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Towards the rank-one singular perturbations theory of self-adjoint operators

К сингулярной теории возмущений ранга один самосопряженных операторов

The perturbation theory is developed in the case when an arbitrary positive self-adjoint operator is perturbed by the projector on a generalized vector. Similar to the well-known problem $-\Delta+\lambda\delta$ we obtain in general situation explicit representations for singularly perturbed operators their resolvents find the point spectrum and an explicit form of the corresponding eigenvectors. Our approach differs from usual ones and based on the self-adjoint extensions theory of semibounded operators.

Развита теория возмущений в случае, когда произвольный положительный самосопряженный оператор возмущен проектором на обобщенный вектор. Аналогично известной задаче $-\Delta+\lambda\delta$ получено явное представление для сингулярно возмущенного оператора и его резольвенты в общей ситуации, найден точечный спектр и явный вид соответствующего собственного вектора. Наш подход отличителен тем, что основан на теории самосопряженных расширений полуограниченных опеpатоов.

Розвинуто теорію збурень у випадку, коли довільний додатній самоспряжений оператор є збуреним проектором на узагальнений вектор. Подібно до відомої задачі $-\Delta + \lambda\delta$ одержано явне представлення збуреного оператора та його резольвенти у загальній ситуації, знайдено точковий спектр та явний вигляд відповідного власного вектора. Наш підхід відрізняється з поміж інших тим що грунтується на теорії самоспряжених розширень напівобмежених операторів.

I n t r o d u c t i o n. The perturbation T of the self-adjoint operator A in the Hilbert space $\mathscr{H}$ is called singular if the set Ker T belongs to the domain $\mathscr{D}(A)$ and is dense in $\mathscr{H}$. A large number of pertrubations of the Laplace operator Δ is given in $L^2(\mathbb{R}^3)$ by the potentials with support with Lebesgue measure zero. The tipical rank-one singular perturbations of $-\Delta$ are given by the Dirac δ-function. A lot of physically interesting models with the singular perturbations have been considered in the fundamental monograph [1].

In this work we attempt to develope a general approach to the singular perturbation problem. Our results are based on the abstract theory of self-adjoint extensions of semibounded symmetric operators [2] and they are analogous to the ones from [1]. We show that for arbitrary rank-one singular perturbations all usual problems may be solved in an explicit form. Namely we give the unique construction for the singularly perturbed operator, describe its domain, obtain the explicit formula for resolvent and finally find the point spectrum and corresponding eigenvector.

T h e r a n k-o n e s i n g u l a r p e r t u r b a t i o n s. Let $A = A^* \geqslant m > 0$ be an unbounded self-adjoint operator in the Hilbert space $\mathscr{H}$. Introduce the rigged Hilbert space

$$\mathscr{H}_- \supset \mathscr{H}_0 \equiv \mathscr{H} \supset \mathscr{H}_+,$$

where $\mathscr{H}_+$ coincides with the domain $\mathscr{D}(A)$ in the norm $\|\varphi\|_+ := \|A\varphi\|$, $\varphi \in \mathscr{D}(A)$, and $\mathscr{H}_-$ is the completion of $\mathscr{H}$ in the norm $\|f\|_- := \|A^{-1}f\|$, $f \in \mathscr{H}$. The duality between $\mathscr{H}_-$ and $\mathscr{H}_+$ we denote as $\langle\omega, \varphi\rangle$ or $\langle\varphi, \omega\rangle$, $\varphi \in \mathscr{H}_+$, $\omega \in \mathscr{H}_-$.

D e f i n i t i o n 1. *A symmetric lower semibounded closed quadratic form γ in $\mathscr{H}_+$ is called a singular perturbation for A if the set*

$$\Phi_0 := \operatorname{Ker} \gamma$$

is dense in $\mathscr{H}$. The singular perturbation γ has rank equal n $(n = 1, 2, \ldots, \infty)$ if

$$\dim \mathcal{N}_0 = n$$

where

$$\mathcal{N}_0 := \mathcal{M}_0^\perp \text{ in } \mathscr{H}, \quad \mathcal{M}_0 = A\Phi_0.$$

The set of all such perturbations we denote by $\mathscr{T}_s^n(A)$.

Denote the mapping $A : \mathscr{H}_+ \to \mathscr{H}$ by $I_{0,+}$, and $I_{+,0} = I_{0,+}^{-1} = A^{-1}$. Denote by $I_{-,0}$ the continuous mapping from $\mathscr{H}$ to $\mathscr{H}_-$ which equals A on $D(A)$, and $I_{0,-} = I_{-,0}^{-1}$. At last set $I_{-,+} := I_{-,0}I_{0,+}$, $I_{+,-} := I_{-,+}^{-1}$. Note (see [3]) that

$$\langle\omega, \varphi\rangle = (I_{+,-}\,\omega, \varphi)_+ = (\omega, I_{-,+}\varphi)_-, \quad \omega \in \mathscr{H}_-, \ \varphi \in \mathscr{H}_+,$$
$$\langle f, \varphi\rangle = (f, \varphi), \ f \in \mathscr{H}, \ \varphi \in \mathscr{H}_+. \tag{1}$$

