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ON SUBHARMONIC EXTENSION AND THE EXTENSION IN THE HARDY-ORLICZ CLASSES

ПРО СУБГАРМОНІЧНЕ ПРОДОВЖЕННЯ ТА ПРОДОВЖЕННЯ В КЛАСАХ ХАРДІ-ОРЛІЧА

The paper contains a generalization of certain previous results on subharmonic extension of functions and on extension of functions in the Hardy - Orlicz classes. We give unified proofs of these results.

Стаття містить узагальнення деяких попередніх результатів про субгармонічне продовження функцій та про продовження в класах Харді-Орліча. Даються уніфіковані доведення результатів.

1. Introduction.

1.1. Fix an integer $n \geq 2$.

By definition, a set $E \subset \mathbb{R}^n$ is polar (in $\mathbb{R}^n$), if for any $a \in \mathbb{R}^n$ there exists a subharmonic function $v \not\equiv -\infty$ in a connected neighborhood U of a with $v(x) = -\infty$ for all $x \in E \cap U$.

1.2. In section 2, we consider the possibility to give further generalizations or refinements to the fundamental classical result due to Brelot stating that closed polar sets are removable for subharmonic functions which are bounded above [2]. Our results, Theorems 2.6 and 2.8, generalize previous results due to Hayman and Kennedy, Lelong, Kuran, Brelot, Tamrazov, and Doob. Theorem 2.8 was proved in a different way in our preprint [3, Theorem 1].

In section 3, we consider the extension in the Hardy-Orlicz classes h_φ and H_φ. In Theorem 3.2 below we show that polar sets (not necessarily closed) are removable in the Hardy-Orlicz classes h_φ. As corollaries, we get results due to Singman, Parreau, and Järvi.

Conway, Dudziak, and Straube stated that a closed set is isometrically removable for the Hardy classes H^p if and only if the set is polar. In section 4, we generalize this result, in a certain sense, for the Hardy-Orlicz classes H_φ.

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1.3. We write $B(a, r)$ for the open ball in $\mathbb{R}^n$ with center a and radius r, and $S(a, r) := \partial B(a, r)$. Let $\sigma_{a,r}$ be the Lebesgue measure in $S(a, r)$. We often write σ instead of $\sigma_{a,r}$ when a and r are clear from the context. We also write $\sigma_n := \sigma(S(0, 1))$.

2. Subharmonic extention.

2.1. Following [4, 5], let us consider the class $\mathfrak{D}_n$ of all sets $X \subset \mathbb{R}^n$ which satisfy the condition :

For each point $a \in X$, there exists a sequence $r_j > 0$, $j = 1, 2, \ldots$, with $r_j \to 0$ as $j \to \infty$, and $\sigma_{a,r_j}(S(a, r_j) \cap X) = 0$ for all $j = 1, 2, \ldots$.

If $D \subset \mathbb{R}^n$ is open and $E \subset \mathbb{R}^n$ is of Lebesgue measure zero, then, clearly, $D \setminus E \in \mathfrak{D}_n$.

Let $G \in \mathfrak{D}_n$ and let $u : G \to [-\infty, \infty)$ be an upper semicontinuous function.

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Then we respectively say that u is:

(Def 1) P-subharmonic and belongs to the class $P(G)$ if for each $a \in G$ and all sufficiently small $r > 0$ with $\sigma_{a,r}(S(a, r) \setminus G) = 0$, we have

$$u(a) \le \frac{1}{\sigma_n r^{n-1}} \int_{S(a,r)} u(y)\, d\sigma(y); \tag{1}$$

(Def 2) Q-subharmonic and belongs to the class $Q(G)$ if for each $a \in G$ there exists a sequence $r_j > 0$, $j = 1, 2, \dots$, such that $r_j \to 0$ as $j \to \infty$, and for all $r = r_j$, $j = 1, 2, \dots$, both $\sigma_{a,r}(S(a, r) \setminus G) = 0$ and (1) hold;

(Def 3) A-subharmonic and belongs to the class $A(G)$ if for each $a \in G$ with $u(a) > -\infty$ there exists a sequence $r_j > 0$, $j = 1, 2, \dots$, such that $r_j \to 0$ as $j \to \infty$, $\sigma_{a,r_j}(S(a, r_j) \setminus G) = 0$ for $j = 1, 2, \dots$, and

$$\limsup_{j\to\infty} \frac{1}{r_j^2}\left[\frac{1}{\sigma_n r_j^{n-1}} \int_{S(a,r_j)} u(y)\, d\sigma_{a,r_j}(y) - u(a)\right] \ge 0. \tag{2}$$

Note that it follows from the upper semicontinuity of the function $u : G \to \to [-\infty, \infty)$ and from the condition $\sigma_{a,r}(S(a, r) \setminus G) = 0$ that the mean value

$$\mathcal{L}(u, a, r) := \frac{1}{\sigma_n r^{n-1}} \int_{S(a,r)} u(y)\, d\sigma_{a,r}(y)$$

is defined either as a real number or as $-\infty$.

