Algebra and Discrete Mathematics
Number 1. **(2004).** pp. 17 – 36

RESEARCH ARTICLE

Minimax sums of posets and the quadratic Tits form

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Communicated by V. V. Kirichenko

Dedicated Michael Ringel on the occasion of his sixtieth birthday

Abstract. Let S be an infinite poset (partially ordered set) and $\mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0}$ consisting of all vectors $z = (z_i)$ with finite number of nonzero coordinates. We call the quadratic Tits form of S (by analogy with the case of a finite poset) the form $q_S : \mathbb{Z}_0^{S\cup 0} \to \mathbb{Z}$ defined by the equality $q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$. In this paper we study the structure of infinite posets with positive Tits form. In particular, there arise posets of specific form which we call minimax sums of posets.

1. Introduction

A (finite or infinite) poset S is called a sum of subposets $A_1 \dots, A_m$ if $A_i \cap A_j = \varnothing$ for any distinct i, j and $S = A_1 \cup \ldots \cup A_m$. If any two elements $a \in A_i$ and $b \in A_j$ are incomparable whenever $i \neq j$, this sum is called direct. We say that the sum $S = A_1 + \ldots + A_m$ is said to be *minimax* (resp. *semiminimax*) if $x < y$ with x and y belonging to different summands implies that x is minimal and (resp. or) y maximal in S.

Let S be a finite poset. Denote by $R_0(S)$ the set of pairs $(x, y) \in S \times S$ with "adjacent" x and y (i.e. comparable x and y such that there is no element z satisfying $x < z < y$ if $x < y$, and $x > z > y$ if $x > y$). If

2000 Mathematics Subject Classification: *15A, 16G.*
Key words and phrases: *poset, minimax sum, the rank of a sum, the Tits form.*

$S = A_1 + \ldots + A_m$, the number

$$r_0(S) = \frac{1}{2}(|R_0(S)| - \sum_{i=1}^{m} |R_0(A_i)|)$$

will be called the *rank of the sum*.

Recall that a linear ordered set is also called a chain (or a chained set).

From the well-known results of V. M. Bondarenko, M. M. Kleiner, L. A. Nazarova and A. G. Zavadskij (see e.g. [1]) it follows that finite posets of various representation types have a number of properties which are naturally formulated in our new terms (and, clearly, could not be observed before).

Theorem 1. *Let S be a minimal poset of infinite (resp. wild) representation type. Then S is a minimax sum of three (resp. three or four) chains, having rank* 0 *or* 1.

Theorem 2. *Let S be a faithful poset of finite representation type. Then S is a minimax sum of $n \leqslant 3$ chains.*

Theorem 3. *Let S be a minimal non-finitely-parameter poset. Then S is a semiminimax sum of three or four chains.*

Theorem 4. *Let S be a faithful poset of polynomial growth (up to isomorphism and duality, the number of such posets is* 110*). Then S is a semiminimax sum of $n \leqslant 4$ chains.*

Notice that in Theorem 2 and 4 the number of the chains is not necessarily equal to the width of S.

We do not know direct proofs of this theorems. But in studying some of properties of the Tits form of posets there appear to be new connections with minimax sums, and we will see one of such connection in this paper.

The quadratic Tits form, introduced by P. Gabriel [2] for quivers and Yu. A. Drozd for posets and a wide class of classification problems (see resp. [3] and [4] plays an important role in representation theory. In particular, there are many results on connections between representation types of various objects and properties of the Tits forms. The reader interested in this theme is referred to the papers of [5], [6], the monographs [1], [7], and, e.g., the papers [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19] (with the bibliographies therein). Our paper is devoted to study the structure of infinite posets with positive Tits form.

2. Formulation of the main result

Let S be an infinite poset and $\mathbb{Z}$ denotes the integer numbers. Denote by $(\mathbb{Z}^{S\cup 0})_0 = \mathbb{Z}_0^{S\cup 0}$ the subset of the cartesian product $\mathbb{Z}^{S\cup 0}$ consisting of all vectors $z = (z_i)$ with finite number of nonzero coordinates. We call *the quadratic Tits form of S* (by analogy with the case of a finite poset) the form $q_S : \mathbb{Z}_0^{S\cup 0} \to \mathbb{Z}$ defined by the equality

$$q_S(z) = z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j,i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i.$$

This form is called *positive* if it take positive values for all $z \in \mathbb{Z}_0^{S\cup 0}$.

In considering a poset $S = (A, \leqslant)$ the set A will not be written and therefore we keep to the following conventions: by a subset of S we mean a subset of A together with the induced order relation (which is denoted by the same symbol $\leqslant$), we write $x \in S$ instead of $x \in A$, etc.

The definition of rank of a sum of finite posets from Introduction is incorrect for infinite ones. Now we introduce the notion of the rank of a sum of posets in the general case (which coincides with those of Introduction in the case of finite posets).

