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A. F. IVANOV (Inst. of Math. Ukr. Acad. of Sci., Kiev), T. KUSANO (Japan)

Oscillation of solutions of second order nonlinear functional differential equations of neutral type

Колеблемость решений нелинейных функционально-дифференциальных уравнений второго порядка нейтрального типа

The oscillation condition of solutions to nonlinear differential equations of the second order of neutral type on the semiaxis $t > 0$ and the properties of nonoscillatory solutions are invastigated.

Исследованы условия осцилляции решений нелинейных дифференциальных уравнений второго порядка нейтрального типа на полуоси $t > 0$, а также свойства неосциллирующих решений.

Досліджені умови осціляції розв'язків нелінійних диференціальних рівнянь другого порядку нейтрального типу на півосі $t > 0$, а також властивості неосцілюючих розв'язків.

1. Introduction. The equation to be considered in this paper is

$$L[x(t) + \lambda x(\tau(t))] + f(t, x(g(t))) = 0, \tag{1}$$

where L is the differential operator

$$L[u(t)] = \frac{d}{dt}\left(\frac{1}{p(t)}\frac{du(t)}{dt}\right).$$

With regard to (1) the following conditions are assumed to hold without further mention:

a) $p : [a, \infty) \to (0, \infty)$ is continuous and satisfies

$$\int_a^\infty p(t)\,dt = \infty; \tag{2}$$

b) λ is a constant with $|\lambda| < 1$;

c) $\tau:[a, \infty)\to\mathbb{R}$ is continuous and strictly increasing, $\tau(t)<t$ for $t\geqq a$ and $\lim_{t\to\infty}\tau(t)=\infty$;

d) $g:[a, \infty)\to\mathbb{R}$ is continuous and $\lim_{t\to\infty} g(t)=\infty$;

e) $f:[a, \infty)\times\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing in the second variable, and $yf(t, y)\geqq 0, \neq 0$ for $y\neq 0$ and $t\geqq a$.

We are interested in the oscillatory and nonoscillatory behavior of solutions of equation (1). By a solution of (1) we mean a continuous function $x:[t, \infty)\to\mathbb{R}$ such that $x(t)+\lambda x(\tau(t))\in\mathcal{D}_L$ and satisfies (1) for all sufficiently large t. Here $\mathcal{D}_L$ denotes the domain of L, i. e. the set of continuous functions $u:[t_u, \infty)\to\mathbb{R}$ such that $u(t)$ and $(1/p(t))(du(t)/dt)$ are continuously differentiable for $t>t_u$. A solution of (1) is said to be oscillatory if it has a sequence of zeros tending to infinity; otherwise it is said to be nonoscillatory.

It is shown that a nonoscillatory solution $x(t)$ of (1) eventually satisfies either $x(t)[x(t)+\lambda x(\tau(t))]<0$, in which case $x(t)$ tends to zero as $t\to\infty$, or $x(t)[x(t)+\lambda x(\tau(t))]>0$, in which case one of the following cases holds:

$$\text{I)}\ 0<\liminf_{t\to\infty}\frac{|x(t)|}{P(t)}\leqq\limsup_{t\to\infty}\frac{|x(t)|}{P(t)}<\infty;$$

$$\text{II)}\ \lim_{t\to\infty}\frac{|x(t)|}{P(t)}=0\ \text{and}\ \lim_{t\to\infty}|x(t)|=\infty;$$

$$\text{III)}\ 0<\liminf_{t\to\infty}|x(t)|\leqq\limsup_{t\to\infty}|x(t)|<\infty;$$

where $P(t)=\int_a^t p(s)\,ds$. Our first task (pt 2) is to obtain conditions under which (1) has nonoscillatory solutions of the above types I, II and III. This is accomplished by solving, with the aid of fixed point techniques, appropriate nonlinear Volterra-like integral equations. Our second task (pt 3) is to present oscillation criteria for equation (1) with strong nonlinear structure. Some of the methods used in the study of oscillation of «ordinary» (or nonneutral) functional differential equations are shown to be also applicable to the «neutral» equation under consideration. Combining the oscillation criteria obtained with nonoscillation results of pt 2, we are able to exhibit certain classes of equations of the form (1) for which the situation of oscillation of all (or almost all) solutions can be completely characterized.

Since the appearance of the paper [1], there has been a growing interest in oscillation theory of neutral functional differential equations; see e. g. [2— 14] and the references cited therein. Most of the literature, however, is concerned with linear equations with constant coefficients and constant deviations, for which oscillation criteria are given in terms of the associated characteristic equations, and very little is known about genuinely nonlinear equations with general deviating arguments. This paper could be regarded as an attempt at a systematic investigation of oscillatory behavior of general neutral equations to which the theory of characteristic equations fails to apply.

2. Existence of nonoscillatory solutions.

A) Classification of nonoscillatory solutions. We introduce the notation:

$$P(t, s)=\int_s^t p(r)dr, \quad s, t\in[a, \infty), \quad P(t)=P(t, a), \tag{3}$$

$$\tau^0(t)=t,\ \tau^j(t)=\tau(\tau^{j-1}(t)),\ \tau^{-j}(t)=\tau^{-1}(\tau^{-(j-1)}(t)),\ j=1, 2, \dots, \tag{4}$$

where $\tau^{-1}(t)$ is the inverse function of $\tau(t)$.