Obviously, $P(G) \subset Q(G) \subset A(G)$. Each of these classes is a positive cone (containing the function identically equal to zero) which is invariant under the operation of adding any function from $P(G)$ to its elements. In particular, $P(G)$ is a semigroup with respect to addition.

The class $A(G)$ on $\mathfrak{D}_n$ was introduced and used in [4, p. 6] and [5, p. 23]. In [6, p. 10] and [7, p. 631], the class $P(G)$ was introduced for those $G \in \mathfrak{D}_n$ which satisfy the following additional condition: For every $a \in G$ and all sufficiently small $r > 0$, we have $\sigma_{a,r}(S(a, r) \setminus G) = 0$.

If $D \subset \mathbb{R}^n$ is open, then $D \in \mathfrak{D}_n$ and each of the classes $A(D)$, $P(D)$, and $Q(D)$ coincides with the class of all functions subharmonic in D (in the classical sense). This is well-known and follows immediately from the Blaschke-Privalov Theorem [8, p. 20] and from the (classical) definition of subharmonicity. Note that if u is subharmonic in D, then u may be identically $-\infty$ on any component of D.

If u is subharmonic in an open set D and $G \in \mathfrak{D}_n$ is its subset, then $u|G \in \in P(G)$.

2.2. Let D be an open set in $\mathbb{R}^n$ and let $E \subset D$ be closed in D and polar. A classical result of Brelot [2] states that if u is subharmonic in $D \setminus E$ and, moreover, locally bounded above in D, i. e., each point $a \in D$ has a neighborhood U_z such that u is bounded above in $U_z \setminus E$, then u has a subharmonic extention u^* to D. This extension $u^* : D \to [-\infty, \infty)$ is given by

$$u^*(x) := \limsup_{y \to x,\ y \in D \setminus E} u(y).$$

Different generalizations and refinements of this result can be found in the works of

Hayman and Kennedy, Lelong, Kuran, Brelot, Armitage, Tamrazov, Doob, and Riihentaus and Tamrazov. References and details are given below.

In Hayman's and Kennedy's formulation [9, Theorem 5. 18] of cited Brelot's result the exceptional set E need not be closed in D but is supposed to be contained in a countable union of closed polar sets, and the subharmonicity of u is replaced by the condition $u \in Q(D \setminus E)$.

In Tamrazov's [6, Theorem A], [7, Theorem 1], [4, Theorem 1], [5, Theorem 1] and Doob's [10, Theorem, p. 60] generalizations, the exceptional set E was supposed to be just polar (without closeness or any other additional assumptions), and the subharmonicity of u was replaced by the condition $u \in P\,(D \setminus E)$ in [6], [7], [10] and by the condition $u \in A\,(D \setminus E)$ in [4], [5], respectively.

On the other hand, supposing that E is closed in D, Lelong [11, Proposition 7], [12, Theorem 3_a], Kuran [13, Theorem 1 and Proposition 2], and Brelot [14, Theorem 2] generalized the cited Brelot's result by replacing the condition that u is locally bounded above in D by certain milder conditions.

Tamrazov [6, Theorem], [7, Theorem 3], [4, Theorem 1], [5, Theorem 1] relaxed the local upper boundedness condition of u as well as its previous generalizations, and under certain weaker assumptions established the results for any polar set E. He also obtained [4], [5] the results for more general functions u defined in $D \setminus E$ in terms of Blaschke-Privalov's inequality, i. e., for $u \in A\,(D \setminus E)$.

A generalization of Tamrazov's result was given in our preprint [3, Theorem 1] where the proof was based partly on the previous Tamrazov's technique and partly on certain considerations based on [15, Theorem 2]. Below, the result of [3, Theorem 1] is formulated as Theorem 2.8.

2.3. Lemma [15, Theorem 2]. *Let D be an open subset of $\mathbb{R}^n$. Assume that, for any λ from some index set Λ, the function v_λ is subharmonic in D. If, for some $p > 0$, there is a function $u \in \mathcal{L}^p_{loc}(D)$ such that $v_\lambda \le u$ for each $\lambda \in \Lambda$, then v^*, the upper semicontinuous regularization of the function $v = \sup_{\lambda \in \Lambda} v_\lambda$, is subharmonic in D. Moreover, the set*

$$\{x \in D : v\,(x) < v^*(x)\} \tag{3}$$

is polar.

In view of [15, Theorem 2] it remains to show that the set (3) is polar. But this follows from the fact that by [15, proof of Theorem 2] the family of functions v_λ, $\lambda \in \Lambda$, is indeed locally bounded above and from [1, Theorem 1 d), p. 26].

The next lemma is a very particular case of the results in [4, Theorem 1], [5, Theorem 1]. We shall prove it in a different way.

