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FULL CASCADES OF SIMPLE PERIODIC ORBITS ON THE INTERVAL[1]

ПОВНИЙ КАСКАД ПЕРІОДИЧНИХ ОРБІТ НА ВІДРІЗКУ

Any continuous interval map of type greater than 2^∞ is shown to have what we call a full cascade of simple periodic orbits. This is used to prove that, for maps of any types, the existence of such a full cascade is equivalent to the existence of an infinite ω-limit set. For maps of type 2^∞, this is equivalent to the existence of a (period doubling) solenoid. Hence, any map of type 2^∞ which is either piecewise monotone (with finite number of pieces) or continuously differentiable has both a full cascade of simple periodic orbits and a solenoid.

Показано, що кожне неперервне відображення відрізка прямої, тип якого більший ніж 2^∞, має повний каскад періодичних орбіт. Це використовується для того, щоб показати, що для відображень довільного типу існування таких повних каскадів еквівалентне існуванню нескінченних ω-граничних множин. Для відображень типу 2^∞ це еквівалентно існуванню (двоперіодичного) соленоїда. Таким чином, довільне відображення типу 2^∞, яке є або кусково-монотонним, або неперервно диференційовним, має повний каскад простих орбіт та соленоїд.

1. Introduction and main results. The notion of a *simple periodic orbit* of period 2^n, $n \in \{0\} \cup \mathbb{N}$, of a continuous map of the interval belongs to Block [1] and can be defined by induction on n. Any periodic orbit of f of period 1 is simple and if P is a periodic orbit of f of period 2^n, where $n > 0$, then P is called simple if the left and right halves of P each form simple orbits of f^2 with period 2^{n-1}. Suppose that f has a periodic orbit of period 2^n for some $n \in \mathbb{N}$, say $n = 3$. In combinatorial dynamics it is well known (see, e.g., [2], Corollary 2.11.2) that then f also has a simple periodic orbit $\{d_1, d_2, \ldots, d_8\}$ of period 2^3 and, further, simple periodic orbits $\{c_1, c_2, c_3, c_4\}$, $\{b_1, b_2\}$, and $\{a\}$ of periods 2^2, 2^1, and 2^0, respectively, whose points "interwind" as follows:

$$d_1 < c_1 < d_2 < b_1 < d_3 < c_2 < d_4 < a < d_5 < c_3 < d_6 < b_2 < d_7 < c_4 < d_8 .$$

Moreover, $f(\{d_1, d_2\}) = \{d_5, d_6\}$ or $\{d_7, d_8\}$ depending on whether $f(c_1) = c_3$ or c_4 and similarly for f-images of $\{d_3, d_4\}$, $\{d_5, d_6\}$, and $\{d_7, d_8\}$. In the present paper, we will refer to this situation as a cascade of simple periodic orbits of f of depth 3 (cf. also Fig. 1). It is also natural to define cascades of infinite depth (we will call them full cascades of simple periodic orbits) and to ask a question which maps have them.

In the present paper, we show that a map of type greater than 2^∞ necessarily has a full cascade of simple periodic orbits and then we use this fact to prove that, for

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maps of any types, the existence of such a cascade is equivalent to the existence of an infinite ω-limit set. For maps of type 2^∞, this is equivalent to the existence of a solenoid. As a consequence, we get that any map of type 2^∞ which is either piecewise monotone or continuously differentiable has a solenoid (in other words, the map is infinitely renormalizable). It seems that this fact, though straightforwardly implied by a combination of some known results, has not been stated explicitly yet (cf. [3] or Remark after Corollary C).

We start with notation and some precise definitions. Let I be a real compact interval and let $C(I)$ be the set of continuous maps from I into itself. Let $\mathbb{N}$ be the set of positive integers. A point $p \in I$ is a *periodic point* of a map $f \in C(I)$ if $f^n(p) = p$ for some $n \in \mathbb{N}$. The *period* of p is the least such integer n, and the *orbit* of p under f is the set $\mathrm{orb}_f(p) = \{f^k(p): k = 0, 1, \ldots, n-1\}$. We refer to such an orbit as a *periodic orbit* of f of period n. Let $P(f)$ denote the set of periodic points of f and let $\mathrm{Per}(f)$ be the set of their periods.

Consider the Sharkovskii ordering of the set $\mathbb{N} \cup \{2^\infty\}$:

$$3 \succ 5 \succ 7 \succ \ldots \succ 2\cdot 3 \succ 2 \cdot 5 \succ 2 \cdot 7 \succ \ldots \succ 4 \cdot 3 \succ 4 \cdot 5 \succ 4 \cdot 7 \succ \ldots \succ \ldots$$

$$\succ 2^n \cdot 3 \succ 2^n \cdot 5 \succ 2^n \cdot 7 \succ \ldots \succ \ldots \succ 2^\infty \succ \ldots \succ 2^n \succ \ldots \succ 4 \succ 2 \succ 1.$$

We will also use the symbol $\succeq$ in the natural way. For $n \in \mathbb{N} \cup \{2^\infty\}$, we denote by $S(n)$ the set $\{k \in \mathbb{N}: n \succeq k\}$ ($S(2^\infty)$ stands for the set $\{1, 2, 4, \ldots, 2^k, \ldots\}$). The Sharkovskii theorem [4] says that *for every $f \in C(I)$, there exists $n \in \mathbb{N} \cup \{2^\infty\}$ such that $\mathrm{Per}(f) = S(n)$; conversely, for every $n \in \mathbb{N} \cup \{2^\infty\}$, there exists $f \in C(I)$ with $\mathrm{Per}(f) = S(n)$* (see [4, 5]). If $\mathrm{Per}(f) = S(n)$, then f is said to be *of type* n. Thus, any map $f \in C(I)$ is of some type and, for every $n \in \mathbb{N} \cup \{2^\infty\}$, there is a map of type n. When speaking of types, we consider them to be ordered by the Sharkovskii ordering. Thus, if a map f is of type 2^∞, then $\mathrm{Per}(f) = \{1, 2, \ldots, 2^k, \ldots\}$, and if f is of type greater than 2^∞, then it has a periodic point with period that is not a power of 2.