Let N denote the set of all nonoscillatory solutions of equation (1). If $x\in N$, then (1) implies that the function $x(t)+\lambda x(\tau(t))$ is eventually of constant sign, so that either

$$x(t)[x(t)+\lambda x(\tau(t))]>0 \text{ for all large } t \tag{5}$$

or

$$x(t)[x(t)+\lambda x(\tau(t))]<0 \text{ for all large } t. \tag{6}$$

Note that (6) holds only when $\lambda < 0$. We denote by N^+ and N^- the sets of all $x \in N$ satisfying (5) and (6), respectively. Thus, $N = N^+$ if $\lambda \geqq 0$ and $N = N^+ N \cup$ if $\lambda < 0$.

All the members of N^- tend to zero as $t \to \infty$. In fact, from (6) we have $|x(t)| \leqslant |\lambda| |x(\tau(t))|$, and hence $|x(\tau^{-n}(t))| \leqq |\lambda|^n |x(t)|$ for t large enough, $n = 1, 2, \ldots$, which implies $\lim\limits_{t\to\infty} x(t) = 0$.

Now consider a member $x(t)$ of N^+. Put

$$y(t) = x(t) + \lambda x(\tau(t)). \tag{7}$$

Then, from (1),

$$y(t)\frac{d}{dt}\left(\frac{1}{p(t)}\frac{dy(t)}{dt}\right) \leqq 0 \text{ for all large } t,$$

and so it can be proved routinely that $y(t)(dy(t)/dt) \geqslant 0$ eventually and one of the following cases holds for $y(t)$:

$$\begin{gathered} \text{I) } \lim_{t\to\infty}[y(t)/P(t)] = \text{const} \neq 0; \\ \text{II) } \lim_{t\to\infty}[y(t)/P(t)] = 0 \quad \text{and} \quad \lim_{t\to\infty} y(t) = \infty \text{ or } -\infty; \\ \text{III) } \lim_{t\to\infty} y(t) = \text{const} \neq 0. \end{gathered} \tag{8}$$

Rewriting (7) as

$$x(t) = y(t) - \lambda x(\tau(t)), \tag{9}$$

we find

$$x(t) = y(t) - \lambda y(\tau(t)) + \lambda^2 x(\tau^2(t)),$$

from which, in view of the nondecreasing property of $|y(t)|$, it follows that

$$|x(t)| \geqq (1 - |\lambda|)\, |y(t)| \tag{10}$$

for all large t, say $t \geqq t_1$. Let $t_2 \geqq t_1$ be such that $\tau(t) \geqq t_1$ for $t \geqq t_2$. Repeated application of (9) yields

$$x(t) = \sum_{j=0}^{n(t)-1} (-\lambda)^j y(\tau^j(t)) + (-\lambda)^{n(t)} x(\tau^{n(t)}(t)),\ t > t_2,$$

where $n(t)$ denotes the least positive integer such that $t_1 < \tau^{n(t)}(t) \geqq t_2$. It then follows that

$$|x(t)| \leqq \frac{|y(t)|}{1-|\lambda|} + \xi_x \text{ for } t \geqq t_2, \tag{11}$$

where $\xi_x > 0$ is a constant. Combining (8) with (10) and (11), we conclude that $x(t)$ satisfies one of the following relations:

$$\begin{gathered} \text{I) } 0 < \liminf_{t\to\infty} \frac{|x(t)|}{P(t)} \leqq \limsup_{t\to\infty} \frac{|x(t)|}{P(t)} < \infty; \\ \text{II) } \lim_{t\to\infty} \frac{|x(t)|}{P(t)} = 0 \text{ and } \lim_{t\to\infty} |x(t)| = \infty; \\ \text{III) } 0 < \liminf_{t\to\infty} |x(t)| \leqq \limsup_{t\to\infty} |x(t)| < \infty. \end{gathered} \tag{12}$$

This is a classification of nonoscillatory solutions of class N^+.

B) Existence of nonoscillatory solutions. We are now concerned with the problem of finding conditions for the existence of nonoscillatory solutions of equation (1). Because of the difficulty in constructing solutions of class N^-, our attention will be restricted to the study of the class N^+.

Theorem 1. *Equation* (1) *has a nonoscillatory solution of type* I *if and only if*

$$\int^{\infty} |f(t,\ kP(g(t)))|\, dt < \infty \tag{13}$$

for some nonzero constant k.

Proof. (The «only if» part.) Let $x(t)$ be a nonoscillatory solution of type I of (1). It is clear that

$$\liminf_{t\to\infty} \frac{|x(g(t))|}{P(g(t))} > 0.$$

Combining this with the inequality

$$\int_T^{\infty} |f(t,\ x(g(t)))|\, dt < \infty,$$

$T > a$ being sufficiently large, which follows from (1), we obtain the desired inequality (13).