2.4. Lemma. *Let D be an open subset of $\mathbb{R}^n$ and $E \subset D$ be polar. Suppose that $u \in A\,(D \setminus E)$. If u is locally bounded above in D, then u has a subharmonic extension u^* to D.*

Proof. Since the problem is local, we may suppose that D is a bounded domain. We may also suppose that $u \not\equiv -\infty$. By [1, Theorem 1 a), p. 24], there exists $\omega \not\equiv -\infty$ subharmonic in D such that $\omega < 0$ and $\omega\,(x) = -\infty$ for all $x \in E$.

For each $k \in N$, we define $u_k : D \to [-\infty, \infty)$ by

$$u_k(x) := \begin{cases} u(x) + \omega(x)/k, & \text{for } x \in D \setminus E; \\ -\infty, & \text{for } x \in E. \end{cases}$$

We show that u_k is subharmonic. By using the facts that ω is upper semicontinuous, $\omega\,(x) = -\infty$ for $x \in E$, and u is locally bounded above in D, we can see that

u_k is upper semicontinuous in D.

Suppose that for some $a \in D$ we have $u_k(a) > -\infty$. Then clearly, $a \in D \setminus E$, $u(a) > -\infty$ and $\omega(a) > -\infty$. Since u_k is upper semicontinuous in D and ω is subharmonic in D, the mean values $\mathcal{L}(u, a, r)$ are defined as real numbers or as $-\infty$, at least for small $r > 0$. In fact, for small $r > 0$,

$$\mathcal{L}(u, a, r) = \mathcal{L}(u_k, a, r) - \frac{1}{k}\mathcal{L}(\omega, a, r). \tag{4}$$

Since u is subharmonic in $D \setminus E$, there is a sequence $r_j > 0$, $j = 1, 2, \dots$, such that $r_j \to 0$ as $j \to \infty$, $\sigma_{a,r_j}(S(a, r_j) \setminus (D \setminus E)) = 0$ for $j = 1, 2, \dots$, and

$$\limsup_{j \to \infty} \frac{1}{r_j^2}[\mathcal{L}(u, a, r_j) - u(a)] \geq 0. \tag{5}$$

By using (4), we get

$$\limsup_{r \to 0} \frac{1}{r^2}[\mathcal{L}(u_k, a, r) - u_k(a)] = \limsup_{r \to 0} \{\frac{1}{r^2}[\mathcal{L}(u, a, r) - u(a)] +$$

$$+ \frac{1}{kr^2}[\mathcal{L}(\omega, a, r) - \omega(a)]\} \geq \limsup_{r \to 0} \frac{1}{r^2}[\mathcal{L}(u, a, r) - u(a)] \geq$$

$$\geq \limsup_{j \to \infty} \frac{1}{r_j^2}[\mathcal{L}(u, a, r_j) - u(a)] \geq 0.$$

where the last inequality follows from (5).

Since this is true for any $a \in D$ with $u_k(a) > -\infty$, it follows from the Blaschke-Privalov Theorem [8, p. 20] that u_k is subharmonic in D.

To show that u has a subharmonic extension u^* to D, we employ (the classical form of) Lemma 2.3 given above as follows. We write $w := \sup_{k \in N} u_k$. Since u is locally bounded above in D and $\omega < 0$, the functions u_k, $k \in N$, are locally uniformly bounded above in D. Thus, by Lemma 2.3, w^*, the upper semicontinuous regularization of w, is subharmonic in D. Moreover, the set $F := \{x \in D : w(x) < w^*(x)\}$ is polar.

It remains to show that $w^*(x) = u(x)$ for all $x \in D \setminus E$. For this purpose, we write $H := \{x \in D : \omega(x) = -\infty\}$ and $E' := E \cup F \cup H$. If $x \in D \setminus E'$, then clearly, $w(x) = u(x)$. Using this fact, the polarity of E', the subharmonicity of w^*, and the upper semicontinuity of u in $D \setminus E$, we find that, for each $x \in D \setminus E$,

$$w^*(x) = \lim_{r \to 0} \frac{1}{\sigma_n r^{n-1}} \int_{S(x,r) \setminus E'} w^*(y)\, d\sigma_{x,r}(y) =$$

$$= \lim_{r \to 0} \frac{1}{\sigma_n r^{n-1}} \int_{S(x,r) \setminus E'} w(y)\, d\sigma_{x,r}(y) =$$

$$= \lim_{r \to 0} \frac{1}{\sigma_n r^{n-1}} \int_{S(x,r)} u(y)\, d\sigma_{x,r}(y) \leq u(x).$$

