Algebra and Discrete Mathematics
Volume **21** (2016). Number 1, pp. 59–68

RESEARCH ARTICLE

Construction of self-dual binary $[2^{2k}, 2^{2k-1}, 2^k]$-codes

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Communicated by V. I. Sushchansky

Abstract. The binary Reed-Muller code $\mathrm{RM}(m-k, m)$ corresponds to the k-th power of the radical of $GF(2)[G]$, where G is an elementary abelian group of order 2^m (see [2]). Self-dual RM-codes (i.e. some powers of the radical of the previously mentioned group algebra) exist only for odd m.

The group algebra approach enables us to find a self-dual code for even $m = 2k$ in the radical of the previously mentioned group algebra with similarly good parameters as the self-dual RM codes.

In the group algebra

$$GF(2)[G] \cong GF(2)[x_1, x_2, \ldots, x_m]/(x_1^2 - 1, x_2^2 - 1, \ldots x_m^2 - 1)$$

we construct self-dual binary $C = [2^{2k}, 2^{2k-1}, 2^k]$ codes with property

$$\mathrm{RM}(k-1, 2k) \subset C \subset \mathrm{RM}(k, 2k)$$

for an arbitrary integer k.

In some cases these codes can be obtained as the direct product of two copies of $\mathrm{RM}(k-1, k)$-codes. For $k \geqslant 2$ the codes constructed are doubly even and for $k = 2$ we get two non-isomorphic $[16, 8, 4]$-codes. If $k > 2$ we have some self-dual codes with good parameters which have not been described yet.

*Research of the first author was partially supported by funding of EU’s FP7/2007-2013 grant No. 318202.
2010 MSC: 94B05, 11T71, 20C05.
Key words and phrases: Reed–Muller code, Generalized Reed–Muller code, radical, self-dual code, group algebra, Jacobson radical.

Introduction and Notation

Let K be a finite field of characteristic p and let V be a vector space over K, and C be a subspace of V. Then C is called a *linear code*. Let $x, y \in C$, then the Hamming weight of x is the number of its non-zero coordinates and the *Hamming distance* of x and y is the weight of $x - y$. The Hamming distance (or weight) of a linear code C is the minimum of all Hamming distances of its codewords.

In the study of binary codes $C \subseteq V$ it is convenient that the space V has an additional algebraic structure. For example, if V is a group algebra $K[G]$, where G is a finite abelian p-group and C is an ideal of such a group algebra, then C is called an *abelian group code*.

The Hamming distance of a linear code determines the ability of error-correcting property of the code. The authors in [6] proved that for any $1 \leqslant d \leqslant \left[\frac{m+1}{2}\right]$ there exists an Abelian 2-group G_d that a power of the radical is a self-dual code with parameters $(2^m, 2^{m-1}, 2^d)$. These codes are ideals in the group algebra $GF(2)[G_d]$ and they are "monomial codes" in the sense of [5] as defined below.

Throughout, p will denote a prime and K a field of p elements. Let $G = \langle g_1 \rangle \times \cdots \times \langle g_m \rangle \cong C_p^m$ be an elementary abelian p-group of order p^m i.e. $K[G]$ is a modular group algebra, then the group algebra $K[G]$ and K^n are isomorphic as vector spaces by the mapping

$$\varphi : K[G] \mapsto K^n, \text{ where } \varphi\left(\sum_{i=1}^{n} a_i g_i\right) \mapsto (a_1, a_2, \ldots, a_n) := \mathbf{c} \in C.$$

Reed-Muller (RM) binary codes were introduced in [12] as binary functions. These codes are frequently used in applications and have good error correcting properties. Now we are looking for self-dual codes in the radical of $K[G]$ with similarly good parameters as the RM codes.
If K is a field of characteristic 2 Berman [2] and in the general case Charpin [3] proved that all Generalized Reed-Muller (GRM) codes coincide with powers of the radical of the modular group algebra of $K[G]$, where G is an elementary abelian p-group. This group algebra is clearly isomorphic with the quotient algebra

$$GF(p)[x_1, x_2, \ldots x_m]/(x_1^p - 1, \ldots x_m^p - 1).$$

Self-dual RM-codes (i.e. some power of the radical of the group algebra $GF(2)[G]$) exist only for odd m. They are $(2^m, 2^{m-1}, 2^{\frac{m+1}{2}})$-codes.