For a sum S of posets $A_1, \dots, A_n$ let $R^<(A_i, A_j)$ denotes the set of pairs $(x, y) \in A_i \times A_j$ with $x < y$. Such a pair (a, b) is said to be *short* if there is no other such a pair (a', b') satisfying $a \leqslant a'$, $b' \leqslant b$. By $R_0^<(A_i, A_j)$ we denote the subset of all short pairs from $R^<(A_i, A_j)$. We call the order $r_0 = r_0(A_1, \dots, A_n)$ of the set

$$R_0(A_1, \dots, A_n) = \bigcup_{\substack{i,j=1\\(i\neq j)}}^{n} R_0^<(A_i, A_j)$$

the *rank of the sum S*. Obviously, a direct sum of posets has the rank 0.

Note that the notion of a sum have been introduced by the internal way (see Introduction). The external way is possible but not natural, however it is natural in some special cases, and in particular when one has a minimax sum of rank 1 of two chains.

A poset with the only pair of incomparable elements will be called an *almost chain* or an *almost chained set*.

Our aim in this paper is to classifying the infinite posets with positive Tits form.

Main theorem. *Let S be an infinite poset. Then the Tits form of S is positive if and only if one of the following condition holds*:

1) *S is a minimax sum of rank $r < 2$ of two chains;*

2) *S is a direct sum of a chain and an almost chain.*

Note that a sum of rank 0 of two chains from condition 1) is the direct sum of these chains; chains in 1) and 2) may be empty.

3. Subsidiary lemmas

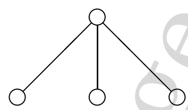
When we determine some poset T, the corresponding order relation is given up to transitivity. In the case when the elements of a poset T are denoted by natural numbers, the order relation is denoted by $\prec$ (to distinguish between the given relation and the natural ordering of the integer numbers); and then the coordinates z_i of a vector $z \in \mathbb{Z}^{T \cup 0}$ will be arranged in the natural way (in increasing order of the integer index $i \in T \cup 0$). If, in addition, we indicate the figure corresponding to the poset T, its points are not indexed, but the reader can easily establish a one-to-one correspondence between elements of T and points of the figure.

Lemma 1. *Let* $T = \{1, 2, 3, 4\}$ (*without comparable* i *and* j *for* $i \neq j$) :

 ;

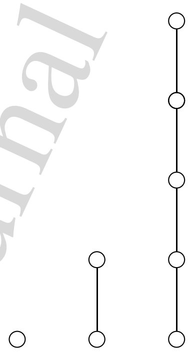
then $q_T(2, 1, 1, 1, 1) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

Lemma 2. *Let* $T = \{1, 2, 3, 4 \,|\, 1 \prec 4, 2 \prec 4, 3 \prec 4\}$:

 ;

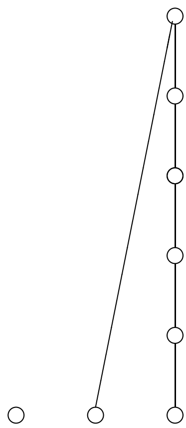
then $q_T(1, 1, 1, 1, -1) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

Lemma 3. *Let* $T = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 2 \prec 3, 4 \prec 5 \prec 6 \prec 7 \prec 8\}$:

 ;

then $q_T(6, 3, 2, 2, 1, 1, 1, 1, 1) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

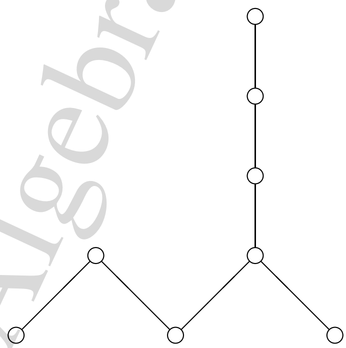
Lemma 4. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$:



;

then $q_T(4,2,3,1,1,1,1,1,-2) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

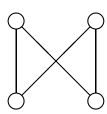
Lemma 5. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 3, 2 \prec 3, 2 \prec 5, 4 \prec 5 \prec 6 \prec 7 \prec 8\}$:



;

then $q_T(0,1,3,-2,2,-1,-1,-1,-1) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

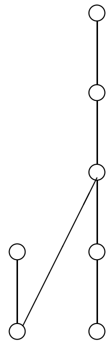
Lemma 6. *Let* $T = \{1,2,3,4 \,|\, 1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}$:



;

then $q_T(0,1,1,-1,-1) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

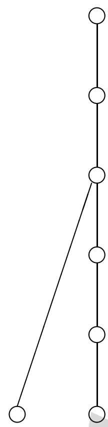
Lemma 7. *Let* $T = \{1,2,3,4,5,6,7 \,|\, 1 \prec 2, 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 5\}$:



;

then $q_T(0,-2,1,-1,-1,1,1,1) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

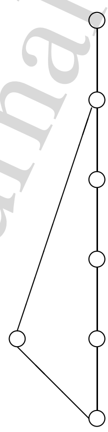
Lemma 8. *Let* $T = \{1,2,3,4,5,6,7 \,|\, 1 \prec 5, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7\}$:



;

then $q_T(1,2,1,1,1,-1,-1,-1) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

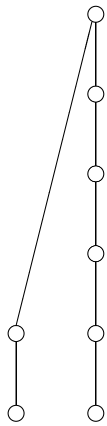
Lemma 9. *Let* $T = \{1,2,3,4,5,6,7 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7, 1 \prec 2 \prec 6\}$:



;

then $q_T(1,-1,2,1,1,1,-1,-1)=0$*, and consequently the form* $q_T(z)$ *is not positive.*

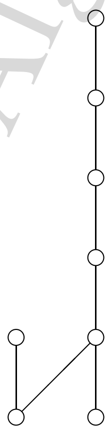
Lemma 10. *Let* $T=\{1,2,3,4,5,6,7,8\,|\,1\prec 2\prec 8, 3\prec 4\prec 5\prec 6\prec 7\prec 8\}:$



;

then $q_T(3,2,2,1,1,1,1,1,-3)=0$*, and consequently the form* $q_T(z)$ *is not positive.*

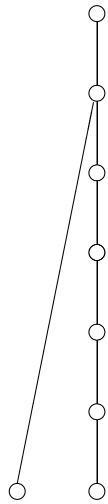
Lemma 11. *Let* $T=\{1,2,3,4,5,6,7,8\,|\,1\prec 2, 3\prec 4\prec 5\prec 6\prec 7\prec 8, 1\prec 4\}:$



;

then $q_T(1,-3,2,-2,1,1,1,1,1)=0$*, and consequently the form* $q_T(z)$ *is not positive.*

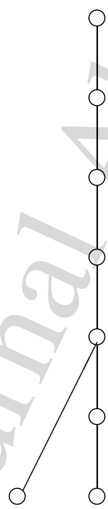
Lemma 12. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 7, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$:



;

then $q_T(2,3,1,1,1,1,1,-2,-2) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

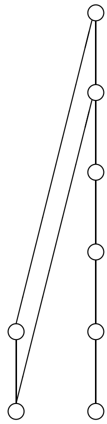
Lemma 13. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 4, 2 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8\}$:



;

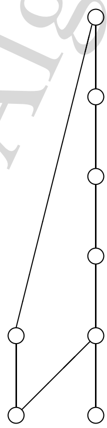
then $q_T(1,3,2,2,-1,-1,-1,-1,-1) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

Lemma 14. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 7\}$:


;

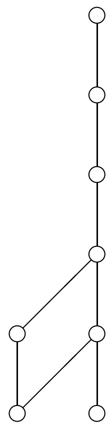
then $q_T(2,2,1,1,1,1,1,-1,-2) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

Lemma 15. *Let* $T = \{1,2,3,4,5,6,7,8 \,|\, 1 \prec 2 \prec 8, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$:


;

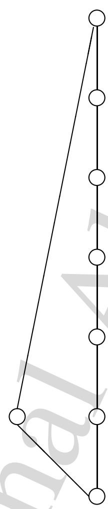
then $q_T(1,-2,2,-1,1,1,1,1,-1) = 0$, *and consequently the form* $q_T(z)$ *is not positive.*

Lemma 16. *Let* $T = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 2 \prec 5, 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 4\}$:



then $q_T(1, 1, 2, 2, 1, -1, -1, -1, -1) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

Lemma 17. *Let* $T = \{1, 2, 3, 4, 5, 6, 7, 8 \,|\, 1 \prec 3 \prec 4 \prec 5 \prec 6 \prec 7 \prec 8, 1 \prec 2 \prec 8\}$:



then $q_T(2, -2, 3, 1, 1, 1, 1, 1, -2) = 0$*, and consequently the form* $q_T(z)$ *is not positive.*

For a poset T, we denote by T^* the poset dual to T; we will always assume that $T^* = T$ as usual sets (then $x < y$ in $T^* = S$ iff $x > y$ in T). Since the Tits forms of a poset T and the dual poset T^* are the same, the dual lemmas (Lemmas 1^* – 17^*), i.e. those with the dual posets T_1^*–T_{17}^* instead of T_1–T_{17} (and the same vectors), are hold.

4. Proof of Main theorem: necessity

We first introduce some definitions and notation.

Given nonempty subsets $X, Y \in S$, we write $X \lhd Y$ if $x < y$ for some $x \in X$, $y \in Y$. and $X \not\lhd Y$ if otherwise. The sum $S = A_1 + \ldots + A_m$ is said to be *one-sided* if the transitive relation on the set of summands $\{A_1, \ldots, A_m\}$ generated by the relation $\lhd$ is an order relation. When the sum S is one-sided and $A_i \lhd A_j$ is not satisfied whenever $i > j$ (resp. $i < j$), we say that the sum is *right* (resp. *left*).