With each singular perturbation $\gamma \in \mathscr{T}_s^n(A)$ in $\mathscr{H}_+$ is associated the self-adjoint operator V_γ:

$$\gamma(\varphi, \psi) = (V_\gamma\varphi, \psi)_+, \quad \varphi, \psi \in \mathscr{D}(V_\gamma) \subseteq Q(\gamma) \subseteq \mathscr{H}_+.$$

Using V_γ we can introduce the operator

$$T_\gamma := I_{-,+}V_\gamma : \mathscr{H}_+ \to \mathscr{H}_-.$$

It is obvious that

$$\Phi_0 := \operatorname{Ker} \gamma = \operatorname{Ker} V_\gamma = \operatorname{Ker} T_\gamma.$$

We can write

$$\mathcal{H}_+ = \Phi_0 \oplus \mathcal{N}_{0,+}, \ \mathcal{N}_{0,+} := I_{+,0}\mathcal{N}_0.$$

So

$$\operatorname{rank} \gamma = \operatorname{rank} V_\gamma = \operatorname{rank} T_\gamma := \dim \mathcal{N}_0 \equiv \dim \mathcal{N}_{0,+}.$$

Now we can give another equivalent definition of singular perturbation.

D e f i n i t i o n 1′. *A linear lower bounded operator $T : \mathcal{H}_+ \to \mathcal{H}_-$ which is closed and Hermitian ($\langle T\varphi, \psi \rangle = \langle \varphi, T\psi \rangle$) is called a singular perturbation for A if the set*

$$\Phi_0 := \operatorname{Ker} T$$

is dense in $\mathcal{H}$. rank $T := \dim R(T) \equiv \dim N_{0,-}$, *where* $N_{0,-} := I_{-,0}N_0$. *If* rank $T = n$ *we write* $T \in T_s^n(A)$.

Between the singular perturbations for A given by quadratic forms γ and the one given by linear operators T there exists a one-to-one correspondence which is fixed by the formula

$$\gamma(\varphi, \psi) = \langle I_{-,+}V_\gamma\varphi, \psi\rangle := \langle T\varphi, \psi\rangle, \ \varphi, \psi \in \mathcal{D}(T) \subseteq Q(\gamma).$$

The following assertion is obvious.

P r o p o s i t i o n 1. *Each rank-one singular perturbation for A is uniquely determined by a vector $\omega \in \mathcal{H}_- \setminus \mathcal{H}$, $\| \omega \|_- = 1$ and by a number $\lambda \in \mathbb{R}^1$, $\lambda \neq 0$:*

$$\gamma(\varphi, \psi) \equiv \gamma_{\lambda,\omega}(\varphi, \psi) = \lambda\langle \varphi, \omega\rangle (\omega, \psi), \ \varphi, \psi \in Q(\gamma),$$

$$T_\gamma\varphi \equiv T_{\lambda,\omega}\varphi = \lambda \langle \varphi, \omega\rangle \omega, \ \varphi \in \mathcal{D}(T) \subseteq Q(\gamma). \tag{2}$$

Let now $\gamma_{\lambda,\omega}$ or $T_{\lambda,\omega}$ be given. The singular perturbed operator formally is defined as

$$A_{\lambda,\omega} = A + T_{\lambda,\omega}.$$

The first problem of the singular perturbation theory is to give the precise definition, of $A_{\lambda,\omega}$ at the self-adjoint operator in $\mathcal{H}$. We shall solve this problem in the particular case which is described below.

Denote $\mathcal{H}_{+2} := \mathcal{H}_+$, and $\mathcal{H}_{+1}$ is the completion of $\mathcal{D}(A)$ in the norm $\| \varphi \|_{+1} := \| A^{1/2}\varphi \|$; $\mathcal{H}_{-2} := \mathcal{H}_-$ and $\mathcal{H}_{-1}$ is the completion of $\mathcal{H}$ in the norm $\| f \|_{-1} := \| A^{-1/2} f \|$. So we have the scale of Hilbert spaces

$$\mathcal{H}_{-2} \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \equiv \mathcal{H} \supset \mathcal{H}_{+1} \supset \mathcal{H}_{+2}.$$

The canonical unitary isomorphism between $\mathcal{H}_j$ and $\mathcal{H}_k$ ($j, k = 0, \pm 1, \pm 2$) we denote by $I_{j,k}$. We write $\gamma_{\lambda,\omega}$ (or $T_{\lambda,\omega}$) $\in \mathcal{T}_{s_1}^1$ if $\gamma_{\lambda,\omega}(T_{\lambda,\omega}) \in \mathcal{T}_s^1$ and the set Φ_0 is dense in $\mathcal{H}_{+1}$.

P r o p o s i t i o n 2. *The rank-one singular perturbation $\gamma_{\lambda,\omega}$ respectively $T_{\lambda,\omega}$ belongs to class $\mathcal{T}_{s_1}^1(A)$ iff*

$$\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}.$$

P r o o f. See Theorem 4.