Whenever we say that $P = \{p_1, p_2, \ldots, p_n\}$ is a periodic orbit of f, we suppose that $p_1 < p_2 < \ldots < p_n$, i.e., we always use the spatial labelling of periodic orbits. Further, for any $k \in \mathbb{N}$ dividing n, define $P(k, i) = \{p_{(i-1)k+1}, p_{(i-1)k+2}, \ldots, p_{ik}\}$, $i = 1, 2, \ldots, n/k$. So the set $P(k, i)$ is the ith k-tuple of points from P. Note that $P(1, i) = \{p_i\}$, $i = 1, 2, \ldots, n$. So, a periodic orbit $P = \{p_1, p_2, \ldots, p_n\}$ of f of period a power of two, $n = 2^m$, $m \in \{0\} \cup \mathbb{N}$, is a *simple periodic orbit* (SPO, for short) if for any $k \in \{0, 1, \ldots, m\}$, $P(2^k, i)$ is a periodic orbit of $f^{2^{m-k}}$ of period 2^k, $i = 1, 2, \ldots, 2^{m-k}$.

A map $f \in C(I)$ is called *piecewise monotone* if there are points $\min I = a_0 < a_1 < \ldots < a_n = \max I$ such that for every $i \in \{1, \ldots, n\}$, the restriction of f to the interval $[a_{i-1}, a_i]$ is (not necessarily strictly) monotone.

When speaking of continuously differentiable maps from $C(I)$, at the endpoints of I, we have one-sided derivatives in mind.

The convex hull of a set $A \subset I$ will be denoted by $\mathrm{conv}A$ and the usual distance of points or sets on the real line by $\mathrm{dist}(.,.)$. The set of all limit points of the trajectory $\{f^n(x)\}_{n=0}^\infty$ of a point x is called the *ω-limit set of x under f* and denoted by $\omega_f(x)$.

Given $f \in C(I)$, a closed subinterval J of I is periodic with period n if $f^n(J) = J$ and $f^k(J) \cap f^l(J) = \emptyset$ for any $0 \le k < l < n$. Further, $S \subset I$ is called a (period doubling or simple) *solenoid* of f if $S = \bigcap_{n=0}^\infty \bigcup_{k=0}^{2^n-1} f^k(I^n)$, where for any n, I^n is

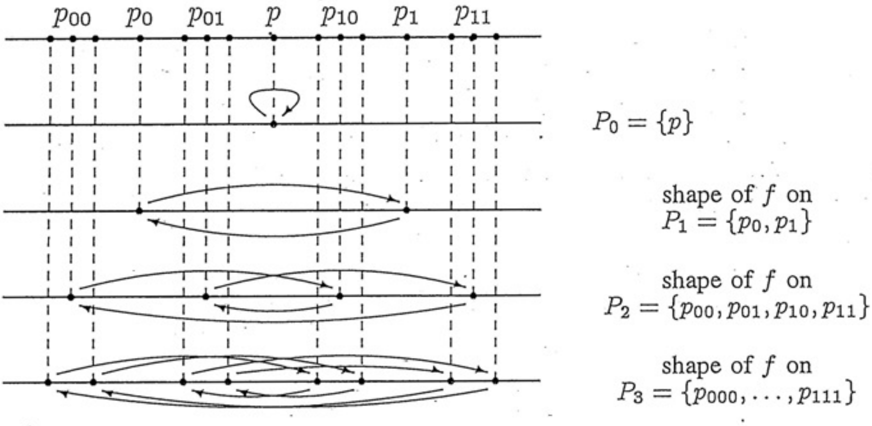

Fig. 1. A CSPO of f of depth $n = 3$.

a periodic interval of period 2^n such that $I^n \supset I^{n+1}$. (See [6] for more information on solenoids.)

Definition. *A cascade of simple periodic orbits (CSPO , for short) of a map $f \in C(I)$ of depth $n \in \mathbb{N}$ is a finite sequence $(P_0, P_1, P_2, \dots, P_n)$ satisfying the following three conditions:*

(i) for $k = 0, 1, \dots, n$, P_k is an SPO of f of period 2^k,

(ii) for $k = 0, 1, \dots, n-1$ and $i = 1, 2, \dots, 2^k$, $(\operatorname{conv} P_{k+1}(2, i)) \cap (P_0 \cup \ldots \cup P_k) = P_k(1, i)$,

(iii) for $k = 0, 1, \dots, n-1$, if $f(P_k(1, i)) = P_k(1, j)$ for some $i, j \in \{1, 2, \dots, 2^k\}$, then $f(P_{k+1}(2, i)) = P_{k+1}(2, j)$. The orbit P_k, $k = 0, 1, \dots, n$, will be called the k-th term of the cascade, P_0 will be also called its initial term, and $(P_0, P_1, \dots, P_k)$, $k \leqslant n$, its initial k-block.

For an example of a CSPO of depth $n = 3$, see Fig. 1. Clearly, if $(P_0, P_1, \dots, P_n)$ is a CSPO of f of depth n and $k < n$, then the initial k-block $(P_0, P_1, \dots, P_k)$ is a CSPO of f of depth k.

Definition. *An infinite sequence $(P_0, P_1, P_2, \dots)$ is a full cascade of simple periodic orbits (FCSPO , for short) of a map $f \in C(I)$ if, for any $n \in \mathbb{N}$, $(P_0, P_1, \dots, P_n)$ is a CSPO of f of depth n.*

Similarly as in the case of a CSPO , we will use the notions of terms and initial blocks of an FCSPO .