(The «if» part.) Assume that (13) holds for some $k > 0$. The case $k < 0$ can be treated similarly. Take a constant $c>0$ such that $c/|\lambda|(1-|\lambda|) \leqq k$ and choose $T > a$ large enough so that

$$T_* = \min\{\tau(T),\ \inf_{t\geqq T} g(t)\} \geqq a \tag{14}$$

and

$$\int_T^{\infty} \left| f\left(t, \frac{c}{|\lambda|(1-|\lambda|)} P(g(t))\right)\right| dt \leqq \frac{(1-|\lambda|)c}{|\lambda|}. \tag{15}$$

Let X denote the set of all continuous functions $x : [T_*,\ \infty) \to \mathbb{R}$ such that

$$\begin{gathered} cP(t,\ T) \leqq x(t) \leqq (c/|\lambda|)P(t,\ T) \text{ for } t \geqq T, \\ x(t) = 0 \text{ for } T_* \leqq t \leqq T; \end{gathered} \tag{16}$$

X is a closed convex subset of the Frechet space $C[T_*,\ \infty)$ of continuous functions on $[T_*,\ \infty)$. Following Ruan [11], we associate to each $x \in X$ the function $\hat{x}$: $[T_*,\ \infty) \to \mathbb{R}$ defined by

$$\hat{x}(t) = \sum_{j=1}^{n(t)-1} (-\lambda)^j x(\tau^j(t)) \text{ for } t > T \text{ and } \hat{x}(t) = 0 \text{ for } T_* \leqq t \leqq T, \tag{17}$$

where $n(t)$ denotes the least positive integer such that $T_* < \tau^{n(t)}(t) \leqq T$. It can be shown that $x \in X$ implies that $x(t) \geqq 0$ for $t \geqq T^*$,

$$\hat{x}(t) \leqq \frac{c}{|\lambda|(1-|\lambda|)} P(t,\ T), \quad t \geqq T^*, \tag{18}$$

and

$$\hat{x}(t) + \lambda\hat{x}(\tau(t)) = x(t), \quad t \geqq T. \tag{19}$$

In fact, (18) and (19) follow readily from (17), and the positivity of $\hat{x}(t)$ from (17) rewritten as

$$\hat{x}(t) = \sum_{j=0}^{m-1} [(-\lambda)^{2j} x(\tau^{2j}(t)) + (-\lambda)^{2j+1} x(\tau^{2j+1}(t))] + (-\lambda)^{2m} x(\tau^{2m}(t))$$

$$\text{for } n(t) = 2m+1,\ m = 0,\ 1,\ \ldots,$$

$$\hat{x}(t) = \sum_{j=0}^{m-1} [(-\lambda)^{2j} x(\tau^{2j}(t)) + (-\lambda)^{2j+1} x(\tau^{2j+1}(t))] \text{ for } n(t) = 2m,$$

$$m = 1,\ 2,\ \ldots,$$

combined with the observation that, in view of (16), $x \in X$ implies for each j

$$(-\lambda)^{2j} x(\tau^{2j}(t)) + (-\lambda)^{2j+1} x(\tau^{2j+1}(t)) = \lambda^{2j}[x(\tau^{2j}(t)) - \lambda x(\tau^{2j+1}(t))] \geqq$$

$$\geqq \lambda^{2j}[cP(\tau^{2j}(t), T) - |\lambda|(c/|\lambda|)P(\tau^{2j+1}(t), T)] =$$

$$= c\lambda^{2j}[P(\tau^{2j}(t), T) - P(\tau^{2j+1}(t), T)] \geqq 0, \ t \geqq T.$$

Let us now define the map $F: X \to C[T_*, \infty)$ by

$$Fx(t) = cP(t, T) + \int_T^t p(s) \int_s^\infty f(r, \hat{x}(g(r)))\, drds \ \text{ for } t \geqq T,$$

$$Fx(t) = 0 \ \text{ for } T_* \leqslant t \leqslant T. \tag{20}$$

If $x \in X$, then since by (18) and (15)

$$\int_T^t p(s) \int_s^\infty f(r, \hat{x}(g(r)))\, drds \leqq$$

$$\leqq \int_T^t p(s)\, ds \cdot \int_T^\infty f\left(r, \frac{c}{|\lambda|(1-|\lambda|)} P(g(r))\right) dr \leqq \frac{(1-|\lambda|)c}{|\lambda|} P(t, T)$$

for $t \leqq T$, we see that $Fx \in X$, showing that F maps X into itself. Furthermore it can easily be proved that F is continuous and $F(X)$ is relatively compact in the $C[T_*, \infty)$-topology. Therefore, by the Schauder-Tychonoff fixed point theorem, there exists an element $x^* \in X$ such that $Fx^* = x^*$, i e.,

$$x^*(t) = cp(t, T) + \int_T^t P(s) \int_s^\infty f(r, \hat{x}^*(g(r)))\, drds, \quad t \leqq T.$$

Since by (19) $\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t)) = x^*(t)$, $t \leqq T$, this can be written as

$$\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t)) = cP(t, T) + \int_T^t p(s) \int_s^\infty f(r, \hat{x}^*(g(r)))\, drds, \quad t \leqq T,$$

when ce by differentiation it follows that the function $\hat{x}^*(t)$ is a solution of equation (1). That $x^*(t)$ is of type I follows from (20). This completes the proof of the «if» part.

Theorem 2. *Equation* (1) *has a nonoscillatory solution of type* III *if and only if*

$$\int^\infty P(t)\, |f(t, k)|\, dt < \infty \tag{21}$$

for some nonzero constant k.

Proof. The «only if» part follows from the observation that a nonoscillatory solution $x(t)$ of type III nesessarily satisfies

$$\int_T^\infty p(t) \int_s^\infty |f(s, x(g(s)))|\, dsdt = \int_T^\infty P(t, T)\, |f(t, x(g(t)))|\, dt < \infty$$

provided $T > a$ is large enough.