For any basis $\{g_1, g_2, \dots g_m\}$ of such a group G consider the algebra isomorphism μ mapping $g_j \mapsto x_j \ \ (1 \leqslant j \leqslant m)$, and therefore we have the algebra isomorphism

$$\mathcal{A}_{p,m} \cong GF(p)[x_1, x_2, \dots, x_m]/(x_1^p - 1, x_2^p - 1, \dots x_m^p - 1),$$

where $GF(p)[x_1, x_2, \dots, x_m]$ denotes the algebra of polynomials in m variables with coefficients in $GF(p)$.

It is known ([7]) that the set of monomial functions ($k_i \in \mathbb{N} \cup 0$)

$$\left\{ \prod_{i=1}^{m} (x_i - 1)^{k_i} \text{ where } 0 \leqslant k_i < p \right\}$$

form a linear basis of the radical $\mathcal{J}_{p,m}$. Clearly the nilpotency index of $\mathcal{J}_{p,m}$ (i.e. the smallest positive integer t, such that $\mathcal{J}_{p,m}^t = 0$) is equal to $t = m(p-1) + 1$.

Introducing the notation

$$X_i = x_i - 1, \ (1 \leqslant i \leqslant m)$$

(which will be used from now on) we have the following isomorphism

$$\mathcal{J}_{p,m} \simeq GF(p)[X_1, X_2, \dots, X_m]/(X_1^p, X_2^p, \dots X_m^p). \tag{1}$$

The k-th power of the radical consists of reduced m-variable (non-constant) polynomials of degree at least k, where $0 \leqslant k \leqslant t - 1$, where $t = m(p-1) + 1$.

$$\mathcal{J}_{p,m}^k = \mathrm{GRM}(t - 1 - k, m) = \langle \prod_{i=1}^{m} (X_i)^{k_i} \mid \sum_{i=1}^{m} k_i \geqslant k \ (0 \leqslant k_i < p) \rangle. \tag{2}$$

Such a basis was exploited by Jennings [7].

By (2) the quotient space $\mathcal{J}_{p,m}^k / \mathcal{J}_{p,m}^{k+1}$ has a basis

$$\left\{ \prod_{i=1}^{m} X_i^{k_i} + \mathcal{J}_{p,m}^{k+1}, \text{ where } 0 \leqslant k_i < p \text{ and } \sum_{i=1}^{m} k_i = k \right\}. \tag{3}$$

Remark 1. It is known [15] that the dual code $C^{\perp}$ of an ideal C in $\mathcal{A}_{p,m}$ coincides with the annihilator of C^*, where C^* is the image of C by the involution $*$ defined on $\mathcal{A}_{p,m}$ by

$$* : \ g \mapsto g^{-1} \text{ for all } g \in G \text{ from } \mathcal{A}_{p,m} \text{ to itself.}$$

The annihilator of $\mathcal{J}_{p,m}^k$ is obviously $\mathcal{J}_{p,m}^{m(p-1)+1-k}$. Thus the dual codes of GRM-codes are GRM-codes and

$$\mathrm{GRM}(k, m)^{\perp} = \mathrm{GRM}(m(p-1) - k - 1, m).$$

It follows that for $m = 2k + 1$ and $p = 2$ the code $\mathrm{GRM}(k, m)$ is self-dual.