We prove the following statement:

Theorem A. *If $f \in C(I)$ is of type greater than 2^∞ then it has an FCSPO*

Then this result will be used in the proof of the following theorem:

Theorem B. *For $f \in C(I)$, the following conditions are equivalent:*

(a) f has an FCSPO ,

(b) f has a Cantor-like ω-limit set,

(c) f has an infinite ω-limit set.

Moreover, these conditions are implied by the condition

(d) f has a solenoid, and if f is of type 2^∞, then so (d) is equivalent to (a)–(c).

***Remark* 1.** Here, "a Cantor-like set" means, of course, "a set homeomorphic to the Cantor set". But we can claim more. Suppose that f has an FCSPO , denote it by $\{P_0, P_1, P_2, \dots\}$, $P_n = \{p_1^{(n)}, p_2^{(n)}, \dots, p_{2^n}^{(n)}\}$, $n = 0, 1, 2, \dots$, and put $P = \bigcup_{n=0}^{\infty} P_n$.

It will be seen from the proof of $(a) \Rightarrow (b)$ that as the Cantor-like ω-limit set from condition (b), we can take the set $C = \overline{P} \setminus P$ and that, for every $n = 1, 2, \ldots$ the set C can be decomposed into 2^n sets $C_1^{(n)}, C_2^{(n)}, \ldots, C_{2^n}^{(n)}$ with disjoint convex hulls such that $\operatorname{conv} C_i^{(n)} \ni p_i^{(n)}$, $i = 1, 2, \ldots, 2^n$ and $f(C_i^{(n)}) = C_j^{(n)}$ whenever $f(p_i^{(n)}) = p_j^{(n)}$, $i, j = 1, 2, \ldots, 2^n$, $i \neq j$. Thus, f is "solenoidal" on that ω-limit set and "agrees" there with that FCSPO . Of course, this does not mean that f must have a solenoid if f is not of type 2^∞.

***Remark* 2.** For some other conditions equivalent to (c), see, e.g., [7] or [8], Proposition VI.10. One of them is that the set $P(f)$ is not closed, which was originally proved by Sharkovskii [5].

Let us discuss Theorem B from the point of view of the type of a map.

First, that if a map $f \in C(I)$ has an SPO of period 2^n, then f obviously has a CSPO of depth n. By [1], all maps of types at most 2^∞ have only periodic orbits which are simple. So if a map is of type 2^n for some $n \in \{0\} \cup \mathbb{N}$, then it has a CSPO of depth n (but no deeper CSPO) and if a map is of type 2^∞, then for any n, it has a CSPO of depth n. Some of the maps of type 2^∞ have even an FCSPO , e.g., the logistic map from $C([0,1])$, namely, $F_{\lambda_*}(x) = \lambda_* x(1-x)$, $\lambda_* \approx 3.569\ldots$. This follows from the well-known fact that this map has an infinite ω-limit set, which is also a solenoid of $F_{\lambda_*}(x)$ (see, e.g., [9]; cf. also Corollary C). But some of maps of type 2^∞ have no FCSPO . An example of such a map $F \in C([0,1])$ can be found in [10]. The map F can be constructed in such a way that $F(1) = 1$ and for every $n = 0, 1, 2, \ldots$, F maps the interval $\left[\frac{n}{n+1}, \frac{n+1}{n+2}\right]$ into itself, and F restricted to this interval is of type 2^n. (With a little care, one can modify the construction of F to get a map which is differentiable on $[0,1]$ and has the derivative $F'(1) = 1$ but F' will not be continuous at the point 1; cf. Corollary C.)

It is well known (see [11], cf. [5]) that if a map f is of type less than 2^∞, i.e., of type 2^n for some $n \in \{0\} \cup \mathbb{N}$, then the ω-limit set of any point under f is a periodic orbit of f and is thus finite. Therefore, in this case, none of conditions (a)–(d) from Theorem B is fulfilled.

If $f \in C(I)$ is of type greater than 2^∞, then, by Theorems A and B, it satisfies conditions (a)–(c). (The fact that a map of type greater than 2^∞ always has an infinite ω-limit set was, of course, well known before [see [5] or use the fact that such a map has a horseshoe [2])]. Condition (d) may or may not be fulfilled. For example, it is known that piecewise linear maps have no solenoids (see [12]; cf. [13]).

Finally, consider a map f of type 2^∞. The examples given above show that f may or may not have an infinite ω-limit set. In [7], it is proved that if, in addition, f is continuously differentiable, then it has an infinite ω-limit set. Later, this result was extended. In [8], Proposition II.28 and Proposition VI.10, it is proved that if f is of type 2^∞ and either piecewise monotone or continuously differentiable, then it has an infinite ω-limit set. (Though the definition of piecewise monotonicity used in [8] is more restrictive than the one used in the present paper, one can see that the proof also remains valid for maps that are piecewise monotone in our sense. Another and very short proof in the piecewise monotone case can be found in [12].) So, a map f of type 2^∞ may or may not satisfy the equivalent conditions (a)–(d) and, by Theorem B and the mentioned results, the following statement is true:

Corollary C. *If $f \in C(I)$ is of type 2^∞ and either piecewise monotone or continuously differentiable, then it satisfies conditions (a)–(d) of Theorem B [in particular, f has a (period doubling) solenoid].*

***Remark* 3.** Hu and Tresser [3] have recently proved that maps of type 2^∞ belonging to a special family of piecewise monotone maps of the class C^3 (including analytic maps) have (period doubling) solenoids. They follow a completely different approach to ours.

The proofs of the results from [8] and [7] we used to get Corollary C (see the paragraph above it) are not easy to follow in the sense that they involve several auxiliary results, some of them being not very well known. In Section 3, we give another proof of the statement that a map of type 2^∞ has an FCSPO if it is piecewise monotone or continuously differentiable.