Let A and B be subsets of a poset S. If there are no comparable elements $x \in A$ and $y \in B$, then A and B will be called *incomparable* (in this case their sum is direct). It is natural to assume that subsets $A \neq \varnothing$ and $B = \varnothing$ are incomparable; and when we say that A and B are not incomparable (or write $x < y$ for some $x \in A$ and $y \in B$), it, in particular, means that $A \neq \varnothing$ and $B \neq \varnothing$).

A subset A of a poset S is said to be *upper* (respectively *lower*) if $x \in A$ whenever $x > y$ (respectively $x < y$) and $y \in A$. The subset of S consisting of all elements x which are comparable to each element $y \in S$ is denoted by S_0; obviously, it is chained. Note that we identify singletons with the elements themselves.

Let S be a poset. By $N_S(x)$, where x is an element of S, we denote the set of all elements of S not comparable to x, and, for a subset X of S, set $N_S(X) = \bigcap_{x \in X} N_S(x)$. An element $x \in S$ is said to be *isolated* if $x \in N_S(S \setminus x)$, or in other words the subsets $\{x\}$ and $S \setminus x$ are incomparable; analogously, a subset A of S is said to be *isolated* if A and $S \setminus A$ are incomparable. Recall that the width of the poset S is the maximum number of pairwise incomparable elements of S; it is denoted by $w(S)$.

We will need a special case of the following proposition.

Proposition 18. *An infinite poset S of finite width m is a sum of chains $S_1, \ldots, S_m$ with S_1 being infinite and each S_i a maximal chained subset in $S_i + \cdots + S_m$.*

Proof. It follows from Theorem 15 [20, p.133] that the poset S is a sum of chained subsets $X_1, \cdots, X_m$. One of these chains, say X_1, is infinite. Let $m > 1$ (the case $m = 1$ is obvious). Set

$$
\begin{aligned}
&X_1^{(1)} = X_1,\\
&X_1^{(2)} = X_1^{(1)} \cup \{x \in X_2 \mid X_1^{(1)} \cup x \,\text{is chained}\},\\
&X_1^{(3)} = X_1^{(2)} \cup \{x \in X_3 \mid X_1^{(2)} \cup x \,\text{is chained}\},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&X_1^{(m)} = X_1^{(m-1)} \cup \{x \in X_m \mid X_1^{(m-1)} \cup x \,\text{is chained}\}.
\end{aligned}
$$

Obviously, $X_1^{(1)} \subseteq X_1^{(2)} \cdots \subseteq X_1^{(m)}$ with all the subsets to be chained. Moreover, it is easy to see that $X_1^{(m)}$ is a maximal chained subset (of S). Indeed, suppose the contrary and let $a \notin X_1^{(m)}$ be an element with the subset $X_1^{(m)} \cup a$ to be chained; and if $a \in X_j$, then $a \in X_1^{(j)}$ (by the definition of $X_1^{(j)}$), and consequently $a \in X_1^{(m)}$; a contradiction.

Thus we can take $X_1^{(m)}$ as S_1. Note that in fact we do not make use of infiniteness in the last arguments; considering the poset $(X_2 \setminus S_1) + \cdots + (X_m \setminus S_1)$ of width $m-1$ and applying induction (on the width), we complete the proof of our statement. □

We proceed now immediately to the proof of the necessity of the theorem; up to the end of the section we assume that S is an infinite poset with positive Tits form. We prove that one of the following condition holds:

I) S is a direct sum of two chained subsets;

II) S is a direct sum of a chained and an almost chained subsets;

III) S is a minimax sum of rank $r = 1$ of two chained subsets.

By Lemma 1 $w(S) < 4$. If $w(S) = 1$, then condition I (with a direct summand to be empty) holds.

We now consider the case $w(S) = 2$. Then by Lemma 6 $S_0 = S_0^- \cup S_0^+$, where the subset S_0^- is lower and the subset S_0^+ upper.

We say that a poset S contains (no) subsets of the form T, where T is a fixed poset, if there are (no) subsets of S isomorphic to T.

We need the following simple statement.

Proposition 19. *A (finite or infinite) poset with positive Tits form is chained or almost chained if and only if it contains no subset isomorphic to one of the following posets:*

a) $\{1, 2, 3\}$ *(without comparable $i \neq j$);*

b) $\{1, 2, 3 \mid 1 \prec 2\}$.

Proof. It is easy to see that a poset P is chained or almost chained if and only if it contains no subset isomorphic to the poset a) or the poset b) or the following one:

c) $\{1, 2, 3, 4 \mid 1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}$.

But when P has positive Tits form, it does not contain subsets of the form *c*). □

Using this proposition, we prove the following lemma.