1. Construction of binary self-dual codes

Let us consider the group algebra

$$\mathcal{A}_{2,m} = GF(2)[x_1, \dots x_m]/(x_1^2 - 1, x_2^2 - 1, \dots x_m^2 - 1) \simeq GF(2)[C_2^m]$$

as a vector space with basis

$$x_1^{a_1} x_2^{a_2} \dots x_m^{a_m},\ a_i \in \{0, 1\}. \tag{4}$$

It is known ([7]) that the radical $\mathcal{J}_{2,m}$ of this group algebra is generated by the monomials $X_i = x_i - 1 = x_i + 1$.

Definition 1 ([5]). The code C in $\mathcal{J}_{2,m}$ (see (1)) is said to be a *monomial code* if it is an ideal in $\mathcal{A}_{2,m}$ generated by some monomials of the form

$$X_1^{k_1} X_2^{k_2} \dots X_m^{k_m},\ \text{where } 0 \leqslant k_i \leqslant 1 \tag{5}$$

The codes we intend to study are monomial codes.

For $p = 2$ using the usual polynomial product in the Boolean monomial $X_1^{k_1} X_2^{k_2} \dots X_m^{k_m}$ $(k_i \in \{0, 1\})$ we have

$$X_1^{k_1} X_2^{k_2} \dots X_m^{k_m} = (x_1 + 1)^{k_1} (x_2 + 1)^{k_2} \dots (x_m + 1)^{k_m}$$

and the Hamming weight in the basis (4) of this monomial equals $\prod_{i=1}^{m}(1+k_i)$.

Example. Let G be an elementary abelian group of order 2^m, $m \geqslant 2$. Define the codes C_j as ideals in $K[G]$ generated by $X_j = x_j - 1$. These codes are binary self-dual $[2^m, 2^{m-1}, 2]$ codes and they are self-dual since $C_j = C_j^\perp = \langle X_j \rangle$. Further, this code is a direct sum of $[2, 1, 2]$-codes. The dimension of the code C_j is 2^{m-1}, the same as the dimension of the radical of the group algebra $GF(2)[H]$, where H is an elementary abelian 2-group of rank $m - 1$. The minimal distance of C_j is $d = 2$. This follows from the fact that the element $X_j = x_j + 1$ is included in the basis of C_j. Thus, C_j is a self-dual $[2^m, 2^{m-1}, 2]$-code.

By Remark 1 one can see that a power of the radical of a modular group algebra is self-dual if and only if the nilpotency index of the radical is even. In our case (when G is elementary abelian of order p^m) the nilpotency index is even if and only if $p = 2$ and m is odd.

If m is odd, the binary RM-codes with parameters $[2^m, 2^{m-1}, 2^{\frac{m+1}{2}}]$ are self-dual and they are the $\frac{m+1}{2}$-th powers of the radical $\mathcal{A}_{2,m}$.

For $m = 2k$ where k is an arbitrary integer, we have a new method to construct a doubly-even class of binary self-dual C codes with parameters $[2^m, 2^{m-1}, 2^k]$. For this code C we have $\mathrm{RM}(k-1, 2k) \subset C \subset \mathrm{RM}(k, 2k)$. In the case of $m = 4$, we get two known extremal $[16, 8, 4]$ codes (listed in [14]) and for $m > 4$ these codes are not extremal. A doubly-even (i.e. its minimum distance is divisible by 4) self-dual code is called extremal, if we have for its minimum distance $d = 4\left[\frac{n}{24}\right] + 4$, where n denotes the code length (see Definition 39 and Lemma 40 in [8]).

To abbreviate the description of our codes, we shall refer to the monomial $X_1^{k_1} \dots X_m^{k_m}$ as the m-tuple $(k_1, k_2, \dots, k_m) \in \{0, 1, \dots, p-1\}^m$ of exponents.

Using Plotkin's construction of RM-codes (see Theorem 2 [13], Ch. 13, § 3) we obtain the following property of RM-codes.