2. Proofs of Theorems A and B. Before passing to the proof of Theorems A and B, we introduce some necessary definitions and notation. If $A, B \subset I$, then $A < B$ means that $a < b$ whenever $a \in A$, $b \in B$. Instead of $\{a\} < B$, we also write $a < B$ and, similarly, $A < b$. If $A < C$ and $C < B$, we say that C lies between A and B or, if $C = \{c\}$, that c lies between A and B. $\overline{A}$ and $\mathrm{int}A$ are the closure and the interior of A. If $f: A \to B$ and $C \subset A$, then $f|_C$ is the restriction of f to C.

For any $j \in \mathbb{N}$, $j = \infty$, we define $\{0,1\}^j$ as the set of finite sequences of 0's and 1's of length j (infinite sequences, respectively). For the convenience of the notation, we will also write $\{0,1\}^0 = \emptyset$. The i-th element of a (finite or infinite) sequence α will be denoted by α_i. If $\alpha \in \{0,1\}^\infty$, $\beta = \alpha|^j \in \{0,1\}^j$ is defined by $\beta_i = \alpha_i$ for any $i = 1, 2, \ldots, j$. If $\alpha \in \{0,1\}^j$, $j \in \mathbb{N}$, we define $\beta = \alpha 0 \in \{0,1\}^{j+1}$ ($\alpha 1$, respectively) by $\beta_i = \alpha_i$ if $i = 1, 2, \ldots, j$ and $\beta_{j+1} = 0$ ($\beta_{j+1} = 1$, respectively). We will also put $\emptyset 0 = 0$, $\emptyset 1 = 1$. If $j \geqslant 2$ ($j = \infty$, respectively) we define $\beta = \sigma(\alpha) \in \{0,1\}^{j-1}$ ($\beta = \sigma(\alpha) \in \{0,1\}^\infty$, respectively) by $\beta_i = \alpha_{i+1}$ for any i. We say that $\alpha \in \{0,1\}^\infty$ is a periodic sequence if there exists $n \in \mathbb{N}$ such that $\alpha_{i+n} = \alpha_i$ for any i. The least integer n with this property will be called the period of α. Finally, let $\alpha, \beta \in \{0,1\}^j$, $j \in \mathbb{N} \cup \{\infty\}$. We say that $\alpha < \beta$ if $\alpha_1 < \beta_1$ or there exists k such that $\alpha_i = \beta_i$ for any $i = 1, 2, \ldots, k$ and either $\sum_{i=1}^k \alpha_i$ is even and $\alpha_{k+1} < \beta_{k+1}$ or $\sum_{i=1}^k \alpha_i$ is odd and $\alpha_{k+1} > \beta_{k+1}$.

Proof of Theorem A. Let $l = 2^n(2p+1)$ be the type of f, $n \geqslant 0$, $p \geqslant 1$. By [14] and Corollary 2.11.2 from [2] (see also [15]), there exists a periodic orbit $P = \{p_1, p_2, \ldots, p_l\}$ with the following properties:

1. For any $r, s \in \mathbb{N}$ such that $rs = l$ and $r = 2^t$ for some $t \in \{0, 1, \ldots, n\}$, $P(s, k)$ is a periodic orbit of f^r with period s, $k = 1, 2, \ldots, r$.

2. There exists $m \in \{1, 2, \ldots, 2^n\}$ such that we can write the elements of $P(2p+1, m)$ as $q_1, q_2, \ldots, q_{2p+1}$ in such a way that $f^{2^n}(q_i) = q_{i+1}$ for any $i = 1, 2, \ldots, 2p$ and $f^{2^n}(q_{2p+1}) = q_1$, and either

$$q_{2p+1} < q_{2p-1} < \ldots < q_3 < q_1 < q_2 < \ldots < q_{2p-2} < q_{2p}$$

or

$$q_{2p} < q_{2p-2} < \ldots < q_2 < q_1 < q_3 < \ldots < q_{2p-1} < q_{2p+1}.$$

Moreover, $f|_{P(2p+1,k)}$ is monotone for every $k \neq m$.

An example of such an orbit can be found in Fig. 2.

Note that in the case $n > 0$, we can easily construct, using property (1), a CSPO $(P_0, P_1, \ldots, P_{n-1})$ such that for the set $\bigcup_{i=0}^{n-1} P_i = \{a_1, a_2, \ldots, a_{2^n-1}\}$, we have $P(2p+1, k) < a_k < P(2p+1, k+1)$ for any $k = 1, 2, \ldots, 2^n - 1$.

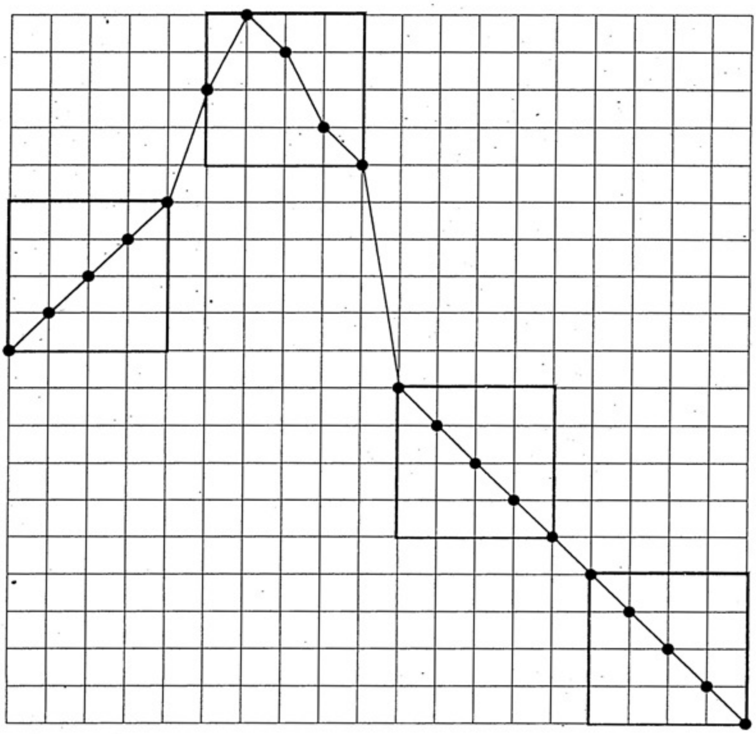

Fig. 2. A periodic orbit P of period 20 for a function f of type 20.