To show that in fact $w^*(x) = u(x)$ for all $x \in D \setminus E$, we suppose, on the contrary, that $w^*(x_0) < u(x_0)$ for some $x_0 \in D \setminus E$. Choose $\alpha \in \mathbb{R}$ such that $w^*(x_0) < \alpha < u(x_0)$. Since w^* is upper semicontinuous in D, there is $r_0 > 0$ such that $B(x_0, r_0) \subset D$ and $w^*(x) < \alpha$ for all $x \in B(x_0, r_0)$. Thus,

$$\limsup_{r\to 0} \frac{1}{r^2} [\mathcal{L}(w^*, x_0, r) - u(x_0)] = -\infty.$$

On the other hand, by using the fact that $w^*(x) = w(x) = u(x)$ for all $x \in D \setminus E'$ and the assumption, we get

$$\limsup_{r\to 0} \frac{1}{r^2} [\mathcal{L}(w^*, x_0, r) - u(x_0)] = \limsup_{r\to 0} \frac{1}{r^2} [\mathcal{L}(u, x_0, r) - u(x_0)] \geq 0,$$

arriving at a contradiction.

The next lemma is a particular case of the result from our preprint [3, Theorem 1] which is below formulated as Theorem 2. 8. Under the more restrictive assumption that v is subharmonic in D, these statements are contained in [4, Theorem 1], [5, Theorem 1].

2. 5. Lemma. *Let D be an open subset of $\mathbb{R}^n$ and $E \subset D$ be polar. Suppose that $u \in A(D\setminus E)$. Suppose that there is a function $v \in A(D\setminus E)$ such that, for each component Ω of D, one has $v \mid \Omega \setminus E \not\equiv -\infty$ and such that, for all $\varepsilon > 0$ and for all $x' \in E$,*

$$\limsup_{x\to x'} [u(x) + \varepsilon v(x)] < \infty. \tag{6}$$

Then u has a subharmonic extension u^* to D.

Proof. Since u and v are upper semicontinuous in $D\setminus E$, each $x \in D\setminus E$ has a neighborhood $U_x \subset D$ such that u and v are bounded above in $U_x \setminus E$. Thus, by Lemma 2. 4, $u \mid U_x \setminus E$ and $v \mid U_x \setminus E$ have subharmonic extensions u_x^* and v_x^* to U_x. Since $x \in D\setminus E$ is arbitrary and E is polar, we can see that u and v have subharmonic extensions u_1 and v_1 to $D\setminus E_1$, where

$$E_1 := D \setminus \bigcup_{x\in D\setminus E} U_x$$

is closed in D. Since $E_1 \subset E$, E_1 is also polar. In the remaining part of the proof, we write, for simplicity of notation, $E = E_1$, $u = u_1$, and $v = v_1$. Since E is polar, (6), clearly, still holds. Moreover, in the remaining part of the proof, we may suppose that D is a domain and that $u \not\equiv -\infty$.

For each $k \in \mathrm{N}$, we write $u_k : D\setminus E \to [-\infty, \infty)$,

$$u_k(x) := u(x) + \frac{1}{k} v(x).$$

By (6) and Lemma 2.4, we find that each u_k, $k \in \mathrm{N}$, has a subharmonic extension u_k^* to D. Since $v = 2(u_1^* - u_2^*)$ almost everywhere in D, we get $v \in \mathcal{L}^1_{\mathrm{loc}}(D)$. Similarly, $u = u_1^* - v$ almost everywhere in D; hence, $u \in \mathcal{L}^1_{loc}(D)$. Therefore, $w := |u| + |v| \in \mathcal{L}^1_{\mathrm{loc}}(D)$, and $u_k^* \leq w$ for all $k \in \mathrm{N}$. Then, for each $m \in \mathrm{N}$, we define the function $w_m : D \to [-\infty, \infty)$, where

$$w_m(x) := \sup\{u_k^*(x) : k \geq m\}.$$

(Note that indeed $w_m : D \to [-\infty, \infty)$, by Lemma 2.3). By Lemma 2.3, we see that w_m^*, the upper semicontinuous regularization of w_m, is subharmonic in D. Since, clearly, $w_m \geq w_{m+1}$, and thus, $w_m^* \geq w_{m+1}^*$ for all $m \in \mathbb{N}$, we can see that $w^* := = \lim_{m\to\infty} w_m^*$ is subharmonic.

It remains to show that w^* is an extension of u. It is clearly sufficient to show that $w^*(x) = u(x)$ for almost all $x \in D\setminus E$. Let us write $F := \{x \in D\setminus E : v(x) = -\infty\}$ and take $x \in D\setminus(E\cup F)$ arbitrarily. If $v(x) > 0$, then $w_m(x) = u(x) + (1/m)v(x)$. Since u and v are subharmonic in the open set $D\setminus E$, we have $w_m^*(x) = w_m(x) = = u(x) + (1/m)v(x)$. Thus, $w_m^*(x) \to u(x)$ as $m \to \infty$. If $v(x) \leq 0$, then $w_m(x) = = u(x) = w^*(x) \to u(x)$ as $m\to\infty$. Therefore, $w^*(x) = u(x)$ for all $x \in D\setminus(E\cup F)$. Since F is polar, the theorem is proved.

