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On classification of pairs of potent linear operators with the simplest annihilation condition

Vitaliy M. Bondarenko, Olena M. Tertychna, Olesya V. Zubaruk

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Abstract. We study the problem of classifying the pairs of linear operators $\mathcal{A}, \mathcal{B}$ (acting on the same vector space), when the both operators are potent and $\mathcal{AB} = 0$. We describe the finite, tame and wild cases and classify the indecomposable pairs of operators in the first two of them.

Introduction

Throughout the paper, k is an algebraic closed field of characteristic $\operatorname{char} k = 0$. All k-vector space are finite-dimensional. Under consideration maps, morphisms, etc., we keep the right-side notation.

We call a Krull-Schmidt category (i.e. an additive k-category with local endomorphism algebras for all indecomposable objects) *of tame* (respectively, *wild*) *type* if so is the problem of classifying its objects up to isomorphism (see precise general definitions in [1]). For formal reasons we exclude the categories of finite type (i.e. with finite number of the isomorphism classes of indecomposable objects) from those of tame type.

In this paper we study the problem of classifying the pairs of annihilating potent linear operators (an operator $\mathcal{C}$ is called potent or, more precisely, s-potent if $\mathcal{C}^s = \mathcal{C}$, where $s > 1$).

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Formulate our problem more precisely and in the category language.

Let $\mathcal{P}(k)$ denotes the *category of pairs of linear operators* acting on the same k-vector space, i.e. the category with objects the triples $\overline{U} = (U, \mathcal{A}, \mathcal{B})$, consisting of a k-vector space U and linear operators $\mathcal{A}, \mathcal{B}$ on U, and with morphisms from $\overline{U} = (U, \mathcal{A}, \mathcal{B})$ to $\overline{U'} = (U', \mathcal{A}', \mathcal{B}')$ the linear maps $X : U \to U'$ such that $\mathcal{A}X = X\mathcal{A}$, $\mathcal{B}X = X\mathcal{B}$. Since it is a Krull-Schmidt category, each object is uniquely determined by its direct summands. For natural numbers $n, m \geqslant 1$, denote by $\mathcal{P}_k^{\circ}(n, m)$ the full subcategory of $\mathcal{P}(k)$ consisting of all triples $(U, \mathcal{A}, \mathcal{B})$ with $\mathcal{A}$ being n-potent, $\mathcal{B}$ being m-potent and $\mathcal{A}\mathcal{B} = 0$. Our aim is to describe the type of every such category and to classify (up to isomorphism) the indecomposable objects in finite and tame cases.

Theorem 1. *A category $\mathcal{P}_k^{\circ}(n, m)$ is of*

- *finite type if $nm < n + m + 3$,*
- *tame type if $nm = n + m + 3$,*
- *wild type if $nm > n + m + 3$.*

With respect to the mentioned classification see section 2.

Note that from Theorems 3.1 and 3.2 of [2] it follows that without the relation $\mathcal{A}\mathcal{B} = 0$ the corresponding overcategory $\mathcal{P}_k(n, m)$ is of tame type if $n = m = 2$ and of wild type otherwise (see more in 3.6 below).

1. Proof of the theorem

We first establish a connection between the categories $\mathcal{P}_k^{\circ}(n, m)$ and the categories of representations of quivers.

Recall the notion of representations of a quiver [3].

Let $Q = (Q_0, Q_1)$ be a finite quiver (directed graph), where Q_0 and Q_1 are the sets of its vertices and arrows, respectively. *A representation of the quiver $Q = (Q_0, Q_1)$ over a field K* is a pair $R = (V, \gamma)$ formed by a collection $V = \{V_x \mid x \in Q_0\}$ of K-vector spaces V_x and a collection $\gamma = \{\gamma_\alpha \mid \alpha : x \to y \text{ runs through } Q_1\}$ of linear maps $\gamma_\alpha : V_x \to V_y$. A morphism from $R = (V, \gamma)$ to $R' = (V', \gamma')$ is given by a collection $\overline{\lambda} = \{\lambda_x \mid x \in Q_0\}$ of linear maps $\lambda_x : V_x \to V'_x$, such that $\gamma_\alpha \lambda_y = \lambda_x {\gamma'}_\alpha$ for any arrow $\alpha : x \to y$. The category of representations of $Q = (Q_0, Q_1)$ over K will be denoted by $\operatorname{rep}_K Q$. It is a Krull-Schmidt category.

