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REMARKS ON SUMMABILITY OF SERIES FORMED FROM DEVIATION PROBABILITIES OF SUMS OF INDEPENDENT IDENTICALLY DISTRIBUTED RANDOM VARIABLES

ЗАУВАЖЕННЯ ДО СУМОВНОСТІ РЯДІВ, УТВОРЕНИХ ЗА ЙМОВІРНОСТЯМИ ВІДХИЛЕННЯ СУМ НЕЗАЛЕЖНИХ ОДНАКОВО РОЗПОДІЛЕНИХ ВИПАДКОВИХ ВЕЛИЧИН

We make some remarks leading to a refinement of the recent work of O. I. Klesov (1993) on the connection between the convergence of $\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon n^{\alpha})$ for every $\varepsilon > 0$ and that of $\sum_{n=1}^{\infty} n\tau_n P(|X_1| \geq \varepsilon n^{\alpha})$ again for every $\varepsilon > 0$.

Одержано результати, що уточнюють недавню роботу О. І. Кльосова (1993) про зв'язок між збіжністю $\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon n^{\alpha})$ для всіх $\varepsilon > 0$ і збіжністю $\sum_{n=1}^{\infty} n\tau_n P(|X_1| \geq \varepsilon n^{\alpha})$ також для всіх $\varepsilon > 0$.

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables. Put $S_n = X_1 + \ldots + X_n$ and fix $\alpha > 1/2$. Starting with Hsu – Robbins [1] and Erdős [2], a number of people considered the connection between the convergence of

$$\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon n^{\alpha}) \tag{1}$$

for every positive ε and that of

$$\sum_{n=1}^{\infty} n\tau_n P(|X_1| \geq \varepsilon n^{\alpha}) \tag{2}$$

again for every positive ε, for various choices of $\tau_n \geq 0$ and $\alpha > 1/2$. Recently, Klesov [3] determined several auxiliary conditions under which the convergence of (2) for all $\varepsilon > 0$ implied the convergence of (1) for all $\varepsilon > 0$, and showed that, under the auxiliary condition that $\lim_{n\to\infty} nP(|X_1| \geq \varepsilon n^{\alpha}) = 0$ for every $\varepsilon > 0$, we have the converse implication.

Our first remark is that we may obtain a partial converse result even in the absence of Klesov's auxiliary condition, and that the auxiliary condition itself may be weakened to the assumption that $\sup_n nP(|X_1| \geq \varepsilon n^{\alpha}) < \infty$ for every $\varepsilon > 0$.

Theorem 1. *Let τ_n be any sequence of non-negative numbers and let a_n be any sequence of real numbers tending to infinity. Suppose that for every $\varepsilon > 0$ we have*

$$\sum_{n=1}^{\infty} \tau_n P(|S_n| \geq \varepsilon a_n) \tag{3}$$

converging. Then for every $\varepsilon > 0$ there is an $M_\varepsilon \subseteq \mathbb{N} = \{1, 2, \ldots\}$ such that

$$\sum_{n \notin M_\varepsilon} n\tau_n P(|X_1| \geq \varepsilon a_n) < \infty \tag{4}$$

while

$$\sum_{n \in M_\varepsilon} \tau_n < \infty. \tag{5}$$

Moreover, we may take $M_\varepsilon = \{n \in \mathbb{N} : nP(|X_1| \geq \varepsilon a_n) > \lambda\}$, *where* λ *is any finite positive number.*

Remark. The conclusion of Theorem 1 is equivalent to asserting that for every $\varepsilon > 0$ we have

$$\sum_{n=1}^{\infty} \tau_n \min(1, nP(|X_1| \geq \varepsilon a_n)) < \infty.$$

***Corollary* 1.** *Under the same conditions as in the Theorem, if we additionally have*

$$\sup_n nP(|X_1| \geq \varepsilon a_n) < \infty, \tag{6}$$

for every positive ε, *it follows that we must in fact have*

$$\sum_{n=1}^{\infty} n\tau_n P(|X_1| \geq \varepsilon a_n) < \infty, \tag{7}$$

for every positive ε.

***Proof of Corollary* 1.** In Theorem 1, take $M_\varepsilon = \{n \in \mathbb{N} : nP(|X_1| \geq \varepsilon a_n) > \lambda\}$, where $\lambda = \sup_n nP(|X_1| \geq \varepsilon a_n)$. Then clearly $M_\varepsilon = \varnothing$ and the Corollary follows.

The following easy proof of our Theorem 1 is largely due to the anonymous referee of a previous version of the present paper and represents a significant simplification of the author's original proof which had employed a more complicated argument due to Erdős [2] in place of the rather simple inequality (10), below.

***Proof of Theorem* 1.** First note that once we find some M_ε satisfying (4) and (5) then $N = \{n \in \mathbb{N} : nP(|X_1| \geq \varepsilon a_n) > \lambda\}$ will also work in its place. For, in light of (4) and (5) it would suffice to verify that

$$\sum_{n \in N^c \setminus M_\varepsilon^c} n\tau_n P(|X_1| \geq \varepsilon a_n) < \infty \tag{8}$$

and

$$\sum_{n \in N \setminus M_\varepsilon} \tau_n < \infty. \tag{9}$$

But (8) follows from (5) together with the inequality $nP(|X_1| \geq \varepsilon a_n) \leq \lambda$, valid for $n \in N^c$. Also, $nP(|X_1| \geq \varepsilon a_n) > \lambda$ for $n \in N$ so that (4) implies that $\sum_{n \in N \setminus M_\varepsilon} \lambda\tau_n < \infty$, which in turn implies (9) since $\lambda > 0$. This completes the proof of the "moreover" part of the Theorem.