2.6. Theorem. *Let D be an open subset of $\mathbb{R}^n$ and let $E \subset D$ be polar. Suppose that $u \in A(D\setminus E)$. Suppose also that there is a function $v \in A(D\setminus E)$ and a nonpositive function $w \in A(D\setminus E)$, such that for each component Ω of D, we have $v \mid \Omega\setminus E \not\equiv -\infty$ and $w \mid \Omega\setminus E \not\equiv -\infty$ and such that, for all $\varepsilon > 0$ and all $x' \in E$,*

$$\limsup_{x\to x'} \frac{u(x)+\varepsilon v(x)}{-w(x)} \leq 0. \tag{7}$$

Then u has a subharmonic extension u^* to D.

Proof. It follows from Lemma 2.4 that w has a subharmonic extension w^* to D. Hence, $w \in P(D\setminus E)$, $h := v + w \in A(D\setminus E)$. For almost all $x \in D\setminus E$, we have $h(x) \neq -\infty$; for every such x, we also have $w(x) \neq -\infty$. It follows from (7) that

$$\frac{u(x)+\varepsilon v(x)}{-w(x)} < \varepsilon,$$

$$u(x) + \varepsilon h(x) < 0,$$

provided that this x is sufficiently close to x'. Consequently, for each $x' \in E$,

$$\limsup_{x\to x'} [u(x) + \varepsilon h(x)] \leq 0.$$

Thus, the result immediately follows from Lemma 2.5.

2.7. ***Remarks***. Choosing $w = 1$, we see that Theorem 2.6 contains [11, Théorème 3_a]. If, moreover, we choose, for example, $v \in A(D\setminus E)$ with $v(x) = -\infty$ for each $x \in E$, then we get that [9, Theorem 5. 18] is contained in Theorem 2.6, as well. On the other hand, by choosing $v = 0$, we find that [13, Theorem 1 and Proposition 2] and [14, Theorem 2, p. 73] are contained in Theorem 2.6.

Let

$$K_n(x) := \begin{cases} \log\dfrac{1}{|x|}, & \text{for } n=2; \\ |x|^{2-n}, & \text{for } n\geq 3. \end{cases}$$

2.8. Theorem [3, Theorem 1]. *Let D be an open subset of $\mathbb{R}^n$ and let $E \subset \subset D$ be polar. Suppose that $u \in A(D\setminus E)$. Suppose further that for each $a \in E$ there is $r_a > 0$ with $B(a, r_a) \subset D$ and a function $v \in A(B(a, r_a)\setminus E)$, $v \not\equiv -\infty$, with the following property: For each ε, $0 < \varepsilon \leq 1$, there is a function $\alpha_{a,\varepsilon}$: $(0, r_a) \to [0, \infty)$ locally bounded and such that $\alpha_{a,\varepsilon}(r) = o(K_n(r))$, as $r \to 0$, and*

$$u(x) + \varepsilon v(x) \leq \alpha_{a,\varepsilon}(|x-a|) \tag{8}$$

for all $x \in B(a, r_a)\setminus E$.

Then u *has the subharmonic extension* u^* *to* D.

Proof. Since the problem is local, we may suppose that $r_a < 1$. By [1, Theorem 1 a), p. 24], there is a superharmonic function ω in $B(a, r_a)$ such that $\omega > 0, \omega \not\equiv \infty$, and $\omega(x) = \infty$ for all $x \in E \cap B(a, r_a)$. Then the function $\omega_1 : B(a, r_a) \to (0, \infty]$, where $\omega_1(x) := K_n(|x-a|) + \omega(x)$, is superharmonic. We show that, for each ε, $0 < \varepsilon \leq 1$, and for each $x' \in E \cap B(a, r_a)$,

$$\limsup_{x \to x'} \frac{u(x)+\varepsilon v(x)}{\omega_1(x)} \leq 0. \tag{9}$$

To show this, we first suppose that $x' \neq a$. Since $\alpha_{a,\varepsilon}$ is locally bounded in $(0, r_a)$, there is $r_{x'} > 0$ and $M < \infty$ such that $\alpha_{a,\varepsilon}(|x-a|) < M$ for all $x \in B(x', r_{x'})$. Then it follows from (8) and the choice of ω that (9) holds. Suppose also that $x' = a$. Since $\omega_1 > 0$, we may suppose that, for each $r > 0$, the set $A_r := \{x \in B(a, r)\setminus E : u(x) + \varepsilon v(x) \geq 0\}$ is nonempty. Namely, otherwise, (9) holds automatically for $x' = a$. Therefore, we have

$$\limsup_{x \to a} \frac{u(x)+\varepsilon v(x)}{\omega_1(x)} = \limsup_{x \to a,\, x \in A_r} \frac{u(x)+\varepsilon v(x)}{\omega_1(x)} \leq \limsup_{x \to a,\, x \in A_r} \frac{u(x)+\varepsilon v(x)}{K_n(|x-a|)}.$$

Since by assumption

$$\limsup_{x \to a} \frac{u(x)+\varepsilon v(x)}{K_n(|x-a|)} \leq 0;$$

we again have (9). Thus, the statement follows from Theorem 2.6.