A quiver Q is said to be *of finite*, *tame* or *wild representation type over K* if the caregory $\operatorname{rep}_K Q$ has respectively finite, tame or wild type. By results of [3] (respectively, [4] and [5]), a connected quiver is of finite (respectively, tame) representation type if and only if it is a Dynkin (respectively, extended Dynkin) graph. Note that by a Dynkin graph we mean a Dynkin diagram with some orientation of edges, and for simplicity denote it in the same way as the Dynkin diagram (analogously for an extended Dynkin graph).

Now we proceed to investigate connections between categories of the forms $\mathcal{P}_k^\circ(n,m)$ and $\operatorname{rep}_k Q$.

We identify a linear map α of $U = U_1 \oplus \dots U_p$ into $V = V_1 \oplus \dots V_q$ with the matrix $(\alpha_{ij})_{i=1}^{p}{}_{j=1}^{q}$, where $\alpha_{ij} : U_i \to V_j$ are the linear maps induced by α; if $p = q$ and the matrix is diagonal, we write $\alpha = \oplus_{i=1}^{p} \alpha_i$. The identity linear operator on W is denoted by 1_W.

For natural numbers $n, m > 1$, denote by $Q(n,m)$ the quiver with set of vertices $Q_0(n,m) = \{1, 2 \dots, n+m\}$ and set of arrows $Q_1(n,m) = \{i \to j \mid j = 1, \dots, n, i = n+1, \dots, n+m\}$. The primitive root of unity of degree s is denoted by ε_s.

Define the functor G_{nm} from $\operatorname{rep}_k Q(n-1, m-1)$ to $\mathcal{P}_k^\circ(n,m)$ as follows. G_{nm} assigns to each object $(V,\gamma) \in \operatorname{rep}_k Q(n-1, m-1)$ the object $(V^\oplus, \mathcal{A}^\gamma, \mathcal{B}^\gamma) \in \mathcal{P}_k^\circ(n,m)$ where $V^\oplus = \oplus_{i=1}^{n+m-2} V_i$, $\mathcal{A}^\gamma_{ij} = \varepsilon_{n-1}^{i} 1_{V_i}$ if $i = j \leqslant n-1$ and $\mathcal{A}^\gamma_{ij} = 0$ if otherwise, $\mathcal{B}^\gamma_{n+i-1,n+i-1} = \varepsilon_{m-1}^{i} 1_{V_{n+i-1}}$ if $i \leqslant m-1$, $\mathcal{B}^\gamma_{n+i-1,j} = \gamma_{ij}$ if $i \leqslant m-1, j \leqslant n-1$, and $\mathcal{B}^\gamma_{pq} = 0$ in all other cases. G_{nm} assigns to each morphism λ of $\operatorname{rep}_k Q(m-1, n-1)$ the morphism $\oplus_{i=1}^{n+m-2} \lambda_i$ of $\mathcal{P}_k^\circ(n,m)$.

Proposition 1. *The functor G_{nm} is full and faithful.*

Proof. It is obvious that G_{nm} is faithful. Prove that it is full. Let δ be a morphism from $(V,\gamma)G_{nm} = (V^\oplus, \mathcal{A}^\gamma, \mathcal{B}^\gamma)$ to $(W,\sigma)G_{nm} = (W^\oplus, \mathcal{A}^\sigma, \mathcal{B}^\sigma)$. In other words, δ is a linear map of $V^\oplus$ into $W^\oplus$ such that $\mathcal{A}^\gamma\delta = \delta\mathcal{A}^\sigma$ and $\mathcal{B}^\gamma\delta = \delta\mathcal{B}^\sigma$. We consider these equalities as matrix ones (see the definition of $V^\oplus$), and the induced by them scalar equalities $(\mathcal{A}^\gamma\delta)_{ij} = (\delta\mathcal{A}^\sigma)_{ij}$ and $(\mathcal{B}^\gamma\delta)_{ij} = (\delta\mathcal{B}^\sigma)_{ij}$ denote, respectively, by $[a,i,j]$ and $[b,i,j]$.

Since $\varepsilon_{n-1}, \varepsilon_{n-1}^2, \dots, \varepsilon_{n-1}^{n-1}$ and 0 are pairwise different elements of the field k, it follows from the equalities $[a,i,j]$ with $i,j \in \{1, \dots, n-1\}$, $i \neq j$, $[a,i,j]$ with $i \in \{1, \dots, n-1\}, j \in \{n, \dots, n+m-2\}$ and $[a,i,j]$ with $i \in \{n, \dots, n+m-2\}, j \in \{1, \dots, n-1\}$ that the block $(\delta_{pq})_{p,q=1}^{n-1}$ of σ (as a matrix) is diagonal and the blocks $(\delta_{pq})_{p=1}^{n-1}{}_{q=n}^{n+m-2}$, $(\delta_{pq})_{p=n}^{n+m-2}{}_{q=1}^{n-1}$ are zero. Then analogously to above, it follows from the equalities $[b,i,j]$ with $i,j \in \{n, \dots, n+m-2\}$, $i \neq j$, that the block $(\delta_{pq})_{p,q=n}^{n+m-2}$ is diagonal. Thus σ (as a matrix) is diagonal, and it is easy to see that the equalities $[b,i,j]$ with $i \in \{n, \dots, n+m-2\}, j \in \{1, \dots, n-1\}$ means that $\overline{\sigma} =$

