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RESEARCH ARTICLE

On lattices, modules and groups with many uniform elements

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Abstract. The uniform dimension, also known as Goldie dimension, can be defined and used not only in the class of modules, but also in large classes of lattices and groups. For considering this dimension it is necessary to involve uniform elements.

In this paper we are going to discuss properties of lattices with many uniform elements. Further, we examine these properties in the case of lattices of submodules and of subgroups. We also formulate some questions related to the subject of this note.

1. Preliminaries

In this section we present basic notions and results on lattices. For convenience we assume that all lattices have 0 and 1. Sublattices with the same 0 will often be called 0-*sublattices.* Further notation and terminology on lattices is similar to that from [6]. However, as in the case of submodules and subgroups, in any lattice L, if $a \leq b$ then the interval $[a, b]$ will be denoted by $[b/a]$. Intervals of the form $[a/0]$ will often be called 0-*intervals.* Our crucial examples are:

Example 1.1. Let R be an associative ring with unity, M a (right) R-module and $\mathrm{L}(M)$ the set of all submodules of M ordered by inclusion. It is known that $\mathrm{L}(M)$ is an algebraic and modular lattice with $0 = 0_R$ and $1 = M_R$.

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Example 1.2. Let G be a group and $\mathrm{L}(G)$ be the set of all subgroups of G ordered by inclusion. Then it is well known that $\mathrm{L}(G)$ is an algebraic lattice with $0 = \langle e \rangle$ and $1 = G$. This lattice need not be modular.

The above examples and study of uniform dimension suggest to consider not only modular lattices. Thus some notions, defined in fact in [12, 18, 11] are of interest here. A lattice L is *balanced* if L satisfies the following quasi-identity:

$$(x \wedge y) \vee ((x \vee y) \wedge z) = 0 \Longrightarrow (y \vee z) \wedge x = 0$$

and L is *strongly balanced* if all nonempty intervals of L are balanced lattices. The last property can be expressed by the following quasi-identity:

$$(x \wedge y) \vee ((x \vee y) \wedge z) = x \wedge y \wedge z \Longrightarrow (y \vee z) \wedge x = x \wedge y \wedge z.$$

By definition we have that strongly balanced lattices form a quasi-variety of all lattices and balanced lattices form a quasi-variety of lattices with 0. Some properties of these quasi-varieties can be found for example in [12, 11, 18]. For us it is important to remember that any modular lattice is strongly balanced and any strongly balanced lattice is balanced, but need not be modular: the pentagon is the smallest nonmodular, strongly balanced lattice. The smallest example of a nonbalanced lattice is the left picture in Figure 1, while the smallest balanced but not strongly balanced lattice is the right picture in Figure 1.

Let L be a lattice and $a \in L$. Then, as in [8, 7, 12], a is called *essential (in L)* if for any $b \in L$ from $a \wedge b = 0$ it follows $b = 0$. Also, $b \in L$ is an *E-complement of a (in L)* if $a \wedge b = 0$ and $a \vee b$ is essential in L. Further, L will be called *E-complemented* if for any $a \in L$ there exists an E-complement of a in L. These notions are of special interest in balanced lattices. For example, from [7, 12] we have:

Proposition 1.3. *Let L be a balanced lattice.*

- *Any pseudocomplement in L is an E-complement.*
- *If L is algebraic then L is E-complemented.*
- *Let L be E-complemented and $a, b, c \in L$ are such that $a \vee b \leq c$ and $a \wedge b = 0$. Then there exists an E-complement b' of a in $[c/0]$ such that $b \leq b'$. In particular, the interval $[c/0]$ is E-complemented.*