Now, by a standard symmetrization argument, it is easy to see that it suffices to prove the rest of the Theorem for symmetric X_1 (it is here that one actually uses the condition that $a_n \to \infty$ which guarantees that $\mu(X_1)/a_n \to 0$ whenever $\mu(X_1)$ is a median of X_1.) See [4] (§ 17.1A) or [5] (Lemma VI.14), together with the proof of [1, Thm. 1], for more information on symmetrization. Thus we may assume that X_1 is symmetric. Then, note that we have

$$P\left(\max_{1\le j\le n}|Y_j|\ge t\right) \ge \frac{\sum_{j=1}^{n} P(|Y_j|\ge t)}{1+\sum_{j=1}^{n} P(|Y_j|\ge t)}$$

whenever the Y_j are independent random variables. This inequality can be found in [6] (Proof of Lemma 3.2) or [7] (Lemma 2.6). Now, by an equality of Lévy type [8] (Prop. 1.1.2), if the Y_j are also symmetric then it follows that

$$2P\left(\left|\sum_{n=1}^{n} Y_j\right|\ge t\right) \ge \frac{\sum_{j=1}^{n} P(|Y_j|\ge t)}{1+\sum_{j=1}^{n} P(|Y_j|\ge t)}. \tag{10}$$

Letting $Y_j = X_j$, and defining

$$M_\varepsilon = \{n\in\mathbb{N} : nP(|X_1|\ge\varepsilon a_n)\ge 1\},$$

we see that (4) and (5) both follow from (3) and (10).

Our second remark begins by noting that the presence of the exceptional sets M_ε is rather natural when one considers the fact that $\sum_{n\in M_\varepsilon}\tau_n P(|S_n|\ge\varepsilon a_n)$ *automatically* converges for any X_1 if M_ε satisfies (5), and hence the fact that it converges contributes no new information.

Then, two of Klesov's results [3] (Theorems 2 and 3) have a slight improvement which brings the necessary and the sufficient conditions closer together. More precisely, we have the following result.

Theorem 2. *Suppose* $\alpha>1/2$. *Suppose that for every* $\varepsilon>0$ *there is a set* $M_\varepsilon\subseteq\mathbb{N}$ *such that* (4) *and* (5) *hold with* $a_n=n^\alpha$. *Assume that* $E[|X_1|^{1/\alpha}]<\infty$. *If* $\alpha\le 1$ *then assume further that* $E[X_1]=0$. *Finally, suppose that at least one of the following auxiliary conditions holds*:

K_2) $\lim_{n\to\infty} n^{-\theta}\tau_n<\infty$ *for some* $\theta>0$, *and* $E[|X_1|^r]<\infty$ *for some* $r>1/\alpha$;

K_3) *there is a slowly varying function* L *such that* $\sum_{n=1}^{\infty}(\tau_n/L(n)^\theta)<\infty$ *for some* $\theta>0$ *and* $E[|X_1|^{1/\alpha}(L(|X_1|^{1/\alpha}))^\nu]<\infty$ *for some* $\nu>0$.

Then (1) *converges for every* $\varepsilon>0$.

Condition K_N comes from Klesov's Theorem N for $N=2,3$. In his Theorem 2 and 3, Klesov [3] had proved the above result under the stronger condition that (2) converges in place of our weaker condition that (4) and (5) hold. Klesov's necessary and Klesov's sufficient conditions for (1) to converge for every $\varepsilon>0$ are brought somewhat closer together by our two Theorems, though they still do not meet.

The proof of Theorem 2 is essentially the same as Klesov's proofs in [3]. For, in order to show that for some particular $\varepsilon=\varepsilon_0>0$ one has (1) converging, the proofs of Klesov's Theorems 2 and 3 can be made to only use (in addition to the auxiliary conditions which we are not changing) the convergence of (2) for some *single* value of $\varepsilon=\varepsilon_1>0$ (ε_1 depending on ε_0). Then, we may simply set τ_n equal to zero for $n\in M_{\varepsilon_1}$, and note that, with this change, (2) will converge (for $\varepsilon=\varepsilon_1$) if (4) holds. Klesov's proofs would then show that (1) converges for $\varepsilon=\varepsilon_0$, providing τ_n is set to zero for $n\in M_{\varepsilon_1}$. But because of (5) and the fact that $P(|S_n|\ge\varepsilon n^t)\le 1$, it follows that (1) would converge for $\varepsilon=\varepsilon_0$ even if those τ_n are not set to zero.

Finally, we would like to remark that Theorem 2 generalizes easily and directly to the not necessarily identically distributed "regular covering" case considered in [9], with very much the same proof. It does not appear to be as simple, however, to extend

Theorem 1 to this more general case, although the methods of [9] (§ 5) might be relevant to the problem of proving a "regular covering" analogue of Corollary 1. At present however, the author only knows that if for *some* $\varepsilon_1 > 0$ we have the analogue of

$$\limsup_{n \to \infty} nP(|X_1| \geq \varepsilon_1 a_n) < \infty$$

then for any $K > 0$ there is an $\varepsilon_2 > 0$ such that $\limsup_{n \to \infty} nP(|X_1| \geq \varepsilon_2 a_n) < K$, and if the analogue (3) converges for *some* $\varepsilon = \varepsilon_3 > 0$ then the methods of [9] (§ 5) allow one to conclude that the analogue of (7) holds for *some* value of $\varepsilon > 0$ (possibly different from ε_3).

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