Lemma 20. *If $w(S) = 2$ and S_0 is infinite, then S is almost chained.*

Proof. Since $w(S) = 2$, the poset S contains no subsets of the form a). It remains to show that S contains no subsets of the form b). Suppose the contrary and fix elements a, b, c such that $b < c$ and a is incomparable with $\{b, c\}$. Since S_0 is infinite, so is S_0^- or S_0^+. We may assume, without loss of generality, that S_0^+ is infinite (otherwise we replace S by S^*). Then the subset of S, consisting of the elements a, b, c and arbitrary elements $d_1 < d_2 < d_3 < d_4 < d_5$ of S_0^+, is isomorphic to the poset T from Lemma 13, a contradiction. □

Thus, if S is of width 2 and S_0 is infinite, then condition II) (with the empty chain) holds.

We now consider the case when S_0 is empty. Then obviously $N_S(x) \neq \varnothing$ for any $x \in S$.

Lemma 21. *If $w(S) = 2$ and $S_0 = \varnothing$, then S is a one-sided minimax sum (of rank $r \geqslant 0$) of two chained subset.*

Proof. We set up S in the form of a sum of two chained subset P and Q with P being infinite (see Proposition 18). We can obviously suppose that $S = P + Q$ is of rank at least 1. Let $a \in P$ and $b \in Q$ be some comparable elements; without loss of generality one may assume that $a < b$ (when $a > b$, we replace S by S^*).

Set

$$\begin{aligned}
&P_1 = \{x \in S \,|\, x > b\} \cap P = \{a\}^> \cap \{x \in P \,|\, x > b\},\\
&P_2 = \{x \in S \,|\, x < b\} \cap P = (\{a\}^> \cap \{x \in P \,|\, x < b\}) \cup (\{a\}^< \cap P) \cup \{a\},\\
&P_3 = \{a\}^> \cap N_S(b).
\end{aligned}$$

Since P is infinite and $P = P_1 \cup P_2 \cup P_3$, one of the subsets P_i is infinite. We first show that the poset P_1 is finite. Suppose the contrary and consider the subset Q_1, consisting of the element b, arbitrary elements $c \in N_S(a), d \in N_S(b)$ and arbitrary elements $e_1 < e_2 < e_3 < e_4 < e_5$ from P_1. When c is comparable to d (then $c < d$), the subset of Q_1, consisting of the elements a, b, c, d, is isomorphic to the poset T from Lemma 6; and when c and d are incomparable, Q_1 is isomorphic to the poset T from Lemma 13. In both cases we get a contradiction. So P_1 is finite.

Show, further, that the subset P_2 is finite too. Suppose the contrary and let $a_i, 1 \leqslant i \leqslant 5$ be elements of P_2 such that $a_1 < a_2 < a_3 < a_4 < a_5$ (note that the inequality $a_5 \leqslant a$ does not necessarily hold). Fix elements $c \in N_S(a_5)$, $d \in N_S(b)$ and consider the subset R_1, consisting of the elements $b, c, d, a_1, a_2, a_3, a_4, a_5$; we may assume that c and d are incomparable (otherwise a_5, b, c and d form a subset isomorphic to T from Lemma 6). If c and a_i are incomparable for each $i = 1, 2, 3, 4$ then R_1 is isomorphic to the poset T^* from Lemma 11*, a contradiction. Otherwise we denote by s the largest $i \in \{1, 2, 3, 4\}$ for which $a_i < b$. Then if $s = 1$, the subset R_1 is isomorphic to the poset T^* from Lemma 15*, and if $s = 4$, R_1 is isomorphic to the poset T^* from Lemma 16*; when $s = 2$, the subset $R_1 \setminus \{d\}$ is isomorphic to the poset T^* from Lemma 9*, and when $s = 3$, the subset $R_1 \setminus \{b\}$ is isomorphic to the poset T^* from Lemma 8*. Again we have a contradiction. Thus, the subsets P_1 and P_2 are finite, and consequently P_3 is infinite.

We now show that the element b is a maximal element of Q (then, since $S_0 = \varnothing$, b is a maximal element of S).

Suppose the contrary and fix an element c of $\{b\}^> \cap Q$. Consider the subset R_2, consisting of the elements a, b, c and arbitrary elements $a_1 < a_2 < a_3 < a_4 < a_5$ of the subset $N_S(b)$. If c and a_i are incomparable for each $i = 1, 2, 3, 4, 5$ then R_2 is isomorphic to the poset T^* from Lemma 10*, a contradiction. Otherwise we denote by s the largest $i \in \{1, 2, 3, 4, 5\}$ for which a_i is comparable to c; then obviously $a_s < c$. When $s = 1$, the subset R_2 is isomorphic to the poset T^* from Lemma 14*; when $s = 4$, the subset R_2 is isomorphic to the poset T^* from Lemma 15*; when $s = 5$, the subset R_2 is isomorphic to the poset T^* from Lemma 17*; when $s = 2$ the subset $R_2 \setminus \{b\}$ is isomorphic to the poset T^* from Lemma 8*; when $s = 3$ the subset $R_2 \setminus \{a\}$ is isomorphic to the poset T^* from Lemma 7*. Again we have a contradiction. Thus, b is a maximal element of Q (and of S).