To prove the «if» part, assume that (21) holds for some $k > 0$. Let $c > 0$ be such that $2c/|\lambda|(1-|\lambda|) \leqq k$. Take $T > a$ so that (14) holds and

$$\int_T^\infty P(t)\, f\left(t, \frac{2c}{|\lambda|(1-|\lambda|)}\right) dt \leqq \frac{(1-|\lambda|)c}{|\lambda|}, \tag{22}$$

and define $X \subset C[T_*, \infty)$ and $F: X \to C[T_*, \infty)$ by

$x = \{x \in C[T_*, \infty) : x(t)$ is nondecreasing and $c \leqq x(t) \leqq c/|\lambda|$ for $t \geqq T^*\}$ and (23)

$$Fx(t) = \frac{c}{|\lambda|} \int_t^\infty p(s) \int_s^\infty f(r, \hat{x}(g(r)))\, dr ds, \quad t \geqq T,$$
$$Fx(t) = \frac{c}{|\lambda|} - \int_T^\infty p(s) \int_s^\infty f(r, \hat{x}(g(r)))\, dr ds, \quad T_* \leqq t \leqq T, \tag{24}$$

where $\hat{x}(t)$ is given by

$$\hat{x}(t) = \sum_{j=0}^{n(t)-1} (-\lambda)^j x(\tau^j(t)) + \frac{(-\lambda)^{n(t)} x(T)}{1+\lambda}, \quad t > T,$$
$$\hat{x}(t) = \frac{x(T)}{1+\lambda}, \quad T_* \leqq t \leqq T, \tag{25}$$

$n(t)$ being the least positive integer such that $T_* < \tau^{n(t)}(t) \leqq T$. Exactly as in the proof of Theorem 1, one can show that if $x \in X$, then $\hat{x}(t) \geqq 0$ for $t \geqq T^*$, (19) holds and

$$\hat{x}(t) \leqq \frac{2c}{|\lambda|(1-|\lambda|)}, \quad t \geqq T^*. \tag{26}$$

From (22), (24) and (26) if follows that F maps X into itself. Since the continuity of F and the relative compactness of $F(X)$ are proved without difficulty, F has a fixed element x^* in $X: x^* = Fx^*$. The associated function $x^*(t)$ then gives a positive solution of (1) which is clearly of type III. If the constant k in (21) is negative, then a parallel argument assures the existence of a negative solution of type III of equation (1). This completes the proof of Theorem 2.

Unlike the solutions of types I and III, a characterization of type II solutions (1) seems to be difficult to establish. Here we have to content oneselves with the following theorem which covers a limited class of equations of the form (1).

Theorem 3. *Suppose that* $-1 < \lambda < 0$. *Equation* (1) *has a nonoscillatory solution of type* II *if*

$$\int^\infty |f(t, kP(g(t)))|\, dt < \infty \tag{27}$$

for some nonzero $k \neq 0$ *and*

$$\int^\infty P(t)|f(t, l)|\, dt = \infty \tag{28}$$

for all nonzero $l \neq 0$.

Proof. We may suppose that $k > 0$. Let $c > 0$ be such that $2c/(1-|\lambda|) < k$ and let $T > a$ be such that (14) holds and

$$\int_T^\infty f\left(t, \frac{2c}{1-|\lambda|}[P(g(t)) + 1]\right) dt \leqq c. \tag{29}$$

Define X to be the set of all $x \in C[T_*, \infty)$ which are nondecreasing and satisfy

$$c \leqq x(t) \leqq c[P(t) + 1], \ t \geqq T; \ x(t) = x(T), \ T_* \leqq t \leqq T, \tag{30}$$

and consider the map

$$Fx(t) = c + \int_T^t p(s) \int_s^\infty f(r, \hat{x}(g(r)))\, dr ds, \quad t \geqq T, \tag{31}$$
$$Fx(t) = c, \quad T_* \leqq t \leqq T,$$

where $\hat{x}(t)$ is defined by (25). Note that $x \in X$ implies

$$0 \leqq \hat{x}(t) \leqq \frac{2c}{1-|\lambda|}[P(t)+1], \quad t \geqq T^*, \tag{32}$$

the first inequality being a trivial consequence of the assumption $\lambda < 0$. Now it is easy to verify that $F(X) \subset X$, and this together with the continuity of F and the relative compactness of $F(X)$ guarantees the existence of a fixed point $x^* \in X$ of F, which satisfies the equation

$$\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t)) = c + \int_T^t p(s)\int_s^\infty f(r, \hat{x}^*(g(r)))\,drds, \quad t \geqq T. \tag{33}$$

Obviously, $x^*(t)$ is a solution of (1). We find from (33)

$$\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t)) \geqq c + \int_T^t p(s)\int_s^\infty f(r, c)\,drds \geqq c + \int_T^t \Big(\int_T^r p(s)\,ds\Big) f(r, c)\,dr \geqq$$

$$\geqq c + \int_T^t P(r, T) f(r, c)\,dr, \quad t \geqq T,$$

which, because of (28), implies $\lim\limits_{t\to\infty}[\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t))] = \infty$. From (30) we also have

$$\frac{1}{p(t)}\frac{d}{dt}[\hat{x}^*(t) + \lambda\hat{x}^*(\tau(t))] = \int_t^\infty f(r, \hat{x}^*(g(r)))\,dr,$$

implying that $\lim\limits_{t\to\infty}[\hat{x}^*(t) + \hat{\lambda}x^*(\tau(t))]/P(t) = 0$. This shows that the solution $\hat{x^*}(t)$ is of type II. Thus the proof is complete.