Let $\tau\colon \{1,\ldots,2^n\} \to \{1,\ldots,2^n\}$ be a bijection such that $f(P(2p+1,\tau(k))) = P(2p+1,\tau(k+1))$ for any $k = 1,2,\ldots,2^n-1$ and $f(P(2p+1,\tau(2^n))) = P(2p+1,\tau(1))$. In what follows, we assume that $m = \tau(2^n)$, $q_1 < q_2$ and (if $n > 0$) $f|_{P(2p+1,\tau(2^n-1))} \circ f|_{P(2p+1,\tau(2^n-2))} \circ \cdots \circ f|_{P(2p+1,\tau(1))}$ is increasing (the other cases are analogous).

For any $i = 1,2$ and $k = 1,2,\ldots,2^n$, we take $q_i^k \in P(2p+1,\tau(k))$ such that $f^{2^n-k}(q_i^k) = q_i$. Then there exists a periodic orbit $P_n = \{b_1, b_2, \ldots, b_{2^n}\}$ of f with period 2^n such that $b_k \in \operatorname{conv}\{q_1^k, q_2^k\}$ for any k. Moreover, for every $j \in \{0\} \cup \mathbb{N}$, $\alpha \in \{0,1\}^j$ and $k = 1,2,\ldots,2^{n+1}$ we can define compact intervals I_α^k with the following properties:

(i) $I_\emptyset^k \cup I_\emptyset^{k+2^n} \subset \operatorname{conv}P(2p+1,\tau(k))$ and b_k lies between $\operatorname{int}I_\emptyset^k$ and $\operatorname{int}I_\emptyset^{k+2^n}$ for any $k = 1,2,\ldots,2^n$.

(ii) $I_{\alpha 0}^k \cup I_{\alpha 1}^k \subset I_\alpha^k$ for any $j \in \{0\} \cup \mathbb{N}$, $\alpha \in \{0,1\}^j$ and $k = 1,2,\ldots,2^{n+1}$.

(iii) Let $\alpha,\beta,\gamma \in \{0,1\}^j$, $j \in \{0\} \cup \mathbb{N}$. Then I_α^k lies between I_β^k and I_γ^k if and only if $I_\alpha^{2^{n+1}}$ lies between $I_\beta^{2^{n+1}}$ and $I_\gamma^{2^{n+1}}$, $k = 1,2,\ldots,2^{n+1}$. Moreover, $I_\alpha^{2^{n+1}} < I_\beta^{2^{n+1}}$ if and only if $\beta < \alpha$.

(iv) For any $j \in \{0\} \cup \mathbb{N}$ and $\alpha \in \{0,1\}^j$, $f(I_\alpha^k) = I_\alpha^{k+1}$ for any $k = 1,2,\ldots,2^{n+1}-1$, and, $f(I_\alpha^{2^{n+1}}) = I_{\sigma(\alpha)}^1$ for any $j \in \mathbb{N}$ and $\alpha \in \{0,1\}^j$.

Now define $I_\alpha^k = \bigcap_{j=1}^\infty I_{\alpha|^j}^k$ for any $\alpha \in \{0,1\}^\infty$ and $k = 1, 2, \ldots, 2^{n+1}$. Since $f(I_\alpha^k) = I_\alpha^{k+1}$ for any $\alpha \in \{0,1\}^\infty$ if $k < 2^{n+1}$, and $f(I_\alpha^{2^{n+1}}) = I_{\sigma(\alpha)}^1$ if α is a periodic sequence of period u and $\alpha_i \neq 0$ for some i, then there exists a periodic point $p_\alpha \in I_\alpha^{2^{n+1}}$ of f with period $2^{n+1}u$.

Let us now return to the logistic map $F_{\lambda_*}(x) = \lambda_* x(1-x)$ introduced earlier. Let $(Q_0, Q_1, Q_2, \ldots)$ be an FCSPO for F_{λ_*}. For any $c \in Q_i$, $i = 0, 1, 2, \ldots$, define its kneading sequence $\theta(c) \in \{0,1\}^\infty$ by $\theta(c)_j = 0$ if $f^{j-1}(c) < 1/2$ and by $\theta(c)_j = 1$ if $f^{j-1}(c) > 1/2$, $j = 1, 2, \ldots$. Note that $\theta(c)$ is well defined and $c < d$ if and only if $\theta(c) < \theta(d)$. Fix $c_i \in Q_i$ for each i and define $P_{n+1+i} = \mathrm{orb}_f(p_{\theta(c_i)})$, $i = 0, 1, 2, \ldots$. Since $(P_{n+1} \cap I_\emptyset^{2^{n+1}}, P_{n+2} \cap I_\emptyset^{2^{n+1}}, \ldots)$ is an FCSPO for $f^{2^{n+1}}$ and $f|_{I_\emptyset^k \cap (\cup_{i=0}^\infty P_{n+1+i})}$ is monotone for any $k = 1, 2, \ldots, 2^{n+1} - 1$; we get [use also (i)] that $(P_0, P_1, P_2, \ldots, P_n, P_{n+1}, P_{n+2}, \ldots)$ is an FCSPO for f.