Further, the element a is a minimal one of P (then, since $S_0 = \varnothing$, a is a minimal element of S). Indeed, if a were not minimal then $\{a\}^< \cap P$ would not be empty and the subset, consisting of the elements a, b, an arbitrary element $c \in \{a\}^< \cap P$ and arbitrary elements $a_1 < a_2 < a_3 < a_4 < a_5$ of the subset $N_S(b)$, would be isomorphic to the poset T^* from Lemma 12*, a contradiction.

Thus, we have proved that b is a maximal element of Q (and of S) and a is a minimal element of P (and of S). Because the elements $a \in P$ and $b \in Q$ such that $a < b$ were chosen to be arbitrary, in fact we have also proved that a and $Q \setminus \{b\}$ (b and $P \setminus \{a\}$) are incomparable; moreover, by Lemma 6 there are no elements $x \in P$ and $y \in Q$ satisfying $x > y$. So S is a one-sided minimax sum of the (chained) subsets P and Q. □

Thus, from Lemma 21 we have that if $w(S) = 2$ and $S_0 = \varnothing$, then condition I) or II) holds.

Finally, we consider the case when S is of width 2 and S_0 is finite, but not empty.

Lemma 22. *If $w(S) = 2$ and S_0 is finite, but not empty, then the poset S is a one-sided minimax sum of an infinite chained and a one-element subsets.*

Proof. Recall that $S_0 = S_0^- \cup S_0^+$ (S_0^- and S_0^+ are, respectively, the lower and the upper parts of S_0). Since $S_1 = S \setminus S_0$ is an infinite subset of width 2, we have, by Lemma 21, that S_1 is a one-sided minimax sum of an infinite chained subset P and an (infinite or finite, but not empty) chained subset Q.

Without loss of generality we may assume that $S_0^+ \neq \varnothing$ (otherwise we replace S by S^*). Moreover, the subset S_0^+ is one-element, otherwise the subset, consisting of elements $d_1 < d_2 < d_3 < d_4 < d_5$ of P, where d_1 is not minimal and d_5 is not maximal in P, and any elements $a \in Q$, $b, c \in S_0^+$ ($b \neq c$), is isomorphic to the poset T from Lemma 12.

We first show that the rank of S_1 (as the one-sided minimax sum of P and Q) is equal to 0. Suppose the contrary and distinguish two cases: $P \lhd Q$ and $Q \lhd P$. In the first case the subset, consisting of a minimal element $a \in P$, a maximal element $b \in Q$, an element $c \in S_0^+$ and arbitrary elements $d_1 < d_2 < d_3 < d_4 < d_5$ from $P \setminus \{a\}$, is isomorphic to the poset T from Lemma 17. In the second case the subset, consisting of a minimal element $a \in Q$, a maximal element $b \in P$, an element $c \in S_0^+$ and arbitrary elements $d_1 < d_2 < d_3 < d_4 < d_5$ from $P \setminus \{b\}$, is isomorphic to the poset T from Lemma 12. In both cases we obtain a contradiction.

So, S_1 is the direct sum of the posets P and Q. Then Q is one-element, otherwise the subset, consisting of elements $a \in S_0^+, b_1, b_2 \in Q (a \neq b)$ and $c_1, c_2, c_3, c_4, c_5 \in P$ ($c_i \neq c_j$ for $i \neq j$), is isomorphic to the poset T from Lemma 10, a contradiction. Further, the subset S_0^- is empty, otherwise the subset, consisting of elements $a \in S_0^+, b \in Q, c \in S_0^-$ and $d_1, d_2, d_3, d_4, d_5 \in P$ ($d_i \neq d_j$ for $i \neq j$), is isomorphic to the poset T from Lemma 17, a contradiction.

From what we have said above, it follows that S is a one-sided minimax sum of an infinite chained and a one-element subsets (having rank 1). $\square$

So, from Lemma 22 we have that if $w(S) = 2$ and S_0 is finite, but not empty, then condition III) holds.

Thus, the proof of the necessity part of the theorem in the case, when the width of S is equal to 2, is completed. More precisely, we have proved that, for an infinite poset S of width 2 with the Tits form being positive, one of the following conditions holds:

I′)=I) S is a direct sum of two chained subsets;

II′) S is an almost chained subset;

III′)=III) S is a minimax sum of rank $r = 1$ of two chained subsets.

It remains for us to consider the case when the width of the poset S is equal to 3. Note that in this case $S_0 = \varnothing$ (by Lemma 2).

We begin with three lemmas.

Lemma 23. *If $w(S) = 3$ and R is an isolated chained subset of S, containing more than one elements, then $S \setminus R$ is an almost chained poset.*

Proof. If the subset R is infinite, then the subset $S' = S \setminus R$ (of width 2) contains no subsets of the form b) (see Proposition 19), otherwise S contains a subset isomorphic to the poset T from Lemma 3, a contradiction; consequently, by Proposition 19, the subset S' is almost chained.