Example 1. Consider the equation

$$\frac{d^2}{dt^2}[x(t) + \lambda x(\tau(t))] + \varphi(t)|x(g(t))|^\gamma \operatorname{sgn} x(g(t)) = 0, \tag{34}$$

where λ, $\tau(t)$ and $g(t)$ are as before, $\gamma > 0$ is a constant and $\varphi(t) \geqq 0$, $\not\equiv 0$ is a continuous function on $[a, \infty)$. This equation is a special case of (1) in which $p(t) = 1$ and $f(t, y) = \varphi(t)|y|^\gamma \operatorname{sgn} y$.

There are three types of asymptotic behavior for nonoscillatory solutions $x(t)$ such that $x(t)[x(t) + \lambda x(\tau(t))] > 0$ for large t, namely:

$$\text{I)}\ 0 < \liminf_{t\to\infty}[|x(t)|/t] \leqq \limsup_{t\to\infty}[|x(t)|/t] < \infty; \tag{35}$$

$$\text{II)}\ \lim_{t\to\infty}[x(t)/t] = 0 \text{ and } \lim_{t\to\infty}|x(t)| = \infty;$$

$$\text{III)}\ 0 < \liminf_{t\to\infty}|x(t)| \leqq \limsup_{t\to\infty}|x(t)| < \infty.$$

Condition (13) and (21) for (34) become

$$\int^\infty [g(t)]^\gamma \varphi(t)\,dt < \infty \tag{36}$$

and

$$\int^\infty t\varphi(t)\,dt < \infty, \tag{37}$$

respectively. Condition (36) [resp. (37)], therefore, is necessary and sufficient for (34) to possess a nonoscillatory solution of type I [resp. type III]. On the other hand, (34) possesses a nonoscillatory solution of type II if $-1 < \lambda < 0$ and

$$\int^\infty [g(t)]^\gamma \varphi(t)\,dt < \infty \text{ and } \int^\infty t\varphi(t)\,dt = \infty. \tag{38}$$

The two integral conditions in (38) are consistent only when $[g(t)]^{\gamma}$ grows less rapidly than t as $t \to \infty$. For example, if $g(t) = t^{\theta}$, $\theta > 0$, then $\theta\gamma$ must be less than unity.

Example 2. Consider the equation

$$\frac{d^2}{dt^2}[x(t) + \mu x(\sigma(t))] + f(t, x(h(t))) = 0 \tag{39}$$

subject to the conditions:

i) μ is a constant with $\mu > 1$;

ii) $\sigma : [a, \infty) \to \mathbb{R}$ is continuously differentiable and strictly increasing, and $\sigma(t) > t$ for $t \geqq a$;

iii) $h : [a, \infty) \to \mathbb{R}$ is continuous and $\lim_{t\to\infty} h(t) = \infty$;

iv) $f : [a, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing in the second variable, and $yf(t, y) \geqq 0$, $\not\equiv 0$ for $y \neq 0$ and $t \geqq a$.

Let $\tau(s)$ denote the inverse function of $\sigma(t)$. The change of the independent variable $s = \sigma(t)$ (or $t = \tau(s)$) transforms (39) into

$$\frac{d}{ds}\left\{\sigma'(\tau(s))\frac{d}{ds}[x(s) + \mu^{-1}x(\tau(s))]\right\} + [\mu\sigma'(\tau(s))]^{-1}f(\tau(s), x(h(\tau(s)))) = 0, \tag{40}$$

which is exactly of the type of equations to which the above theory is applicable.

Applying Theorems 1 and 2 to (40) and then returning to the orginal variable t, we have the following results:

a) equation (39) has a nonoscillatory solution $x(t)$ such that

$$0 < \liminf_{t\to\infty} [|x(t)|/\tau(t)] \leqq \limsup_{t\to\infty} [|x(t)|/\tau(t)] < \infty$$

if and only if

$$\int^{\infty} |f(t, k\tau(h(t)))|\, dt < \infty \text{ for some } k \neq 0; \tag{41}$$

b) equation (39) has a nonoscillatory solution $x(t)$ such that

$$0 < \liminf_{t\to\infty} |x(t)| \leqq \limsup_{t\to\infty} |x(t)| < \infty$$

if and only if

$$\int t|f(t, k)|\, dt < \infty \text{ for some } k \neq 0. \tag{42}$$

3. Oscillation of all solutions. It is natural to ask if one can characterize the situation in which all solutions of equation (1) are oscillatory, or equivalently, there is no nonoscillatory solution of (1). Our purpose here is to give a partial answer to this question by restricting ourselves to the case where (1) is strongly superlinear or strongly sublinear in the sense defined below.

Definition. *Equation* (1) *is said to be strongly superlinear if there is a constant* $\alpha > 1$ *such that* $|y|^{-\alpha}|f(t, y)|$ *is nondecreasing in* $|y|$ *for each fixed* $t \geqq a$ *: equation* (1) *is said to be strongly sublinear if there is a constant* β, $0 < \beta < 1$, *such that* $|y|^{-\beta}|f(t, y)|$ *is nonincreasing in* $|y|$ *for each fixed* $t \geqq a$.

Theorem 4. *Suppose that either* (1) *is strongly superlinear and*

$$\int^{\infty} P(g_*(t))|f(t, k)|\, dt = \infty \text{ for all } k \neq 0, \tag{43}$$

of (1) *is strongly sublinear and*

$$\int^{\infty} |f(t, kP(g_*(t)))|\, dt = \infty \text{ for all } k \neq 0 \tag{44}$$

where $g_*(t) = \min\{t, g(t)\}$, $t \geqq a$.