$(\sigma_1, \ldots, \sigma_{n+m-2})$ is a morphism between the objects (V, γ) and (W, σ) of the category $\operatorname{rep}_k Q(n-1, m-1)$. Since $\sigma = \overline{\sigma} G_{nm}$, the fullness of G_{nm} is proved. □

Proposition 2. *Each object of $\mathcal{P}_k^{\circ}(n, m)$ is isomorphic to an object of the form $RG_{nm} \oplus (W, 0, 0)$, where R is an object of $\operatorname{rep}_k Q(n-1, m-1)$, W is a k-vector space of dimension $d \geqslant 0$.*

Proof. Let $T = (U, \mathcal{A}, \mathcal{B})$ be an objects of the category $\mathcal{P}_k^{\circ}(n, m)$. Since the roots $\varepsilon_{n-1}, \ldots, \varepsilon_{n-1}^{n-1}$ and 0 of the polynomial $x^n - x$ are pairwise different, we can assume (by the theorem on the Jordan canonical form) that $U = U_1 \oplus \ldots \oplus U_{n-1} \oplus U_0$ with $U_s = \operatorname{Ker}(\mathcal{A} - \varepsilon_{n-1}^s 1_U)$ and $U_0 = \operatorname{Ker} \mathcal{A}$; then $\mathcal{A} = \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{n-1} \oplus \mathcal{A}_0$ with $\mathcal{A}_s : U_s \to U_s$ to be the scalar operator $\varepsilon_{n-1}^s 1_{U_s}$ and $\mathcal{A}_0 : U_0 \to U_0$ to be zero (here $s = 1, \ldots, n-1$). From $\mathcal{A}\mathcal{B} = 0$ it follows that $U_1 \oplus \ldots \oplus U_{n-1} \in \operatorname{Ker} \mathcal{B}$, and consequently we have (since $\mathcal{B}^m = \mathcal{B}$) that the operator $\mathcal{B}_0 : U_0 \to U_0$, induced by $\mathcal{B}$, satisfies the equality $\mathcal{B}_0^m = \mathcal{B}_0$. Then, analogously as above, $U_0 = U_n \oplus \ldots \oplus U_{n+m-2} \oplus W$ with $U_{n+s-1} = \operatorname{Ker}(\mathcal{B}_0 - \varepsilon_{m-1}^s 1_{U_0})$, $s = 1, \ldots, m-1$, and $W = \operatorname{Ker} \mathcal{B}_0$. Besides, it follows from $\mathcal{B}^m = \mathcal{B}$ that $W_0 \in \operatorname{Ker} \mathcal{B}$.

Thus, $U = U_1 \oplus \ldots \oplus U_{n+m-2} \oplus W$ and now the operators $\mathcal{A}, \mathcal{B}$ are uniquely defined by the maps $\mathcal{B}_{ij} : U_i \to U_j$ with i and j running from n to $n+m-2$ and from 1 to $n-1$, respectively. The representation R of the quiver $Q(n-1, m-1)$, corresponding to these maps, satisfies the required condition, i. e. $T = RG_{nm} \oplus (W, 0, 0)$. □

Denote by $\widehat{\mathcal{P}}_k^{\circ}(n, m)$ the full subcategory of $\mathcal{P}_k^{\circ}(n, m)$ consisting of all objects that have no objects $(W, 0, 0)$, with $W \neq 0$, as direct summands.

We have as an immediate consequence of Propositions 1 and 2 the following statement.

Theorem 2. *The functor G_{nm}, viewed as a functor from the category $\operatorname{rep}_k Q(n-1, m-1)$ to the category $\widehat{\mathcal{P}}_k^{\circ}(n, m)$, is an equivalence of categories.*

Using this theorem it is easy to show by the standard method that the types of categories $\mathcal{P}_k^{\circ}(n, m)$ and $\operatorname{rep}_k Q(n-1, m-1)$ coincide.

Now Theorem 1 follows from the simple facts that $Q = Q(n-1, m-1)$ is a Dynkin graph iff either $n = 2, m = 2, 3, 4$, or vice versa, $n = 2, 3, 4, m = 2$ (then $Q = A_2, A_3, D_4$, respectively), and an extended Dynkin graph iff either $n = 2, m = 5$, or vice versa, $n = 5, m = 2$ (then $Q = \widetilde{D}_4$), or $n = m = 3$ (then $Q = \widetilde{A}_3$).