To prove Theorem B we need the following statement:

Lemma 1. *Let $f \in C(I)$, $(P_0, P_1, P_2, \ldots)$ be an FCSPO of f, let $P = \bigcup_{i=0}^\infty P_i$, and let $\pi_i^r = \bigcup_{k=0}^\infty P_{r+k}(2^k, i)$, $r \in \mathbb{N}$, $i = 1, 2, \ldots, 2^r$. Then*

1) *every point $x \in P$ is isolated in P;*

2) *if $a \in \overline{P} \setminus P$ and $\varepsilon > 0$, then there are $r \in \mathbb{N}$ and $i \in \{1, 2, \ldots, 2^r\}$ such that $(a - \varepsilon, a + \varepsilon) \supset \pi_i^r$.*

Proof. Note that, for any r, the sets π_i^r, $i = 1, 2, \ldots, 2^r$, are permuted by f (they form a cycle of sets for f).

1) Let $P_0 = \{p\}$, $P_1 = \{p_0, p_1\}$, and $P_2 = \{p_{00}, p_{01}, p_{10}, p_{11}\}$. Since P_2 is an SPO , we cannot simultaneously have $f(p_{01}) = p_{11}$ and $f(p_{10}) = p_{00}$. Without loss of generality, we can assume that $f(p_{01}) = p_{10}$. Then $f(p_{10}) = p_{00}$ and so $f(\pi_3^2) = \pi_1^2 < p_0$ and $f(\pi_2^2) = \pi_3^2$. In view of the fact that p is a fixed point of f, this implies that $\inf \pi_3^2 > p$ and, hence, $\sup \pi_2^2 < p$. Hence, the point p is isolated in P.

Further, note that $(\{p_0\}, P_2(2,1), P_3(2^2,1), \ldots)$ is an FCSPO of f^2 and, therefore, the same argument as above shows that the point p_0 is isolated in the set $\{z \in P : z < p\}$ and, consequently, in the set P. For an analogous reason, p_1 is isolated in P.

It is now easy to see that, by induction, one can prove that all points from P are isolated in P.

2) Since $a \in \overline{P} \setminus P$, $(a - \varepsilon, a + \varepsilon) \cap P$ contains periodic points q_1, q_2 of f with different periods $2^u < 2^v$. Then for some i, $(a - \varepsilon, a + \varepsilon) \supset \pi_i^{v+1}$. Denote $v + 1 = r$.

Proof of Theorem B. "$(d) \Rightarrow (a)$" Since a periodic interval of f with period 2^n contains a periodic point of period 2^n, this can be seen from the definition of the solenoid.

"$(a) \Rightarrow (b)$" The nonempty set $\overline{P} \setminus P$ is closed by Lemma 1(1) and dense in itself by Lemma 1(2). Finally, $\overline{P} \setminus P$ is obviously nowhere dense. Thus, $\overline{P} \setminus P$ is Cantor-like.

Now take any $x \in \overline{P} \setminus P$. We are going to prove that $\omega_f(x)$ is infinite. First, it follows from Lemma 1 (2) that $\omega_f(x) \ni a$ for any $a \in \overline{P} \setminus P$ (note that for any r, x belongs to some $\overline{\pi_j^r}$ [cf. Lemma 1(1)] and that the sets π_i^r, $i = 1, 2, \ldots, 2^r$ form a cycle of sets for f). Thus $\omega_f(x) \supset \overline{P} \setminus P$ and so $\omega_f(x)$ is infinite. Moreover, $\omega_f(x) = \overline{P} \setminus P$. In fact, $\overline{P}$ is closed, $f(\overline{P}) \subset \overline{P}$, and so $\omega_f(x) \subset \overline{P}$ and $\omega_f(x)$, being infinite, cannot contain any point from P which is, by Lemma 1 (1), isolated in P.

"$(b) \Rightarrow (c)$" is trivial. We prove "$(c) \Rightarrow (a)$". A map f having an infinite ω-limit set is either of type greater than 2^∞ or of type 2^∞. In the former case, we use Theorem A. In the latter case, recall a well-known result implicit in several Sharkovskii's papers and proved in [16] stating that every infinite ω-limit set of a map of type 2^∞ is contained in a solenoid. Now use the implication $(d) \Rightarrow (a)$ proved above.

To complete the proof it suffices to prove "$(c) \Rightarrow (d)$" provided that f is of type 2^∞. But this was shown in the proof of $(c) \Rightarrow (a)$.

3. Appendix. Here, we give another proof of the following result (see the last paragraph in Section 1):

Proposition D. *If $f \in C(I)$ is of type 2^∞ and either piecewise monotone or continuously differentiable then it has an FCSPO .*

The proof will be "constructive" in some sense: We will build the FCSPO of f step by step.

We start with definitions.

We say that two periodic orbits of f of the same period n, $P = \{p_1, p_2, \ldots, p_n\}$ and $Q = \{q_1, q_2, \ldots, q_n\}$, have the same *oriented pattern* if for each $r, s \in \{1, 2, \ldots, n\}$, $f(q_r) = q_s$ if and only if $f(p_r) = p_s$.

A CSPO $(P_0, P_1, \ldots, P_n)$ of f of depth n is said to be *arbitrarily extendable* if for every $k > n$, there is a CSPO of f of depth k such that $(P_0, P_1, \ldots, P_n)$ is its initial n-block.