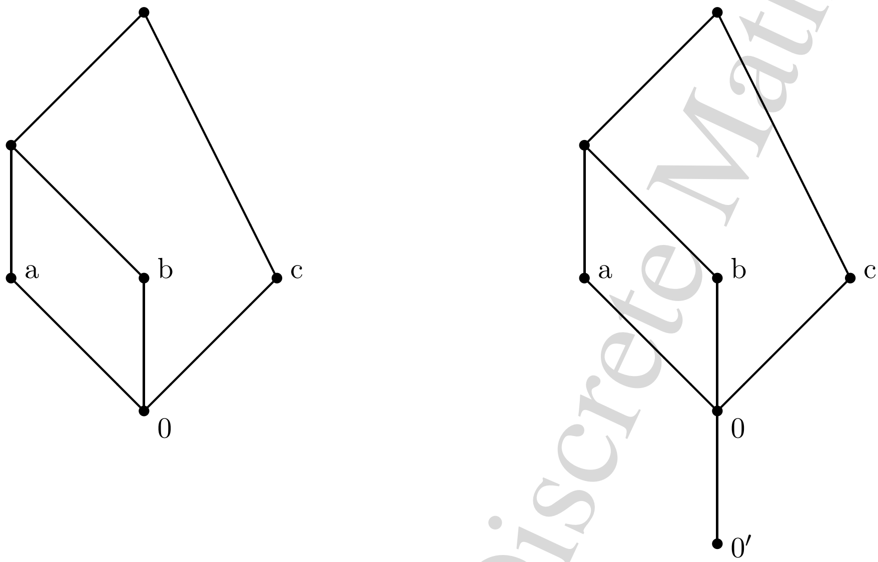

Figure 1: Some small lattices

Note here that in the left lattice from Figure 1 the element b is a pseudo-complement of a, but it is not an E-complement. The following question seems to be not very difficult.

Question 1. *Let L be a lattice satisfying the following condition: For any $c \in L$ and $a, b \in [c/0]$ such that $a \wedge b = 0$ there exists an E-complement b' of a in $[c/0]$ with $b \leq b'$. Is L a balanced lattice?*

The Ascending (Descending) Chain Condition for any partially ordered set will be abbreviated by ACC (DCC respectively). If L and M are lattices, then *trivial extension of L by M* will mean here the set $L \cup (M \setminus \{0\})$ with the relations $\leq$ from L and M and with $l < m$ for any $l \in L, m \in M \setminus \{0\}$ (see pictures on Figure 1 as a simple illustration).

2. Uniform elements

In this section lattices having many uniform elements will be considered. Recall, following [13, 8, 12], that if L is a lattice and $u \in L$ then u is *uniform (in L)* if $u \neq 0$ and any $0 \neq x \in [u/0]$ is essential in this interval. Moreover, L is *uniform* if $0 \neq 1$ and 1 is a uniform element in L. Obviously an element $u \in L$ is uniform if and only if the interval $[u/0]$ is a uniform lattice and any nonzero element of a uniform lattice is uniform. In particular atoms are uniform elements.

It is evident that any chain is a uniform lattice and any uniform lattice is balanced, but it need not be strongly balanced.

Example 2.1. Let L be a uniform lattice and let M be a nonbalanced lattice. Then the trivial extension of L by M is a uniform, but not a strongly balanced lattice.

As in [14, 12], a lattice L will be named *locally uniform* if any 0-interval in L contains a uniform element. Thus every atomic lattice, in particular any finite lattice, is locally uniform.

Clearly any 0-interval of locally uniform lattice is locally uniform and any lattice with DCC for elements is locally uniform too. On the other hand, in [10, 9, 16] there are examples of 0-sublattices of distributive locally uniform lattices, being not locally uniform. We also have

Proposition 2.2. *Let L be an E-complemented balanced lattice with ACC for E-complements. Then L is locally uniform.*

Proof. Let $0 \neq b \in L$. We have to find a uniform element in $[b/0]$. From Proposition 1.3 and the assumption we know that this interval is E-complemented. Hence, we can assume that $b = 1$ and $[b/0] = L$.

If L is uniform then there is nothing to prove. Let L be not uniform. Denote by X the set of all E-complements of nonzero elements. By assumption this set is not empty and contains a maximal element, say c. Hence there exists $0 \neq a \in L$ such that $a \wedge c = 0$ and $a \vee c$ is essential in L. Let $d, e \in [a/0]$ be such that $d \neq 0$ and $d \wedge e = 0$. Evidently $(d \vee e) \wedge c = 0$. Thus, by balancedness of L, we have $d \wedge (e \vee c) = 0$. Hence, by Proposition 1.3, there exists an E-complement f of d in L such that $e \vee c \leq f$. Hence $f \in X$. Now maximality of c in X implies that $f = c$. Thus $e = 0$, because $c \wedge e = 0$ and $e \leq c$. Hence a has to be uniform, as required. □

The above result can, by Proposition 1.3, be applied to balanced lattices with ACC for elements. However the assumption on balancedness is indispensable. Indeed, in [10] a lattice with ACC for elements, but having no uniform elements is constructed. It is also known (see Proposition 3.5 below) that without the ACC condition the above proposition need not be true for modular algebraic lattices.