Lemma 1. *If m is even and $m = 2k$, then $\mathrm{RM}(k-1, m) = \mathcal{J}_{2,m}^{k+1}$ contains a proper subspace which is isomorphic to $\mathrm{RM}(k-1, m-1)$.*

Proof. Recall, that the set of monomials in the basis (2) of $\mathcal{J}_{2,m}^{k+1}$ is invariant under the permutations of the variables X_i, i.e. the set of binary m-tuples $(k_1, k_2, \dots, k_m)$ assigned to the basis (2) is invariant under the permutation of all elements of the symmetric group S_m. Take the basis elements with $k_m = 1$. Then the monomials $X_1^{k_1} \dots X_m^{k_m}$ of degree m can be projected by $\pi : (k_1, k_2, \dots, k_{m-1}, 1) \mapsto (k_1, k_2, \dots, k_{m-1})$. In this way we get a basis of $\mathcal{J}_{2,m-1}^{k} \cong \mathrm{RM}(k-1, m-1)$. □

For $m = 2k$ denote the set of all k-subsets of $\{1, 2, \dots, 2k\}$ by X. The elements of X can be described by binary sequences $(k_1, k_2, \dots, k_m)$ consisting of k '0'-s and k '1'-s in any order. Clearly, the cardinality of the set X is $\binom{2k}{k}$.

We say that the subset Y of binary m-tuples in X is *complement free* if $y \in Y$ implies $\mathbf{1} - y \notin Y$, where $\mathbf{1} = (1, 1, \dots, 1)$. Denote the set of monomials corresponding to the set of exponents in X by $\mathcal{X}$. Denote the set with maximum number of pairwise orthogonal monomials in $\mathcal{X}$ by $\mathcal{Y}$ and their corresponding exponents in X by Y.

Example. For $m = 6$ the quotient space $\mathcal{J}_{2,m}^3 / \mathcal{J}_{2,m}^4$ has a basis with $\binom{6}{3} = 20$ elements, where the binary 6-tuples corresponding to the coset representative monomials (the set X) are listed in pairs of complements:

$$\begin{array}{ll}
(1,1,1,0,0,0) & (0,0,0,1,1,1)\\
(1,1,0,1,0,0) & (0,0,1,0,1,1)\\
(1,1,0,0,1,0) & (0,0,1,1,0,1)\\
(1,1,0,0,0,1) & (0,0,1,1,1,0)\\
(1,0,1,1,0,0) & (0,1,0,0,1,1)\\
(1,0,1,0,1,0) & (0,1,0,1,0,1)\\
(1,0,1,0,0,1) & (0,1,0,1,1,0)\\
(1,0,0,1,1,0) & (0,1,1,0,0,1)\\
(1,0,0,1,0,1) & (0,1,1,0,1,0)\\
(1,0,0,0,1,1) & (0,1,1,1,0,0)
\end{array}$$

and we have $2^{\frac{1}{2}\binom{6}{3}} = 2^{10}$ complement-free sets. For example the following complement free sets Y and $\mathcal{Y}$ of 10 elements:

Y	$\mathcal{Y}$
$(1,1,1,0,0,0)$,	$X_1X_2X_3$
$(0,0,1,0,1,1)$,	$X_3X_5X_6$
$(1,1,0,0,1,0)$,	$X_1X_2X_5$
$(0,0,1,1,1,0)$,	$X_3X_4X_5$
$(1,0,1,1,0,0)$,	$X_1X_3X_4$
$(0,1,0,1,0,1)$,	$X_2X_4X_6$
$(0,1,0,1,1,0)$,	$X_2X_4X_5$
$(0,1,1,0,0,1)$,	$X_2X_3X_6$
$(1,0,0,1,0,1)$,	$X_1X_4X_6$
$(1,0,0,0,1,1)$,	$X_1X_5X_6$