Lemma 2 *(left version). Let $f \in C(I)$ and let $(\{x^{(n)}\}, P_1^{(n)}, \ldots, P_{k_n}^{(n)})$, $n = 0, 1, 2, \ldots$, be a CSPO of f. Let x_0 be a point with $\{x^{(n)}: n = 0, 1, 2, \ldots\} < x_0$ and $x^{(n)} \to x_0$ as $n \to \infty$. Denote $R^{(n)} = \{z \in P_1^{(n)} \cup \ldots \cup P_{k_n}^{(n)}: z > x^{(n)}\}$, $n = 0, 1, 2, \ldots$, and suppose that, for infinitely many n's, $R^{(n)}$ contains a point less than x_0. Then f is neither piecewise monotone nor continuously differentiable.*

Proof. Let the assumptions be satisfied. Without loss of generality, we can assume that a point $r^{(n)} \in R^{(n)}$ less than x_0 exists for every n. Using the fact that $x^{(n)} \to x_0$, one can find a sequence of indices $(n_i)_{i=1}^\infty$ such that

$$x^{(0)} < r^{(0)} < x^{(n_1)} < r^{(n_1)} < \ldots < x^{(n_i)} < r^{(n_i)} < \ldots .$$

Since we have $f(r^{(n)}) < x^{(n)}$ for every n, f cannot be piecewise monotone.

Now suppose that f is differentiable. Then for every i, there are points $z^{(n_i)}$ and $y^{(n_i)}$ such that $x^{(n_i)} < z^{(n_i)} < r^{(n_i)} < y^{(n_i)} < x^{(n_{i+1})}$ and $f'(z^{(n_i)}) < 0$, $f'(y^{(n_i)}) > 1$. Since $z^{(n_i)} \to x_0$ and $y^{(n_i)} \to x_0$ as $i \to \infty$, the derivative of f is discontinuous at the point x_0.

Obviously, the *right version* of this lemma also holds. In it, write $\{x^{(n)}: n = 0, 1, 2, \ldots\} > x_0$ and suppose that, for infinitely many n's, $L^{(n)} = \{z \in P_1^{(n)} \cup \ldots \cup P_{k_n}^{(n)}: z < x^{(n)}\}$ contains a point greater than x_0.

Lemma 3. *Let $f \in C(I)$ be either piecewise monotone or continuously differentiable and let $n \in \{0\} \cup \mathbb{N}$. Let $(P_0, P_1, \ldots, P_n, P_{n+1}^{(i)}, P_{n+2}^{(i)})$, $i = 0, 1, 2, \ldots$, be a CSPO of f. Then*

1) dist $\left(P_n, \bigcup_{i=0}^\infty P_{n+1}^{(i)}\right) > 0$;

2) if $a^{(i)} \in P_{n+1}^{(i)}$, $i = 0, 1, 2, \ldots$, and $a^{(i)} \to a_0$ as $i \to \infty$, then a_0 is a periodic point of f with period 2^{n+1}. Moreover, have for sufficiently large i's, the orbits $P_{n+1}^{(i)}$ the same oriented pattern as $orb_f(a_0)$, and $(P_0, P_1, \ldots, P_n, orb_f(a_0))$ is a CSPO of f.

Proof. 1) This holds trivially if the sequence $(P_{n+1}^{(i)})_{i=0}^\infty$ contains only finitely many mutually different orbits. So suppose this is not the case. Then we may assume that all orbits $P_{n+1}^{(i)}$, $i = 0, 1, 2, \ldots$, are mutually different.

Consider the orbit $P_n = \{p_1, p_2, \ldots, p_{2^n}\}$. To prove 1) it suffices to show that $\text{dist}(p_k, \bigcup_{i=0}^\infty P_{n+1}^{(i)}(2, k)) > 0$ for $k = 1, 2, \ldots, 2^n$. So fix some k and suppose on the contrary that the mentioned distance is zero. Note that $(\{p_k\}, P_{n+1}^{(i)}(2,k), P_{n+2}^{(i)}(4,k))$, $i = 0,1,2,\ldots$, are CSPO of $g = f^{2^n}$ of depth 2. To simplify the notation, denote $x_0 = p_k$, $a_i = \min P_{n+1}^{(i)}(2,k)$, and $b_i = \max P_{n+1}^{(i)}(2,k)$, $i = 0,1,2,\ldots$. We have $\operatorname{dist}(x_0, \{a_i : i = 0,1,\ldots\}) = 0$ or $\operatorname{dist}(x_0, \{b_i : i = 0,1,\ldots\}) = 0$. Suppose that the first equality holds and denote $c_i = \min P_{n+2}^{(i)}(4,k)$ and $d_i = \min(P_{n+2}^{(i)}(4,k) \setminus \{c_i\})$, $i = 0,1,2,\ldots$. The map $h = g^2$ is either piecewise monotone or continuously differentiable and $(\{a_i\}, \{c_i, d_i\})$, $i = 0,1,2,\ldots$ are CSPO of h of depth 1 with $c_i < a_i < d_i < x_0$ for every i. Finally, note that some subsequence of $(a_i)_{i=0}^{\infty}$ converges to x_0. By Lemma 2, h is neither piecewise monotone nor continuously differentiable. We arrive at contradiction.

2) Since $f^{2^{n+1}}(a^{(i)}) = a^{(i)}$, $i = 0,1,2,\ldots$, we have $f^{2^{n+1}}(a_0) = a_0$ and so the period of a_0 divides 2^{n+1}. It follows from 1) and the fact that P_n consists of 2^n points that $\bigcup_{i=0}^{\infty} P_{n+1}^{(i)}$ is a subset of the union of $1+2^n$ closed intervals disjoint with P_n. It follows from the structure of cascades of SPO that for every i, $P_{n+1}^{(i)}$ intersects each of the mentioned intervals. It is now easy to see that the orbit of a_0 contains at least $1 + 2^n$ points. Therefore, the period of a_0 is 2^{n+1}.

Further, since $P_{n+1}^{(i)} = \operatorname{orb}_f(a^{(i)})$ converges to $\operatorname{orb}_f(a_0)$ as $i \to \infty$, for sufficiently large i's, the orbits $P_{n+1}^{(i)}$ have the same oriented pattern as $\operatorname{orb}_f(a_0)$. Finally, it is easy to see that, since for any i, $(P_0, \ldots, P_n, P_{n+1}^{(i)})$ is a CSPO of f, $(P_0, \ldots, P_n, \operatorname{orb}_f(a_0))$ is also a CSPO of f.