One might expect that in any algebraic lattice the interval generated by all uniform elements is locally uniform, but it is not true, even for lattices of submodules (see Example 3.9 below). Only under very strong conditions this is true.

Proposition 2.3. *Let L be an algebraic and distributive lattice and let $a \in L$ be the join of all uniform elements. Then the interval $[a/0]$ is a locally uniform lattice.*

Proof. Let $0 \neq b \in [a/0]$. We can assume that b is compact. Thus, by definition of a, we can assume that there are uniform elements $u_1, \ldots, u_n$ such that $b \leq \bigvee_{i=1}^{n} u_i$. Hence, by distributivity, $b = \bigvee_{i=1}^{n}(b \wedge u_i)$. This means that for some i the meet $b \wedge u_i$ is nonzero and is uniform, because it is smaller than u_i. □

One can observe the following fact about extensions, covering the case of trivial extensions:

Proposition 2.4. *Let L be a lattice and $a \in L$.*

- *If a is essential in L and the interval $[a/0]$ is locally uniform then L is locally uniform;*
- *If a is a modular element and the intervals $[a/0]$ and $[1/a]$ are locally uniform then L is locally uniform.*

As in [16], a lattice L will be called *strongly locally uniform* if any its nontrivial interval is locally uniform. Thus every strongly atomic lattice is strongly locally uniform. Hence, lattices satisfying DCC for elements are strongly locally uniform. Chains form another class of strongly locally uniform lattices.

Immediately from the definition any interval of a strongly locally uniform lattice is strongly locally uniform. On the other hand, in [10, 9, 16] there are examples of distributive, strongly locally uniform lattices containing 0-sublattices being even not uniform. From [10] we have:

Proposition 2.5. *Let L be a lattice with ACC for elements. If L is strongly balanced then L is strongly locally uniform.*

Lemma 2.6. *Let L be a lattice and $a \in L$ be an element such that any $b \in [a/0]$ is modular in L. If the intervals $[a/0]$ and $[1/a]$ are strongly locally uniform then L is strongly locally uniform.*

As a consequence we obtain

Theorem 2.7. *Let L be a complete and modular lattice. Then there exists an element $a \in L$ such that the interval $[a/0]$ is strongly locally uniform and contains all strongly locally uniform 0-intervals of L. Moreover, the lattice $[1/a]$ has no strongly locally uniform interval of the form $[b/a]$.*

Proof. Clearly the union of any increasing chain of strongly locally uniform 0-intervals in L is strongly locally uniform. Hence, by Zorn's Lemma, we can take a maximal interval, say $[a/0]$, of this type. Using Lemma 2.6 one can check that the interval $[1/a]$ has no nontrivial strongly locally uniform intervals of the form $[b/a]$.

Let $c \in L$ be such that the interval $[c/0]$ is strongly locally uniform. Then, by definition, the interval $[c/(a \wedge c)]$ is also strongly locally uniform. Hence, by modularity of L, the interval $[(a \vee c)/a]$ is strongly locally uniform too. By previous part of the proof we then have $c \leq a$, as required. □

Question 2. *Can the above result be proved for strongly balanced algebraic lattices?*

Our interest to study lattices, modules and groups with many uniform elements was motivated by the following result from [9, 12, 11]:

Theorem 2.8. *Let L be a lattice. Then the (strong) uniform dimension is defined for L if and only if L is (strongly) balanced and (strongly) locally uniform.*

3. Modules

In this section we discuss modules with lattices of submodules having many uniform elements. For our convenience modules will often be named according to properties of lattices of their submodules.

There exists a rich theory of chain modules and of uniform modules, often depending on particular properties of the basic ring (see [13]). Injective hulls are very important in this theory. In particular we have

Proposition 3.1. *Let M be a module with the injective hull $\overline{M}$. Then M is uniform or locally uniform if and only if $\overline{M}$ shares the same property.*

From results of previous sections and modularity of lattices of the form $\mathrm{L}(M)$ we have that a submodule of a (strongly) locally uniform module is (strongly) locally uniform and homomorphic image of a strongly locally uniform module is strongly locally uniform. However, (see Example 3.9 below), a homomorphic image of a locally uniform module need not be locally uniform. We only have

Proposition 3.2. *For any ring R the class of all locally uniform R-modules is closed under direct sums. Thus, there exists a locally uniform and faithful R-module.*

Proof. Let M_1, M_2 be locally uniform R-modules and $0 \neq X \subseteq M_1 \oplus M_2$ be a submodule. If $X \cap M_1 = 0$ then X contains a uniform submodule isomorphic to a submodule of M_2. If $X \cap M_1 \neq 0$ then this intersection, and X too, contains a uniform submodule. Hence $M_1 \oplus M_2$ is locally uniform. The same conclusion follows easily for direct sums of any family of locally uniform modules.