Then, if $0 < \lambda < 1$, *every solution of* (1) *is oscillatory, while if* $-1 < \lambda < 0$, *every solution of* (1) *is either oscillatory or tends to zero as* $t \to \infty$.

Proof. Note that the conclusion of the theorem is equivalent to saying that, regardless of the sign of λ, the class $N+$ is empty for (1), that is, (1) has no nonoscillatory solution $x(t)$ such that $x(t)[x(t)+\lambda x(\tau(t))]>0$ for all large t.

Let $x \in N+$ be a solution of (1). We may suppose without loss of generality that $x(t)$ is eventurally positive. Put $y(t)=x(t)+\lambda x(\tau(t))$. From the observations in pt 2, A we see that there exist positive constants $t_0>a$, c_1 and c_2 such that $y(t)$ is positive and nondecreasing for $t \geqq t_0$,

$$c_1 \leqq y(t) \leqq c_2 P(t), \quad t \geqq t_0, \tag{45}$$

and

$$(1-|\lambda|)\, y(t) \leqq x(t), \quad t \geqq t_0. \tag{46}$$

Let $t_1>t_0$ be such that $\min\{\tau(t_1), \inf_{t \geqq t_1} g_*(t)\} \geqq t_0$. Form (1) and (46) it follows that $z(t)=(1-|\lambda|)\, y(t)$ satisfies

$$\frac{d}{dt}\left(\frac{1}{p(t)} \frac{dz(t)}{dt}\right)+(1-|\lambda|)\, f(t, z(g_*(t))) \leqq 0, \quad t \geqq t_1. \tag{47}$$

i) Let (1) be strongly superlinear. Since, by (45), $z(g_*(t)) \geqq (1-|\lambda|)c_1 = k_1$ for $t \geqq t_1$, the strongly superlinearity (with exponent $\alpha>1$) implies

$$[z(g_*(t))]^{-\alpha} f(t, z(g_*(t))) \geqq k_1^{-\alpha} f(t, k_1), \quad t \geqq t_1. \tag{48}$$

Using (48) in the inequality

$$\frac{1}{p(t)} \frac{dz(t)}{dt} \geqq (1-|\lambda|) \int_t^{\infty} f(s, z(g_*(s)))\, ds, \quad t \geqq t_1,$$

which follows readily form (47), we obtain

$$\frac{dz(t)}{dt} \geqq (1-|\lambda|)\, k_1^{-\alpha} p(t) \int_t^{\infty} [z(g_*(s))]^{\alpha} f(s, k_1)\, ds, \quad t \geqq t_1.$$

Let t_2 be such that $g_*(t) \geqq t_1$ for $t \geqq t_2$. Integrating the above inequality divided by $[z(t)]^{\alpha}$ over $[t_1, t_3]$, $t_3>t_2$, we obtain

$$\int_{t_1}^{t_3} [z(t)]^{-\alpha} \frac{dz(t)}{dt}\, dt \geqq (1-|\lambda|) k_1^{-\alpha} \int_{t_1}^{t_3} \frac{p(t)}{[z(t)]^{\alpha}} \int_t^{t_3} [z(g_*(s)))]^{\alpha} f(s, k_1)\, ds dt =$$

$$= (1-|\lambda|) k^{-\alpha} \int_{t_1}^{t_3} \int_{t_2}^{s} p(t) \left[\frac{z(g_*(s))}{z(t)}\right]^{\alpha} f(s, k_1)\, dt ds \geqq$$

$$\geqq (1-|\lambda|)\, k_1^{-\alpha} \int_{t_2}^{t_3} \int_{t_1}^{g_*(s)} p(t) \left[\frac{z(g_*(s))}{z(t)}\right]^{\alpha} f(s, k_1)\, dt ds \geqq$$

$$\geqq (1-|\lambda|)\, k_1^{-\alpha} \int_{t_2}^{t_3} P(g_*(s), t_1)\, f(s, k_1)\, ds,$$

where use was made of the fact that $z(g_*(s)) \geqq z(t)$ for $g_*(s) \geqq t$. Letting $t_3 \to \infty$ in the above, we have

$$\int_{t_2}^{\infty} P(g_*(s), t_1)\, f(s, k_1)\, ds \leqq \frac{k_1}{(1-|\lambda|)(\alpha-1)} [z(t_2)]^{1-\alpha} < \infty,$$

which contradicts (43).

ii) Let (1) be strongly sublinear. Since, by (45), $z(g_*(t)) \leqq k_2 P(g_*(t))$ for $t \geqq t_1$, where $k_2=(1-|\lambda|)\, c_2$, the sublinearity (with exponent $\beta<1$) implies

$$f(t, z(g_*(t))) \geqq \left[\frac{z(g_*(t))}{k_2 P(g_*(t))}\right]^{\beta} f(t, k_2 P(g_*(t))), \quad t \geqq t_1. \tag{49}$$