Lemma 4. *Let $f \in C(I)$ be of type 2^∞ and either piecewise monotone or continuously differentiable. Then there is a fixed point x_0 of f such that $(\{x_0\})$ is an arbitrarily extendable CSPO of f.*

Proof. Since f is of type 2^∞, it has an SPO of period 2^n for any $n \in \{0\} \cup \mathbb{N}$ and so there is a CSPO of f of depth n, $(\{x^{(n)}\}, P_1^{(n)}, \ldots, P_n^{(n)})$.

If the set $\{x^{(n)} : n = 0,1,2,\ldots\}$ is finite, then there is a point x_0 such that $x_0 = x^{(n)}$ for infinitely many n's and so $(\{x_0\})$ is an arbitrarily extendable CSPO of f.

Therefore, assume that set to be infinite. Then there is a monotone, say increasing, sequence of points from that set and so it has a limit, say x_0, which is a fixed point of f. For every n, denote $R^{(n)} = \{z \in P_1^{(n)} \cup \ldots \cup P_n^{(n)} : z > x^{(n)}\}$. Since f is either piecewise monotone or continuously differentiable, Lemma 2 shows that, for infinitely many n's, $R^{(n)} > x_0$. (In fact, this holds for all but finitely many such n's which correspond to the points $x^{(n)}$ belonging to the mentioned monotone sequence.) But then $(\{x_0\})$ is an arbitrarily extendable CSPO of f.

Lemma 5. *Let $f \in C(I)$ be either piecewise monotone or continuously differentiable and let $n \in \{0\} \cup \mathbb{N}$. Let $(P_0, P_1, \ldots, P_n)$ be an arbitrarily extendable CSPO of f. Then there exists an SPO P_{n+1} of f of period 2^{n+1} such that $(P_0, P_1, \ldots, P_n, P_{n+1})$ is an arbitrarily extendable CSPO of f.*

Proof. Since the CSPO $(P_0, P_1, \ldots, P_n)$ is assumed to be arbitrarily extendable, for any $i \in \mathbb{N}$ there is a CSPO of f of the form $(P_0, P_1, \ldots, P_n, P_{n+1}^{(i)}, P_{n+2}^{(i)}, \ldots, P_{n+i}^{(i)})$.

If there is an orbit P_{n+1} with $P_{n+1} = P_{n+1}^{(i)}$ for infinitely many i's, then $(P_0, \ldots, P_n, P_{n+1})$ is an arbitrarily extendable CSPO of f. So assume that such an orbit P_{n+1} does not exist. Then the set $\{P_{n+1}^{(i)} : i = 1,2,\ldots\}$ is infinite.

Denote $P_{n+1}^{(i)} = \{p_1^{(i)}, p_2^{(i)}, \ldots, p_{2^{n+1}}^{(i)}\}$ and consider a strictly increasing sequence i_k, $k = 1,2,\ldots$, of positive integers such that for each $r \in \{1,2,\ldots,2^{n+1}\}$, the sequence $(p_r^{(i_k)})_{k=1}^{\infty}$ is strictly monotone and so convergent to some point p_r. By Lemma 3 (2),

$P_{n+1} = \{p_1, p_2, \ldots, p_{2^{n+1}}\}$ is a periodic orbit of f with period 2^{n+1} such that it has the same oriented pattern as $P_{n+1}^{(i_k)}$ for all k greater than some k_0 and $(P_0, \ldots, P_n, P_{n+1})$ is a CSPO of f. We are going to prove that this CSPO of f is arbitrarily extendable.

Take $g = f^{2^{n+1}}$ and any $r \in \{1, 2, \ldots, 2^{n+1}\}$. The points p_r and $p_r^{(i_k)}$, $k = 1, 2, \ldots$, are fixed points of g, and $p_r^{(i_k)}$ monotonically converge to p_r as $k \to \infty$. Further, for any k, $(\{p_r^{(i_k)}\}, P_{n+2}^{(i_k)}(2, r), P_{n+3}^{(i_k)}(2^2, r), \ldots, P_{n+i_k}^{(i_k)}(2^{i_k-1}, r))$ is a CSPO of g of depth $i_k - 1$. Since g is either piecewise monotone or continuously differentiable, it follows from Lemma 2 or its right version that there exists $k(r)$ such that, for every $k > k(r)$, p_r lies between the sets $L_k = \{z \in P_{n+2}^{(i_k)}(2, r) \cup \ldots \cup P_{n+i_k}^{(i_k)}(2^{i_k-1}, r) : z < p_r^{(i_k)}\}$ and $R_k = \{z \in P_{n+2}^{(i_k)}(2, r) \cup \ldots \cup P_{n+i_k}^{(i_k)}(2^{i_k-1}, r) : z > p_r^{(i_k)}\}$. So, for any $k > \max\{k_0, \max\{k(r) : r = 1, 2, \ldots, 2^{n+1}\}\}$, $(P_0, \ldots, P_n, P_{n+1}, P_{n+2}^{(i_k)}, \ldots, P_{n+i_k}^{(i_k)})$ is a CSPO of f. Thus, $(P_0, \ldots, P_n, P_{n+1})$ is arbitrarily extendable, which completes the proof.

***Remark* 4.** There are examples showing that Lemmas 3, 4, and 5 do not hold without the assumption of piecewise monotonicity or continuous differentiability of f.

Proof of Proposition D. Use Lemma 4 and Lemma 5 to construct, by induction, an infinite sequence $(\{x_0\}, P_1, \ldots, P_n, \ldots)$ that is an FCSPO of f.

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