Now, for any $0 \neq r \in R$ let I_r be a maximal right ideal of R not containing r, and $M_r = R/I_r$. Then it is easy to check that all modules M_r are uniform and the module $M = \bigoplus_{r \in R} M_r$ is a faithful R-module. By the first part of the proof M is locally uniform. □

We also have

Proposition 3.3. *Let R be a ring. Then the following conditions are equivalent:*

1. *Any R-module is locally uniform;*
2. *Any R-module is strongly locally uniform;*
3. *Any cyclic R-module is locally uniform;*
4. *Any nonzero injective R-module contains a nonzero injective indecomposable submodule.*

Proof. Let M be an R-module. It is evident that any interval in the lattice $\mathrm{L}(M)$ is isomorphic to $\mathrm{L}(N)$ for a homomorphic image N of a submodule of M. Hence, if any R-module is locally uniform then any R-module is strongly locally uniform. Now the result follows by standard arguments. □

Let us call a ring R an *LU-ring* if any R-module is locally uniform. Certainly right noetherian rings are LU-rings. More generally, combining Corollary 2.10 from [4] with Proposition 2.6(i) in the same paper we obtain the following result, noticed in [9]

Theorem 3.4. *Any ring with right Gabriel dimension is an LU-ring.*

On the other hand we have the following result, in fact due to Goldie

Theorem 3.5. *Let R be a domain. Then the following conditions are equivalent:*

1. *The module R_R is uniform;*
2. *The module R_R is locally uniform;*
3. *The module R_R contains a uniform submodule;*
4. *R is a right Ore domain.*

Proof. Let the module R_R contain a uniform submodule X and let $0 \neq x \in X$. Then the module $xR \subseteq X$ is uniform and isomorphic to R_R, because R is a domain. In this way $3 \Rightarrow 1$. Now it is easy to complete the proof. □

From the above result we know that any commutative domain is an LU-ring, but there are examples of commutative semiprime rings without this property. It is enough to take a commutative regular (in the sense of von Neumann) ring without minimal idempotents, as in Example 2 in [9].

As a consequence of results from [5] we know that any right chain ring has right Krull dimension if and only if it has right Gabriel dimension. Hence in [4, Example 10,10] it is shown in fact an example of a commutative local valuation domain R without Gabriel dimension. However this domain is an LU-ring by Proposition 3.3. Using the same arguments one can see that any homomorphic image of R, different from the residue field, is also an LU-ring without Gabriel dimension, but has only one prime ideal. Thus it would be interesting to find an intrinsic characterization of the class of all LU-rings, not covered by Proposition 3.3. Results saying that some well known classes of rings contain only LU-rings are also of interest.

For arbitrary ring, using Theorem 2.7, we obtain:

Theorem 3.6. *Let M be a module. Then there exists the largest submodule $\mathrm{S}(M)$ which is strongly locally uniform. Moreover, $\mathrm{S}(M/\mathrm{S}(M)) = 0$.*

As a consequence of the above result and earlier observations we have

Corollary 3.7. *Let R be a ring. Then the class of all strongly locally uniform modules is closed under unions of chains, taking submodules, homomorphic images and extensions.*

The above corollary means that for any ring R to be strongly locally uniform is a hereditary torsion in the category of all R-modules. This torsion is certainly nonzero, because there are simple R-modules, but it can be proper.

Example 3.8. Let F be a field and let A be an F-algebra which is a domain but not a right Ore domain, for example the free associative algebra with more than one generator. Let V be the algebra A considered as a vector space over F and let $E = End_F(V)$. Then $A \subset E$ by right multiplications. Let $B \subset E$ be the set of all endomorphisms of finite rank and $R = A + B$. Then it is known, and easy, that B is an ideal essential in $L(R_R)$. Moreover, B is the right socle of R and $R/B \simeq A$. Hence, by Theorem 3.5, $S(R_R) = B$.