On the other hand, using the nonincreasing property of $(1/p(t))(dz(t)/dt)$, we have

$$z(t) \geqq \int_{t_1}^{t} p(s)\frac{1}{p(s)}\frac{dz(s)}{ds}ds \geqq \frac{P(t, t_1)}{p(t)}\frac{dz(t)}{dt}, \quad t \geqq t_1$$

and hence

$$\frac{1}{p(t)}\frac{dz(t)}{dt} \leqq \frac{z(g_*(t))}{P(g_*(t), t_1)}, \quad t \geqq t_2, \tag{50}$$

where $t_2 > t_1$ is chosen so that $g_*(t) > t_1$ for $t \geqq t_2$. From (47), (49) and (50) it follows that

$$\frac{d}{dt}\left\{-\left(\frac{1}{p(t)}\frac{dz(t)}{dt}\right)^{1-\beta}\right\} \geqq (1-|\lambda|)(1-\beta)\left(\frac{1}{p(t)}\frac{dz(t)}{dt}\right)^{-\beta} f(t, z(g_*(t)) \geqq$$

$$\geqq (1-|\lambda|)(1-\beta)k_2^{-\beta}\left(\frac{P(g_*(t), t_1)}{P(g_*(t))}\right)^{\beta} f(t, k_2 P(g_*(t))) \tag{51}$$

for $t \geqq t_2$. Integrating (51) over $[t_2, \infty)$ and noting that $P(g_*(t), t_1)/P(g_*(t)) \to 1$ as $t \to \infty$, we see that

$$\int_{t_2}^{\infty} f(s, k_2 q(g_*(s)) P(g_*(s))) ds < \infty,$$

which contradicts (44). This completes the proof.

Let us consider the strongly superlinear equation (1) and compare (43) with the condition

$$\int^{\infty} P(t)|f,(t, k)| dt = \infty \text{ for all } k \neq 0 \tag{52}$$

which, by Theorem 2, is necessary for all solutions of (1) to be oscillatory. These conditions are in general different, but they may become equivalent, in which case (43) or (52) gives a necessary and sufficient condition for the oscillation of all solutions of (1) in the sense of the conclusion of Theorem 4. Thus we have the following result.

Theorem 5. *Let* (1) *be strongly superlinear. Suppose that*

$$\limsup_{t\to\infty} \frac{P(t)}{P(g_*(t))} < \infty. \tag{53}$$

Then, (52) *is a necessary and sufficient condition in order that, for* $0 < \lambda < 1$, *every solution of* (1) *be oscillatory, and, for* $-1 < \lambda < 0$, *every solution of* (1) *be either oscillatory or tend to zero as* $t \to \infty$.

Likewise, by comparing (44) with the condition

$$\int^{\infty} |f(t, kP(g(t)))| dt = \infty \text{ for all } k \neq 0, \tag{54}$$

we are able to obtain a characterization of oscillation of all solutions for the strongly sublinear equation (1).

Theorem 6. *Let* (1) *be strongly sublinear. Suppose that*

$$\limsup_{t\to\infty} \frac{P(g(t))}{P(g_*(t))} < \infty. \tag{55}$$

Then, (54) *is a necessary and sufficient condition in order that, for* $0 < \lambda < 1$, *every solution of* (1) *be oscillatory, and, for* $-1 < \lambda < 0$, *every solution be either oscillatory or tend to zero as* $t \to \infty$.

Example 3. We take up equation (34) again:

$$\frac{d^2}{dt^2}[x(t) + \lambda x(\tau(t))] + \varphi(t)|x(g(t))|^{\gamma} \operatorname{sgn} x(g(t)) = 0, \tag{56}$$

where λ, $\tau(t)$ and $g(t)$ are as in (1), $\gamma > 0$ is a constant and $\varphi(t) \geqq 0$, $\not\equiv 0$ is a continuous function on $[a, \infty)$.

Conditions (52) and (54) for this equation take the forms

$$\int^{\infty} t\varphi(t)\,dt = \infty \tag{57}$$

and

$$\int^{\infty} [g(t)]^{\gamma}\varphi(t)\,dt = \infty, \tag{58}$$

respectively, and conditions (53) and (55) reduce to

$$\limsup_{t\to\infty} [t/g_*(t)] < \infty \tag{59}$$

and

$$\limsup_{t\to\infty} [g(t)/g_*(t)] < \infty, \tag{60}$$

respectively. Therefore, we have the following results form Theorems 5 and 6.

i) Let $\gamma > 1$ and suppose that (59) holds. Then, the oscillation of solutions of (56) (in the sense of the theorems) takes place if and only if (57) is satisfied.

ii) Let $0 < \gamma < 1$ and suppose that (60) holds. Then, the oscillation of solutions of (56) takes place if and only if (58) is satisfied.

We observe that both (59) and (60) hold if $g(t)$ satisfies

$$0 < \liminf_{t\to\infty} [g(t)/t] \leqq \limsup_{t\to\infty} [g(t)/t] < \infty,$$

in which case (58) is equivalent to the condition

$$\int^{\infty} t^{\gamma}\varphi(t)\,dt = \infty.$$

Example 4. Consider the following particular case of equation (56):

$$\frac{d^2}{dt^2}[x(t) - \lambda x(t-\tau)] + \varphi(t)|x(t-\sigma)|^{\gamma}\operatorname{sgn} x(t-\sigma) = 0, \tag{61}$$

where $\varphi(t) = (\lambda e^{\tau} - 1)\, e^{-\gamma\sigma} e^{(\gamma-1)t}$. We assume that λ, τ, σ and γ are constants such that $0 < \lambda < 1$, $\tau > 0$, $\gamma > 1$ and $\lambda e^{\tau} > 1$. Then, the function $\varphi(t)$ satisfies (57), so that, by Theorem 4, every solution of (61) is either oscillatory or tends to zero as $t \to \infty$. As is easily checked, (61) has a nonoscillatory solution $x(t) = e^{-t}$ tending to zero as $t \to \infty$. This example shows that, in case the constant λ is negative, condition (44) is not sufficient to preclude the possibility $N^- \neq \varnothing$ for the superlinear equation (1). An interesting problem is to obtain sharp conditions garanteeing the oscillation of all solutions (i. e. $N^+ = N^- = \varnothing$) for both superlinear and sublinear cases of (1) with $\lambda < 0$.