2. The classification of the indecomposable pairs of annihilating potent operators

The functor G_{nm} allows to obtain a classification of indecomposable objects (up to isomorphism) of any category $\mathcal{P}_k^\circ(n, m)$ of finite and tame types (see Theorem 2). To do this, it is need to take representatives of the classes of isomorphic indecomposable objects (one from each class) of the category $\operatorname{rep}_k Q$ with $Q = Q(n-1, m-1)$ and apply to them the functor G_{nm} (as a result we get all representatives of the classes of isomorphic indecomposable objects of $\mathcal{P}_k^\circ(n, m)$, except $(k, 0, 0)$). Such (of the most simple form) representatives are well-known: see [3] for $Q = A_2, A_3, D_4$ (our cases of finite type) and [4, 5] for $Q = \widetilde{A}_3, \widetilde{D}_4$ (our cases of tame type).

3. Remarks

3.1. All the above results are true if k is any field of characteristic 0 and $\varepsilon_n, \varepsilon_m \in k$.

3.2. All the above results are true if k is an algebraic closed field of characteristic $p \neq 0$, which does not divide nm.

3.3. All the above results are true if k is as in 3.2, but does not necessarily algebraically closed, and $\varepsilon_n, \varepsilon_m \in k$.

3.4. All the above results are true if k is an algebraic closed field of any characteristic and $\mathcal{A}, \mathcal{B}$ satisfy, respectively, polynomials $\varphi(x)$ and $\psi(x)$ of degrees n and m without multiple roots such that $\varphi(0) = 0$, $\psi(0) = 0$ (without the last condition the problem is trivial).

3.5. Theorem 1 is true if k is any field of any characteristic and $\mathcal{A}, \mathcal{B}$ satisfy, respectively, any fixed separable polynomials $\varphi(x)$ and $\psi(x)$ of degrees n and m such that $\varphi(0) = 0$, $\psi(0) = 0$ (see the definitions in [1]).

3.6. Classifying the pairs of idempotent operators. As the first author pointed out, the following classification of the pairs of idempotent operators (the objects of $\mathcal{P}_k(2, 2)$) follows from [2, Section 3] and [4].

One will adhere to the matrix language. The field k is assumed to be any algebraic closed (otherwise, it is necessary to replace the below Jordan blocks in 1) by indecomposable Frobenius companion ones).

Let $J_m(\lambda)$ denotes the (upper) $m \times m$ Jordan block with diagonal entries λ, E_m the $m \times m$ identity matrix. Define ${}^0E_m$ (respectively, ${}_0E_m$) as E_m with added null first column (respectively, last row). For an $m \times m$ matrix X, put $X^+ = X$, $X^- = E_m - X$, and for a pair of $m \times m$ matrices $P = (X, Y)$ and $\mu, \nu \in \{+, -\}$, put $P^{\mu\nu} = (X^\mu, Y^\nu)$. Finally, for matrices A, B with the same number of rows, introduce the squared matrices

$$F[A,B] = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}, \qquad S[A,B] = \begin{pmatrix} 0 & 0 \\ A & B \end{pmatrix}.$$

Theorem 3. *The set of all pairs of matrices over k of the forms*

1) $P = (F[E_n, E_n], S[J_n(\lambda), E_n])$, $\lambda \in k \setminus 0$,

2) $P^{\mu\nu}$ *for* $P = (F[E_n, E_n], S[J_n(0), E_n])$ *and* $\mu, \nu \in \{+, -\}$,

3) $P^{\mu\nu}$ *for* $P = (F[E_n, {}_0E_{n-1}], S[{}^0E_{n-1}, E_{n-1}])$ *and* $\mu, \nu \in \{+, -\}$,

where n runs through the natural numbers, is a complete set of pairwise nonsimilar indecomposable pairs of idempotent matrices over k.

Note that this classification implies those of the pairs of involutory matrices (the representations of the infinite dihedral group) if $\operatorname{char} k \neq 2$.

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Contact information

V. M. Bondarenko Institute of Mathematics, NAS, Kyiv, Ukraine
E-Mail(s): `vitalij.bond@gmail.com`

O. M. Tertychna Vadim Hetman Kyiv National Economic University, Kiev, Ukraine
E-Mail(s): `olena-tertychna@mail.ru`

O. V. Zubaruk Kyiv National Taras Shevchenko University, Volodymyrska str., 64, 01033 Kyiv, Ukraine
E-Mail(s): `Sambrinka@ukr.net`

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