Now let $W \subset V$ be a subspace of codimension 1 and let $M = \{r \in R \mid Wr = 0\}$. Certainly $M \subset B$ and M is a maximal submodule of B. Let $X = R/M$. Then one can check that X is uniform, because B/M is an essential submodule in X, but is not strongly locally uniform, because $X/(B/M) \simeq R/B$. In this way we obtained that $S(X) = B/M \neq X$.

Now we show that a module generated by two uniform submodules need not be locally uniform.

Example 3.9. Let X be a uniform module, and let $P \subset X$ be a submodule such that X/P is not locally uniform. Let $N = \{(p,p) \mid p \in P\}$ and $Y = (X \times X)/N$. Then Y is generated by uniform submodules, natural images of $X \times 0$ and $0 \times X$. On the other hand, Y contains a natural image of X embedded diagonally into $X \times X$. This image is isomorphic to X/P and is not locally uniform. Hence Y is not locally uniform.

In connection with just indicated examples the following question seems to be interesting

Question 3. *Let R be a ring, but not an LU-ring. Is there a uniform R-module M with* $S(M) = 0$?

4. Groups

In this section groups with lattices of subgroups having a lot of uniform elements are discussed. Results exhibiting difference from lattices of the form $L(M)$ will be of special interest. Groups will often be named according to properties of lattices of their subgroups.

It is well known ([17]) that a group G is a chain if and only if there exists a prime p such that G is a subgroup of the Prüfer group C_{p^∞}.

It is evident that any uniform group is either a p-group or is torsion free. However the structure of all uniform groups is complicated and is not known yet. To discuss details let us agree that the following groups are *standard uniform groups:*

- subgroups of Prüfer groups C_{p^∞} for all primes p,
- subgroups of the additive group $\mathbb{Q}$ of the rational numbers,
- subgroups of the infinite quaternion group Q_{2^∞}.

It is visible that any standard uniform group has an abelian subgroup of index at most 2. Hence, it would be interesting to know if uniform groups, maybe with some finiteness conditions, have to be standard. In this direction, among others, the following results are proved in [1, 10]:

Theorem 4.1. *Let G be a locally virtually solvable group. If G is uniform then G is a standard uniform group. In particular, any locally linear uniform group is standard uniform.*

Proposition 4.2. *Let G be a periodic locally residually finite group. If G is uniform then G is a standard uniform group.*

Question 4. *Let G be a torsion free and locally residually finite uniform group. Is G a standard uniform group?*

Examples of nonstandard uniform groups were constructed by Adian, Ol'shanskii and others. Some of these examples are presented in [15, §31].

Let us recall the notion invented (under another name) over 30 years ago by S.N. Černikov in [2]: A group G is *locally graded* if any finitely generated subgroup $\langle e\rangle < H \leq G$ contains a subgroup of finite index. In connection with the above results the following question is of interest:

Question 5. *Let G be a locally graded uniform group. Is G a standard uniform group?*

Any cyclic group is certainly locally uniform. Hence we have the following immediate observation

Proposition 4.3. *Let G be a group. Then G is locally uniform.*

Situation with strongly locally uniform groups is complicated. Certainly any abelian group is strongly locally uniform, but in [10] there are examples of metabelian, but not strongly locally uniform groups. Among them are locally finite groups. Thus we know that an extension of a strongly locally uniform group by another strongly locally uniform group need not be strongly locally uniform.

In [10, 16] some strongly locally uniform groups are described:

Theorem 4.4. *Let G be a group. Then G is strongly locally uniform in any of the following cases:*

1. *G satisfies DCC for subgroups,*
2. *G is strongly balanced and satisfies ACC for subgroups,*
3. *G is strongly balanced and locally finite,*
4. *G satisfies the normalizer condition,*

5. G is generalized nilpotent.

In [16] it was shown that some cyclic extensions of periodic abelian groups are strongly locally uniform, and even strongly atomic. As an immediate consequence of Lemma 2.6 we have a related result.

Theorem 4.5. *Let G be a group and $N \subseteq G$ a normal subgroup such that every subgroup of N is modular in G. If N and G/N are strongly locally uniform groups, then G is also strongly locally uniform.*

Corollary 4.6. *Let $N \subseteq G$ be a subgroup such that any subgroup of N is normal in G. If G/N is strongly locally uniform then G is strongly locally uniform too.*

Question 6. *Let G be a strongly balanced group. Is G strongly locally uniform?*

For locally finite groups the affirmative answer follows from [16]. On the other hand, by examples from [10] we know that for arbitrary lattices the answer is in the negative.

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