2.9. ***Remark.*** For other types of subharmonic extension results see [14, Theorems 4 and 5] and [16, Corollary, p. 56].

3. On the extension in the Hardy-Orlicz classes.

3.1. Let $\varphi : [-\infty, \infty) \to [0, \infty)$ be a *strongly convex* function, i. e., nondecreasing and such that $\varphi | \mathbb{R}$ is convex, $\varphi(-\infty) = \lim_{t \to -\infty} \varphi(t)$, and $\varphi(t)/t \to \infty$ as $t \to \infty$.

Let $G \in \mathcal{D}_n$. We say that a function $w : G \to (0, \infty]$ *is suitable for* G if the union of all components Γ of G with $w | \Gamma \not\equiv \infty$ is dense in G. Denote $h_\varphi(G) := \{u \in A(G) : \varphi \circ u$ has in G a majorant w, suitable for G, with $-w \in A(G)\}$.

We say that a function w is φ*-admissible* for a function $u \in h_\varphi(G)$, if w is defined on G and is a majorant for $\varphi \circ u$, suitable for G, with $-w \in A(G)$.

Let D be an open subset of $\mathbb{R}^n$ and let $E \subset D$ be closed in D and polar. Singman [17, Corollary, p. 300] has shown that each $u \in h_\varphi(D\setminus E)$ has an extension $u^* \in h_\varphi(D)$. For a similar result in the special case where $\varphi(t) = (t^+)^p, p > 1$, see [18, Theorem 2.5]. Here, $t^+ := \max\{t, 0\}$, as usual. Theorem 3.2 below generalizes Singman's result by allowing E to be nonclosed in D, too.

3.2. Theorem. *Let* D *be an open subset of* $\mathbb{R}^n$ *and* $E \subset D$ *be polar. Let* $\varphi : [-\infty, \infty) \to [0, \infty)$ *be strongly convex. Then every* $u \in h_\varphi(D\setminus E)$ *has an extension* $u^* \in h_\varphi(D)$, *and every function* w, φ*-admissible for* u, *has a superhar-*

monic extension w_* to D which is φ-admissible for u^*.

Proof. Since $u \in h_\varphi(D \setminus E)$, there exists a function w suitable for $D \setminus E$, with $-w \in A(D \setminus E)$ and

$$\varphi \circ u \leq w. \tag{10}$$

Since $\varphi \geq 0$, it follows from Lemma 2.4 that w has a superharmonic extension w_* to D. Clearly, $w_* \not\equiv \infty$ on any component of D, so w_* is suitable for D.

For any fixed $\varepsilon > 0$, we have $u_\varepsilon := u - \varepsilon w \leq u - \varepsilon\varphi \circ u \leq u$ in $D \setminus E$. For $t > 0$, the quantity $t - \varepsilon\varphi(t) = t(1 - \varepsilon\varphi(t)/t)$ tends to $-\infty$ as $t \to \infty$. Hence, u_ε is bounded above in $D \setminus E$. Therefore, we can apply Lemma 2.5 to u and $v := -w$ and state that u has a subharmonic extension u^* to D.

Since u^* is subharmonic, $\varphi \circ u^*$ is also subharmonic.

For any fixed point $a \in E$ and sufficiently small $r > 0$, we have $\sigma_{a,r}(S(a, r) \setminus (D \setminus E)) = 0$, $\sigma_{a,r}(S(a, r) \cap E) = 0$. By using these facts and (10), we get

$$(\varphi \circ u^*)(a) \leq \mathcal{L}(\varphi \circ u^*, a, r) = \mathcal{L}(\varphi \circ u, a, r) \leq \mathcal{L}(w, a, r) = \mathcal{L}(w_*, a, r) \leq w_*(a).$$

Together with (10) this again yields $\varphi \circ u^* \leq w_*$. Since w_* is suitable for D, so $u^* \in h_\varphi(D)$. The theorem is proved.

3.3. In order to give two corollaries, we recall the definitions of the Hardy-Orlicz class H_φ and of the Smirnov class S. For this purpose, let $\varphi : [-\infty, \infty) \to [0, \infty)$ be a strongly convex function and let Ω be an open subset of $\mathbb{C}^n$, $n \geq 1$, identified with $\mathbb{R}^{2n}$. We then write

$$H_\varphi(\Omega) := \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic, } \varphi \circ \log|f| \text{ has a superharmonic majorant in } \Omega \text{ which is } \not\equiv \infty \text{ on each component of } \Omega\}$$

for the *Hardy-Orlicz class*. If $\varphi(t) = e^{pt}, p > 0$, then we get the familiar Hardy class $H^p(\Omega)$. The *Smirnov class* is defined by

$$S(\Omega) = \{f : \Omega \to \mathbb{C} : f \text{ is holomorphic, there is a strongly convex function } \varphi : [-\infty, \infty) \to [0, \infty) \text{ such that } \varphi \circ \log^+|f| \text{ has a superharmonic majorant which is } \not\equiv \infty \text{ on each component of } \Omega\}.$$

Compare [19] with [20, Theorem 1 and Corollary 3].