Example 5. Consider the equation (39) again:

$$\frac{d^2}{dt^2}[x(t) + \mu x(\sigma(t))] + f(t, x(h(t))) = 0 \tag{62}$$

under the same assumptions as in Example 2; in particular, $\mu > 1$ and $\sigma(t) > t$. As is known, (62) is equivalent to the neutral equation

$$\frac{d}{ds}\left\{\sigma'(\tau)(s))\frac{d}{ds}[x(s) + \mu^{-1}x(\tau(s))]\right\} + [\mu\sigma'(\tau(s))]^{-1} f(\tau(s), x(h(\tau(s)))) = 0, \tag{63}$$

where $\tau(s)$ is the inverse function of $\sigma(t)$.

According to Theorem 4, the oscillation of the strongly superlinear equation (63) is guaranteed by the condition

$$\int^{\infty} \tau((h \circ \tau)_*(s))|f(\tau(s), k)|\,ds = \infty \ \text{ for all } \ k \neq 0,$$

and that of the strongly sublinear equation is guaranteed by the condition

$$\int^{\infty} |f(\tau(s),\ k\tau((h\circ\tau)_*(s)))|\, ds = \infty \text{ for all } \quad k\neq 0.$$

Since $(h\circ\tau)_*(s) \geqq (h_*\circ\tau)(s)$, the above conditions are implied, respectively, by

$$\int^{\infty} \tau((h_*\circ\tau)(s))|f(\tau(s),\ k)|\, ds = \infty \text{ for all } \quad k\neq 0$$

and

$$\int^{\infty} |f(\tau(s),\ k\tau(h_*\circ\tau)(s)))|\, dt = \infty \text{ for all } \quad k\neq 0,$$

which can be rewritten as

$$\int^{\infty} \tau(h_*(t))|f(t,\ k)|\, dt = \infty \text{ for all } \quad k\neq 0 \tag{64}$$

and

$$\int^{\infty} |f(t,\ k\tau(h_*(t)))|\, dt = \infty \text{ for all } \quad k\neq 0. \tag{65}$$

Comparing (64) and (65) with the conditions

$$\int^{\infty} t|f(t,\ k)|\, dt = \infty \text{ for all } \quad k\neq 0 \tag{66}$$

and

$$\int^{\infty} |f(t,\ k\tau(h(t)))|\, dt = \infty \text{ for all } \quad k\neq 0 \tag{67}$$

which are necessary conditions for oscillation of (62) (cf. Example 2), we conclude that i) under the hypothesis

$$0 < \liminf_{t\to\infty} [\tau(h_*(t))/t] \leqq \limsup_{t\to\infty} [\tau(h_*(t))/t] < \infty \tag{68}$$

all solutions of the strongly superlinear equation (62) are oscillatory if and only if (66) holds, and ii) under the hypothesis

$$\limsup_{t\to\infty} [\tau(h(t))/\tau(h_*(t))] < \infty$$

all solutions of the strongly sublinear equation (62) are oscillatory if and only if (67) holds.

Note that (67) holds if $h(t) = h_*(t)$, that is, $h(t)$ is a retarded argument. Both (68) and (67) are satisfied if, for example,

$$\limsup_{t\to\infty} [\sigma(t)/t] < \infty$$

and

$$0 < \liminf_{t\to\infty} [h(t)/t] \leqq \limsup_{t\to\infty} [h(t)/t] < \infty.$$

R e m a r k s. i) We note that existence theorems analogous to the «if» parts of Theorems 1 and 3 can be established even for the case where the nonlinear term $f(t, y)$ in (1) does not satisfy the sign condition $yf(t, y) \geqq 0$. In fact, under the supposition that there exists a continuous function $f^*: [a, \infty) \times \times [0, \infty) \to [0)$, which is nondecreasing in the second variable and such that

$$|f(t,\ y)| \leqslant f^*(t,\ |y|) \text{ for } (t,\ y) \in [a,\ \infty) \times \mathbb{R},$$

it can be proved without difficulty that (1) possesses a nonoscillatory solution $x(t)$ satisfying (12), I) or (12), III) if

$$\int^{\infty} f^*(t,\ kP(g(t)))dt < \infty \text{ for some } k > 0$$

or

$$\int^{\infty} P(t) f^*(t, k)\, dt < \infty \text{ for some } k > 0.$$

ii) The results of this paper can be extended to equations of the form

$$L[x(t) + \lambda x(\tau(t))] + f(t, x(g_1(t)), \ldots, x(g_N(t))) = 0, \qquad (69)$$

where $f(t, y_1, \ldots, y_N)$ is continuous on $[a, \infty) \times \mathbb{R}^N$ and nondecreasing in each y_i, $1 \leqq i \leqq N$, and satisfies

$$y_1 f(t, y_1, \ldots, y_N) \geqq 0 \text{ for all } (t, y_1, \ldots, y_N) \text{ with } y_1 y_i \geqq 0, \ 1 \leqslant i \leqslant N.$$

There is no difficulty in formulating and proving analogues of Theorems 1 — 3 for equation (69). To obtain oscillation criteria for (69) it suffices to define strong superlinearity and sublinearity in terms of $f(t, y, \ldots, y)$ and use the function $g_*(t) = \min\{t, g_1(t), \ldots, g_N(t)\}$.

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