Theorem 1. *Let C be a binary code with $\mathrm{RM}(k-1,2k) \subset C \subset \mathrm{RM}(k,2k)$ with the following basis of the factorspace $C/\mathrm{RM}(k-1,2k)$*

$$\left\{ \prod_{i=1}^{m} X_i^{k_i} + \mathrm{RM}(k-1,2k), \ \textit{where } k_i \in \{0,1\} \textit{ and } \sum_{i=1}^{m} k_i = k \right\}, \qquad (6)$$

where the set of the exponents $(k_1, k_2, \dots, k_m)$ is a maximal (with cardinality $2^{\frac{1}{2}\binom{2k}{k}}$) complement free subset of X. Then C forms a $[2^{2k}, 2^{2k-1}, 2^k]$ self-dual doubly-even code.

Proof. Let G be an elementary abelian group of order 2^m, where $m = 2k,\ k \geqslant 2$. By the group algebra representation of RM-codes and the definition of C we have the relation $\mathcal{J}_{2,m}^{k+1} \subset C \subset \mathcal{J}_{2,m}^{k}$.

For $m = 2k$ the set $\mathcal{X}$ is the set of coset representatives of the quotient space $\mathcal{J}_{2,m}^{k}/\mathcal{J}_{2,m}^{k+1}$, i.e. the set of monomials satisfying (6).

Clearly, two monomials $X_1^{k_1}X_2^{k_2}\dots X_m^{k_m}$ and $X_1^{l_1}X_2^{l_2}\dots X_m^{l_m}$ are orthogonal, i.e. their product is zero, if for some $i : 1 \leqslant i \leqslant m$ we have $k_i = l_i$.

Thus, the elements in the radical corresponding to these monomials are orthogonal if their exponent m-tuples belong to a complement free set.

The m-tuples $(k_1, k_2 \dots k_m)$ have to be complement free in Y, otherwise the corresponding monomials in $\mathcal{Y}$ are not orthogonal. Clearly Y is a complement free subset of X (given by (4)) with cardinality $\frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$.

By definition, $C = \langle\, \mathcal{J}_{2,m}^{k+1} \cup \mathcal{Y}\,\rangle$ is a subspace of the radical $\mathcal{J}_{2,m}$ of the group algebra $\mathcal{A}_{2,m}$ generated by the union of $\mathcal{J}_{2,m}^{k+1}$ and $\mathcal{Y}$. For the dimension of C we have

$$\dim(C) = \dim(\mathrm{RM}(k-1, m)) + \frac{1}{2}\binom{2k}{k} = 1 + \sum_{i=1}^{k-1}\binom{2k}{i} + \frac{1}{2}\binom{2k}{k} = 2^{2k-1}.$$

It follows that C is self-dual. Since a binary self-dual code contains a word of weight 2 if and only if the generator matrix has two equal columns, we have our self-dual code to be doubly-even.

Each monomial in $\mathcal{Y}$ has the same weight 2^k, that is the minimal distance of C. Using the identities for the monomials involved in the basis of our codes

$$x_i(x_j + 1) = (x_i + 1)(x_j + 1) + (x_j + 1) \text{ and } (x_i + 1)^2 = 0,$$

we easily obtain that C (which is subspace of $\mathcal{J}_{2,m}$) is an ideal in the group algebra $GF(2)[G]$. □

Theorem 2. *Let Y and $\mathcal{Y}$ be sets defined above and let C be the code defined in Theorem 1. Suppose that $k_i = 0$ for some $i : 1 \leqslant i \leqslant m$ in each element of the subset Y, (i.e. the variable X_i is missing in each monomial of $\mathcal{Y}$). Then we have the isomorphism*

$$C \simeq \mathrm{RM}(k-1, 2k-1) \oplus \mathrm{RM}(k-1, 2k-1).$$

Proof. The elements of $\mathcal{Y}$ are of the form

$$X_1^{k_1}\dots X_m^{k_m} = (x_1 + 1)^{k_1}(x_2 + 1)^{k_2}\dots(x_m + 1)^{k_m}, \text{ where } \sum_{i=1}^{m} k_i = k$$