3.4. Corollary. *Let Ω be an open subset of $\mathbb{C}^n$, $n \geq 1$. Let $E \subset \Omega$ be closed in Ω and polar. Let $\varphi : [-\infty, \infty) \to [0, \infty)$ be strongly convex. Then every $f \in H_\varphi(\Omega \setminus E)$ has an extension $f^* \in H_\varphi(\Omega)$ and every function w, φ-admissible for $\log|f|$, has a superharmonic extension w_* to Ω which is φ-admissible for $\log|f^*|$.*

3.5. Corollary. *Let Ω and E be as in Corollary 3.4. Then every $f \in S(\Omega \setminus E)$ has an extension $f^* \in S(\Omega)$.*

3.6. *Remarks.* If $\varphi(t) = e^{pt}, p > 0$, then the result of Corollary 3.4 is due to Parreau [21, Théorème 20] for $n = 1$ and to Järvi [22, Theorem 1] for $n \geq 1$. See also [20, Corollary 2], [23, Theorem 4], [24, Theorem 8], [13, Theorem 5], and [25, Theorem 3. 2].

Tumarkin and Havinson [26, Theorem 1] proved the result of Corollary 3.5 for $n =$

$=1$ and E being compact. The general case for $n=1$ was later proved by Yamashita [20, Corollary 1].

4. On isometric removability.

4.1. Let Ω be a domain in the space $\mathbb{C}^n$, $n \geq 1$. Suppose that $E \subset \Omega$ is closed in Ω and $\Omega \setminus E$ is a domain. Conway, Dudziak, and Straube [27, Corollary] showed that if, moreover, Ω is bounded (and thus, both $H^p(\Omega)$ and $H^p(\Omega \setminus E)$ contain nonconstant functions), then E is *isometrically removable* for functions in the Hardy classes H^p, $p > 0$, if and only if E is polar. This means, say, that the following three conditions are equivalent:

(a) E is polar.

(b) For every p, $0 < p < \infty$, the restriction map $H^p(\Omega) \to H^p(\Omega \setminus E)$ is a surjective isometry.

(c) There is a value of p, $0 < p < \infty$, and a nonconstant function $f \in H^p(\Omega \setminus E)$ that has an extention $f^* \in H^p(\Omega)$ with

$$\|f^*\|_{H^p(\Omega)} = \|f\|_{H^p(\Omega \setminus E)}.$$

Here, we use the standard norm: If Ω_1 is a domain in $\mathbb{C}^n$, $n \geq 1$, if $a \in \Omega_1$ and $f \in H^p(\Omega_1)$, then

$$\|f\|_{H^p(\Omega_1)} = u_f(a), \tag{11}$$

where u_f is the least harmonic majorant of the subharmonic function $|f|^p$. Above, in (c), the norming point a can be chosen arbitrarily from $\Omega \setminus E$. Note that (11) gives a real norm only if $p \geq 1$. For more details, see [27, pp. 270–271].

In Corollary 4. 4 below, we generalize, in a certain sense, the above Hardy class H^p result by Conway, Dudziak, and Straube to the Hardy-Orlicz class H_φ. Our proof differs from the argument in [27, pp. 269–271].

We put the main part of the argument into the following theorem:

4.2. Theorem. *Let Ω be a domain in $\mathbb{C}^n$, $n \geq 1$, and let $E \subset \Omega$ be closed in Ω and such that $\Omega \setminus E$ is a domain. Let $\varphi : [-\infty, \infty) \to [0, \infty)$ be strongly convex.*

Then the condition

(a′) $\{E$ *is polar*$\}$

implies the condition

(b′) $\{$*Each* $f \in H_\varphi(\Omega \setminus E)$ *has an extension* $f^* \in H_\varphi(\Omega)$ *with* $u_{f^*} = u_f$ *in* $\Omega \setminus E\}$,

where u_f (u_{f^*}, *respectively*) *is the least superharmonic majorant of* $\varphi \circ \log|f|$ *in* $\Omega \setminus E$ (*respectively*, $\varphi \circ \log|f^*|$ *in* Ω).

Conversely, the condition

(c′) $\{$*There is* $f \in H_\varphi(\Omega \setminus E), f \not\equiv 0,$ *for which* $u_f - \varphi \circ \log|f|$ *has a superharmonic extension* $w \geq 0$ *to* Ω *that is not harmonic*$\}$.

implies (a′).