If R is finite and S'_0 infinite, then S' is almost chained by Lemma 20. The case, when R and S'_0 are finite, is impossible, since by Lemmas 21 and 22 the set S' contains a subset isomorphic to the poset $\{1, 2, 3, 4, 5, 6 \,|\, 2 < 3 < 4 < 5 < 6\}$, and hence S contains a subset isomorphic to the poset T from Lemma 3. □

Lemma 24. *If $w(S) = 3$, then S cannot be a one-sided minimax sum of chained and almost chained subsets, having nonzero rank.*

Proof. Suppose the contrary, and let P, Q be corresponding chained and almost chained subsets, respectively. Without loss of generality we may assume that $P \lhd Q$ (otherwise we replace S by S^*); denote by a the only minimal element of P. Then Q has two maximal elements, because otherwise the subset, consisting of the minimal element a, the only maximal element of Q and two incomparable elements of Q, is isomorphic to the poset T from Lemma 2, a contradiction. Denote these maximal elements by b and c. Since $P \lhd Q$, we have $a < b$ or $a < c$; let $a < b$. When the elements a and c are comparable, S contains a subset isomorphic to the poset T^* from Lemma 2^*, if P is infinite, and a subset isomorphic to the poset T from Lemma 6, if Q is infinite; when a and c are incomparable, S contains a subset isomorphic to the poset T^* from Lemma 4^*, if P is infinite, and a subset isomorphic to the poset T^* from Lemma 11^*, if Q is infinite. In all cases we obtain a contradiction, thus proving the lemma. □

Lemma 25. *Let R be an infinite maximal chained subset of S. Then any subset T of S, which does not intersect R and such that $R + T$ is almost chained, consist of a one element.*

Indeed, if T consisted of more than one elements then, by definition the almost chained poset, all but one of them would be comparable to each element of R, a contradiction with maximality of R.

Continuing the proof (of the necessity of the theorem), we represent S as a sum of chained subsets S_1, S_2 and S_3 with S_1 being infinite and maximal chained (see Proposition 18). Denote by S_{ij}, where $i < j$ $(i, j = 1, 2, 3)$, the subset $S_i + S_j$. From what we proved above, it follows that each of the infinite posets (of width 2) S_{12} and S_{13} satisfies one of conditions I$'$),II$'$), III$'$).

We first show that in fact each of the poset S_{12} and S_{13} satisfies condition I$'$) or II$'$).

Suppose contrary and, for instance, let condition III$'$) holds for S_{12}. Then S_{12} is a one-sided minimax sum of S_1 and S_2; we may assume that $S_1 \lhd S_2$ (if $S_2 \lhd S_1$, we replace S by S^*); denote by a the only minimal element of S_1. If S_{13} satisfies condition I$'$) or condition III$'$) with $S_1 \lhd S_3$, then $S_1 \setminus \{a\}$ is an isolated subset of $S \setminus \{a\}$, and hence by Lemma 23 the subset $S_{23} = S_{23} \cap (S \setminus \{a\})$ is almost chained; but then S is a one-sided minimax sum of the chained subset S_1 and almost chained subset S_{23}, a contradiction with Lemma 24. In the case when S_{13} satisfies condition III$'$) with $S_3 \lhd S_1$, the subset S_1 has also the only maximal element which we denote by b. Then $S_1 \setminus \{a, b\}$ is an isolated subset of $S \setminus \{a, b\}$, and hence by Lemma 23 the subset $S_{23} = S_{23} \cap (S \setminus \{a, b\})$ is almost chained; denote by c the only minimal element of S_3 and by d the only maximal element of S_2. It is easy to see that S contains a subset isomorphic to the poset T from Lemma 6, if $c < d$, and a subset isomorphic to the poset T from Lemma 4, if c and d are incomparable. Again we have a contradiction. Finally, we consider the case when S_{13} satisfies condition II$'$). By Lemma 25 the poset S_3 consist of one element, say c; let d denotes the only element of S_1 which is incomparable with c. By Lemma 24 the element c is comparable to some element of S_2; then $S_2' = \{x \in S_2 \,|\, c < x\} \neq \varnothing$ or $S_2'' = \{x \in S_2 \,|\, c > x\} \neq \varnothing$. First, let $S_2' = \{x \in S_2 \,|\, c < x\} \neq \varnothing$; fix some element b in it. Obviously, c is minimal either in S_{13} or in $S_{13} \setminus \{a\}$ (otherwise S_{12} is not a minimax sum of S_1 and S_2). In the first case the subset of S, consisting of the elements $a = d, c$, the maximal element of S_2 and an arbitrary element of $S_1 \setminus \{a\}$, is isomorphic to the poset T from Lemma 6, a contradiction. And in the second case, the element b is maximal in S_2 (because S_{12} is a one-sided minimax sum of S_1 and S_2) and the element c is incomparable with the subset $S_2 \setminus \{b\}$; but then the (infinite) poset $S \setminus \{a\}$ is a one-sided minimax sum of the almost chained subset $S_{13} \setminus \{a\}$ and the chained subset S_2, a contradiction with Lemma 24. Suppose now that $S_2' = \varnothing$, and $S_2'' \neq \varnothing$; then S_2'' does not contain the maximal element e of S_2

(otherwise $w(S) = 2$), and the subset, consisting of the elements a, c, e and an arbitrary element of S''_2, is isomorphic to the poset T from Lemma 6, a contradiction.