and their weight is 2^k. Project the set of monomials with $k_i = 0$ in $C = \langle\, \mathcal{J}_{2,m}^{k+1} \bigcup \mathcal{Y} \,\rangle$ onto the monomials $X_1^{k_1}, \dots, X_{i-1}^{k_{i-1}}, X_{i+1}^{k_{i+1}}, \dots, X_m^{k_m}$. The image C_1 of this projection is a self-dual RM$(k-1, 2k-1)$-code with parameters $[2^{2k-1}, 2^{2k-2}, 2^k]$.

By Lemma 1 the elements of the basis of $J_{2,m}^{k+1}$ with $k_i = 1$ generate a subspace C_2 which is isomorphic to RM$(k-1, 2k-1)$. The intersection of C_1 and C_2 is empty. Therefore $C \simeq C_1 \oplus C_2$ and the statement follows. □

Remark 2. In particular, by Theorem 1 we get $[16, 8, 4]$ self-dual codes for $m = 4$. These codes are extremal doubly-even codes. Using the SAGE computer algebra software we may check easily the classification of binary self-dual codes listed in [14].

There are two cases:

1) If $k_i = 0$ for some $i : 1 \leqslant i \leqslant m$ in each element of the set Y, then we get the direct sum $E_8 \oplus E_8$, where E_8 is the extended Hamming code.
2) otherwise we get an indecomposable $[16, 8, 4]$ code (which is denoted by E_{16} in [14]).

These codes are formally self-dual. Both classes have the following weight function:

$$z^{16} + 28z^{12} + 198z^8 + 28z^4 + 1$$

Remark 3. It is known that for each odd $m > 1$ there exists a self-dual affine-invariant code of length 2^m over $GF(2)$, which is not a self-dual RM-code [4].
The factor space $\mathcal{J}_{p,m}^k / \mathcal{J}_{p,m}^{k+1}$ is an irreducible $AGL(m, GF(p))$ module. Thus the code C is not affine invariant (see [1] Theorem 4.17) as the powers of the radical of $\mathcal{A}_{p,m}$ are. The code C cannot be an extended cyclic code by Corollary 1 in [4].

Remark 4. Using the inclusion-exclusion principle a formula can be given for the dimension of the $RM(k+1, m)$-code (see for example in [1] Theorem 5.5). If $p = 2$ and $0 \leqslant k \leqslant m$, then we have

$$\dim C = \tfrac{1}{2}\tbinom{2k}{k} + \sum_{i=k+1}^{m} \sum_{j=0}^{2k} (-1)^j \tbinom{2k}{j} \tbinom{2k-2j+i-1}{i-2j} = \sum_{i=k+1}^{m} \tbinom{2k}{i} + \tfrac{1}{2}\tbinom{2k}{k},$$

where $i - 2j \geqslant 0$.

The codes constructed in the current paper are worth to be studied further. Already for $k = 2$ we get two non-isomorphic codes with the same parameters. It would be interesting to determine all classes of codes up to isomorphism for each arbitrary integer k and to determine their automorphism group. The code C in Theorem 1 is not affine-invariant and first computations show that the automorphism group of C with $k_i = 0$ differs from the automorphism group of C with $k_i = 1$ for some $1 \leqslant i \leqslant m$.

We can formulate the following open questions about the code C of Theorem 1:

1) Does there exist a classification for all complement-free sets for arbitrary even m?
2) How many non-equivalent (in any sense) self-dual binary codes exist for fixed m and p?
3) Compare the automorphism groups of the codes C defined in Theorem 1 with the automorphism group of RM-codes.
4) Find decoding algorithms for C.

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Received by the editors: 21.09.2015
and in final form 16.12.2015.