1, 2\}\}. \tag{5}$$

It is easy to verify that the function $f : V(G) \to \{-1, +1\}$ defined by $f(x) = +1$ for $x \in \{u_1, \ldots, u_{\frac{m}{2}+2}\} \cup \{v_1, \ldots, v_{\frac{n}{2}+2}\}$ and $f(x) = -1$ otherwise, is a TSDF of D of weight 8 and so $\mathrm{dom}^*_s(K_{m,n}) \leqslant 8$.

Case 2: assume that m is even and n is odd. If $m \leqslant 6$, then orient the edges of $K_{m,n}$ such that the resulting digraph has the arc set

$$A(D) = \{(u_i, v_j), (v_j, u_s) \mid 1 \leqslant i \leqslant \frac{m}{2},\ 1 \leqslant j \leqslant n,\ \frac{m}{2} + 1 \leqslant s \leqslant m\}.$$

It is easy to see that the function $f : V(G) \to \{-1, +1\}$ defined by $f(x) = +1$ for $x \in V_1 \cup \{v_1, \ldots, v_{\frac{n+1}{2}}\}$ and $f(x) = -1$ otherwise, is an TSDF of D of weight m and so $\mathrm{dom}^*_s(K_{m,n}) \leqslant m+1$. Let now $m \geqslant 8$. Assume that D is an orientation of $K_{m,n}$ such that (5) holds. Define $f : V(G) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in \{u_1, \ldots, u_{\frac{m}{2}+2}\} \cup \{v_1, \ldots, v_{\frac{n+1}{2}+2}\}$ and $f(x) = -1$ otherwise. It is easy to verify that f is a TSDF of D of weight 9 and so $\mathrm{dom}^*_s(K_{m,n}) \leqslant 9$.

Case 3: assume that m is odd and n is even. If $m \leqslant 7$, then let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (v_j, u_s) \mid 1 \leqslant i \leqslant \frac{m+1}{2}, 1 \leqslant j \leqslant n, \frac{m+1}{2} + 1 \leqslant s \leqslant m\},$$

and define $f : V(G) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in V_1 \cup \{v_1, \ldots, v_{\frac{n}{2}}\}$ and $f(x) = -1$ otherwise. It is easy to see that f is a TSDF of D of weight m and so $\mathrm{dom}^*_s(K_{m,n}) \leqslant m$. Let now $m \geqslant 9$ and let D be an orientation of $K_{m,n}$ such that (5) holds. Define $f : V(G) \to \{-1, +1\}$ by $f(x) = +1$ for $x \in \{u_1, \ldots, u_{\frac{m+1}{2}+2}\} \cup \{v_1, \ldots, v_{\frac{n}{2}+2}\}$ and $f(x) = -1$ otherwise. Obviously, f is a TSDF of D of weight 9 implying that $\mathrm{dom}^*_s(K_{m,n}) \leqslant 9$.

Case 4: assume that m and n are both odd. If $m \leqslant 7$, then orient the edges of $K_{m,n}$ such that the resulting digraph has the arc set

$$A(D) = \{(u_i, v_j), (v_j, u_s) \mid 1 \leqslant i \leqslant \frac{m+1}{2},\ 1 \leqslant j \leqslant n,\ \frac{m+1}{2} + 1 \leqslant s \leqslant m\}.$$

It is easy to see that the function $f : V(G) \to \{-1,+1\}$ defined by $f(x) = +1$ for $x \in V_1 \cup \{v_1, \ldots, v_{\frac{n+1}{2}}\}$ and $f(x) = -1$ otherwise, is a TSDF of D of weight $m+1$ and so $\mathrm{dom}_s^*(K_{m,n}) \leqslant m+1$. Let now $m \geqslant 9$ and let D be an orientation of $K_{m,n}$ such that

$$A(D) = \{(u_i, v_j), (v_r, u_s), (v_l, u_t) \mid i, j, t \notin \{1,2\},\ 1 \leqslant s \leqslant n,\ r, l \in \{1,2\}\}.$$

Define $f : V(G) \to \{-1,+1\}$ by $f(x) = +1$ for $x \in \{u_1, \ldots, u_{\frac{m+1}{2}+2}\} \cup \{v_1, \ldots, v_{\frac{n+1}{2}+2}\}$ and $f(x) = -1$ otherwise. It is easy to see that f is a TSDF of D of weight 10 and so $\mathrm{dom}_s^*(K_{m,n}) \leqslant 10$.
Now the result follows by Corollary 31. □

References

[1] S. Arumugam, K. Ebadi and L. Sathikala, *Twin domination and twin irredundance in digraphs*, Appl. Anal. Discrete Math. **7** (2013), 275–284.

[2] M. Atapour, R. Hajypory, S.M. Sheikholeslami and L. Volkmann, *The signed k-domination number of directed graphs*, Cent. Eur. J. Math. **8** (2010), 1048-1057.

[3] G. Chartrand, P. Dankelmann, M. Schultz and H.C. Swart, *Twin domination in digraphs*, Ars Combin. **67** (2003), 105–114.

[4] J.E. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, *Signed domination in graphs*. In Y. Alavi and A.J. Schwenk, editors, *Graph Theory, Combinatorics, and Applications, Proc. 7th Internat. Conf. Combinatorics, Graph Theory, Applications*, volume 1, pages 311-322. John Wiley and Sons, Inc., 1995.

[5] M.A. Henning and P.J. Slater, *Inequalities relating domination parameters in cubic graphs*, Discrete Math. **158** (1996), 87-98.

[6] H. Karami, A. Khodkar and S.M. Sheikholeslami, *Lower bounds on signed domination number of a digraph*, Discrete Math. **309** (2009), 2567–2570.

[7] C. Wang *The signed k-domination numbers in graphs*, Ars Combin. **106** (2012), 205-211.

[8] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.

[9] B. Zelinka, *Signed domination numbers of directed graphs*, Czechoslovak Math. J. **55** (2005), 479–482.

[10] Z. Zhang, B. Xu, Y. Li and L. Liu, *A note on the lower bounds of signed domination number of a graph*, Discrete Math. **195** (1999), 295–298.

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