So each of the posets S_{12} and S_{13} satisfies I$'$) or II$'$).

When both S_{12} and S_{13} satisfy condition I$'$), S_1 is incomparable with S_{23}, and by Lemma 23 the poset S_{23} is almost chained; hence S satisfies condition II) of the theorem.

Show, further, that the case, when both S_{12} and S_{13} satisfy condition II$'$), is impossible. Assume the contrary. Then by Lemma 25 each of the posets S_2 and S_3 consist of a one element, say a and c, respectively; the only incomparable with a (respectively, c) element of S_1 is denoted by b (respectively, d); because $w(S) = 2$, the element c is incomparable with the elements a and b (hence $d = b$). And it is easy to see from this that S contains a subset isomorphic to the poset T from Lemma 2, or to the dual poset T^*. We obtain a contradiction.

Thus, it remains to consider the case when one of the subset S_{12}, S_{13}, say S_{12}, satisfies condition I$'$), and the other one, i.e. S_{13}, satisfies condition II$'$). Then, as in the previous case, S_3 consist of a one element — c; the only incomparable with c element of S_1 is denote again by d.

Show that c is incomparable with S_2. Assume the contrary. Without loss of generality we may assume that $c < a$ for some $a \in S_2$ (otherwise we replace S by S^*). Then the subset $\{c, d\}^>$ is infinite, because otherwise so is the subset $\{c, d\}^<$ (since S_1 is infinite) and then the subset of S, consisting of the elements a, c, d and any five elements of $\{c, d\}^<$, is isomorphic to the poset T^* from Lemma 13*. Further, since $w(S) = 3$, the subset $N_S(c) \cap S_2$ is nonempty; fix some element b in it. And the subset of S, consisting of the elements a, b, c, d and any four elements of $\{c, d\}^>$, is isomorphic to the poset T from Lemma 5, a contradiction. So c is incomparable with S_2, and then S satisfies condition II).

The proof of the necessity part of the theorem is completed.

5. Proof of Main theorem: sufficiency

Suppose first that an infinite poset S is the direct sum of two chains or of a chain and an almost chain, and show that the Tits form $q_S(z)$ is positive. Obviously, it suffices to do it for the finite posets $P = P_{m,n-m} = \{-m, -m+1, \ldots, -1, -0, +0, 1, 2, \ldots, m, m+1, \ldots, n \,|\, -m \prec -m+1 \prec \ldots \prec -1 \prec -0 \prec 1 \prec 2 \prec \ldots \prec m, -1 \prec +0 \prec 1, m+1 \prec \ldots \prec n\}$, where n and $m < n$ are an arbitrary natural number. Moreover, since for each $z \in \mathbb{Z}^{P \cup 0}$, one has the (easily verifiable) equality $q_P(z) = q_Q(z')$, where $Q = P_{m,0}$, $z'_0 = z_0 - \sum_{s=m+1}^{n} z_s$, $z'_s = z_s$ for $s = -m, -m+1, \ldots, -1, -0, +0, 1, 2, \ldots, m$ and $z'_s = -z_s$ for $s = m+1, \ldots, n$, it suffices to consider the (almost chained) posets $Q = P_{m,0}$. The fact that the Tits form of a poset $Q = P_{m,0}$ is positive follows from the following (easily verifiable) equality: $2q_Q(z) = z_0^2 + \sum_{i=-m}^{-1} z_i^2 + \sum_{i=1}^{m} z_i^2 + (z_{-0} - z_{+0})^2 + (z_0 - \sum_{j \in Q} z_j)^2$.

It remains to prove that the Tits form $q_S(z)$ is positive for S to be a minimax sum of two chained posets, having rank 1. Obviously, it suffices to do it for the finite posets $R = R_n = \{1, 2 \ldots, 2n \,|\, 1 \prec 2 \cdots \prec n, n+1 \prec \cdots \prec 2n, 1 \prec 2n\}$, J¤Γ $n > 1$. Denote by $R' = R'_n$ the poset $(R \setminus \{2n\}) \cup \{-1\}$, where $-1 < i$ for $i = 2, \ldots, n$; it is the direct sum of a chained and an almost chained subsets. The fact that the Tits form $q_R(z)$ is positive follows from the following (easily verifiable) equality: $q_R(z) = q_{R'}(z')$, where $z'_0 = z_0 - z_{2n}$, $z'_i = z_i$ for $i = 1, \ldots, n, n+1, \ldots, 2n-1$, $z'_{-1} = -z_{2n}$.

The proof of the sufficiency of the theorem is completed.

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Received by the editors: 18.11.2003
and final form in 09.02.2004.