Proof. Recall first that u_f is harmonic.

To show that (a′) $\Rightarrow$ (b′), suppose that E is polar. By Corollary 3.4, each $f \in H_\varphi(\Omega \setminus E)$ has an extension $f^* \in H_\varphi(\Omega)$ and u_f has a superharmonic extension $(u_f)_*$ to Ω which is φ-admissible for $\log|f^*|$. Thus, $(u_f)_* \geq u_{f^*}$ in Ω. On the other hand, $u_f \leq u_{f^*}$ in $\Omega \setminus E$ and E is polar which implies that $(u_f)_* \leq u_{f^*}$ in Ω. Thus, $u_{f^*} = (u_f)_*$ in Ω and $u_{f^*} = (u_f)_* = u_f$ in $\Omega \setminus E$, proving the implication (a′) $\Rightarrow$ (b′).

To show that (c′) ⇒ (a′), we take f and w as in (c′). Obviously, $w > 0$. Let us write

$$R_w^E := \inf\{u : u \text{ is superharmonic in } \Omega,\ u \geq 0,\ u \geq w \text{ in } E\}.$$

Then $(R_w^E)_*$, the lower semicontinuous regularization of R_w^E, satisfies the inequality $(R_w^E)_* \geq 0$ and is superharmonic in Ω, by Lemma 2.3 (or, of course, directly by [1, Theorem 1, p. 14]). By [1, Proposition 1 c), p. 17], $(R_w^E)_*$ and, hence, also $u_f - (R_w^E)_*$ are harmonic in $\Omega \setminus E$. Since $(R_w^E)_* \leq w$, we have $u_f - (R_w^E)_* \geq \varphi \circ \log|f|$ in $\Omega \setminus E$. It follows from the definition of u_f that $(R_w^E)_* = 0$ in $\Omega \setminus E$, and, thus, $(R_w^E)_* = 0$ in Ω. Since we also have $w > 0$, it follows from [1, Theorem 1 b), p. 24] that E is polar (in the case $n = 1$ we may certainly suppose that Ω is bounded), concluding the proof.

4.3. Corollary. *The statements of Theorem* 4.2 *remain true if condition* (c′) *in its formulation is changed by the following condition:*

(c″) $\{$*There exists* $f \in H_\varphi(\Omega \setminus E)$, $f \not\equiv 0$, *with an extension* $f^* \in H_\varphi(\Omega)$ *for which* $\varphi \circ \log|f^*|$ *is not harmonic and* $u_{f^*}(a) = u_f(a)$ *at some point* $a \in \Omega \setminus E\}$.

Proof. It is sufficient to show that (c″) ⇒ (a′). We take f and a as in (c″). Clearly, $u_f \leq u_{f^*}$ in $\Omega \setminus E$. Since $u_{f^*}(a) = u_f(a)$, we have $u_f = u_{f^*}$ in $\Omega \setminus E$. Thus, u_f has a harmonic extension $(u_f)_*$ to Ω which is equal to u_{f^*}. In Ω, the function $w := u_{f^*} - \varphi \circ \log|f^*|$ is superharmonic but not harmonic and $w \geq 0$. We can also see that $w \mid \Omega \setminus E = u_f - \varphi \circ \log|f|$. Hence, (c″) ⇒ (c′) ⇒ (a′), concluding the proof.

One can get various corollaries just by providing that the function w from (c′) is not harmonic. As an example we give:

4.4. Corollary. *The statements of Theorem* 4.2 *remain valid if the condition* (c′) *in its formulation is changed by the following condition*:

(c‴) $\{\varphi \mid \mathbb{R}$ *is twice differentiable with* $\varphi'' > 0$ *and there exists a nonconstant function* $f \in H_\varphi(\Omega \setminus E)$ *for which* $u_f - \varphi \circ \log|f|$ *has a superharmonic extension* $w \geq 0$ to $\Omega\}$.

Proof. It is sufficient to show that for each nonconstant $g \in H_\varphi(\Omega \setminus E)$ the subharmonic function $u = \varphi \circ \log|g|$ is not harmonic. Suppose, on the contrary, that u is harmonic. Then by the minimum principle we see that $g(z) \neq 0$ for all $z \in \Omega \setminus E$. Then $h := \log|g|$ is (pluri)harmonic in $\Omega \setminus E$. By an easy computation, we get

$$\Delta u(z) = \varphi''(h(z)) \sum_{j=1}^{n} \left[\left(\frac{\partial h(z)}{\partial x_j} \right)^2 + \left(\frac{\partial h(z)}{\partial y_j} \right)^2 \right].$$

This is clearly a contradiction, since $\varphi''(t) > 0$ for all $t \in \mathbb{R}$ and g is nonconstant.

Note that if we choose $\varphi(t) = e^{pt}$, then Corollary 4.4 gives the cited result of Conway, Dudziak, and Straube.

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