T h e p a r a m e t r i z a t i o n o f t h e s i n g u l a r l y p e r t u r b e d o p e r a t o r s. To the notion of the singular perturbation one can come from another side. If for a pair of self-adjoint operators A and $\tilde{A}$ in $\mathcal{H}$ there exists a dense set $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(\tilde{A})$ on which these operators coincide then the difference $\tilde{A} - A$ makes only sense as an unclosable quadratic form in $\mathcal{H}$. In such case it is naturally to sonsider both A and $\tilde{A}$ as the singularly perturbed operators with respect to one another. The corresponding perturbation value is equal to zero on the dense set $\mathcal{D}$ in $\mathcal{H}$.

D e f i n i t i o n 2. *The self-adjoint operator $\tilde{A}$ in $\mathcal{H}$ is called singularly perturbed with respect to A if the set*

$$\mathcal{D} := \{f \in \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A}) \mid Af = \widetilde{A}f\} \tag{3}$$

it dense in $\mathcal{H}$. The set of all such operators we denote as $\mathcal{A}_s(A)$.

Let $A \in \mathcal{A}_s(A)$ be fixed. Introduce for the pair A, $\tilde{A}$ their common symmetric operator

$$\mathring{A} := A \mid \mathcal{D} \equiv \widetilde{A} \mid \mathcal{D}.$$

Obviously $\mathring{A}$ is a closed operator. We write $\widetilde{A} \in \mathcal{A}_s^n(A)$ if the deficiency indexes of $\mathring{A}$

$$n \pm (\mathring{A}) = n, \; n = 1, 2, \ldots, \infty.$$

If in addition $\widetilde{A}$ is boundary invertible and the Friedrichs extension of $\mathring{A}$ coincides with A, i. e.

$$R(\widetilde{A}) = \mathcal{H} = R(A), \; (\mathring{A})_F \equiv A_\infty = A$$

then we write $\tilde{A} \in \mathcal{A}_{s_1}^n(A)$.

The following theorem contains the main result of the work.

Theorem 1. *Between sets $\mathcal{T}_{s_2}^1(A)$ and $A_{s_1}^1(A)$ there exists a bijective correspondence:*

$$\mathcal{T}_{s_1}^1(A) \ni T_{\lambda,\omega} \Leftrightarrow A_{\lambda,\omega} \equiv \widetilde{A} \in \mathcal{A}_{s_1}^1(A).$$

The correspondence can be explicitely described as follows. For each singular perturbation $T_{\lambda,\omega} \in \mathcal{T}_{s_1}^1(A)$ ($\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$, $\|\omega\|_{-2} = 1$, $\lambda \in \mathbb{R}^1$, $\lambda \neq 0$ the singularly perturbed operator $A_{\lambda,\omega}$ is defined as

$$A_{\lambda,\omega}\, g = Af, \; g \in \mathcal{D}(A_{\lambda,\omega}), \tag{4}$$

$$\mathcal{D}(A_{\lambda,\omega}) \equiv \mathcal{D}_{\lambda,\omega} := \{g \in \mathcal{H} \mid g = f + \lambda^{-1} \langle f, \omega \rangle \eta, \; f \in \mathcal{D}(A)\} \tag{5}$$

where $\eta = I_{0,-2}\omega$. From (4), (5) it follows that $A_{\lambda,\omega} \in \mathcal{A}_{s_1}^1(A)$. Conversely for each $\widetilde{A} \in \mathcal{A}_{s_1}^1(A)$ the corresponding singular perturbation $T_{\lambda,\omega}$ is defined by the vector $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ and by the number $\lambda \in \mathbb{R}^1$ which are given as follows

$$\omega = I_{-2,0}\eta, \; \|\eta\| = 1, \; \eta \in N_0 := M_0^\perp, \; M_0 = \mathring{A}\mathcal{D}, \tag{6}$$

$$\lambda^{-1} := ((\widetilde{A}^{-1} - A^{-1})\eta, \eta) \tag{7}$$

(6) *implies that $T_{\lambda,\omega} \in \mathcal{T}_{s_1}^1(A)$. Moreover the operator $A_{\lambda,\omega}$ which is constructed starting from* (4), (5) *coincides with $\widetilde{A}$.*

Proof. Let $T_{\lambda,\omega} \in \mathcal{T}_{s_1}^1$, $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}$, $\lambda \in \mathbb{R}^1$, $\lambda \neq 0$ be given. First we show that the set $\mathcal{D}_{\lambda,\omega}$ is dense in $\mathcal{H}$. Actually even its subset

$$\mathcal{D} := \{f \in \mathcal{D}(A) \mid Af \perp \eta\} \tag{3'}$$

is dense in $\mathcal{H}$. Note that $Af \perp \eta$ is equivalent to $\langle f, \omega \rangle = 0$ as $\eta = I_{0,-2}\omega$. Let now $\psi \in \mathcal{H}$ be a vector such that $(f, \psi) = 0$ for all $f \in \mathcal{D}$. Then $(Af, A^{-1}\psi) = 0$ also. It means that $A^{-1}\psi = c\eta$, $c \in \mathbb{C}$. If $c \neq 0$ then the vector η must belong to the domain $\mathcal{D}(A)$ and $A\eta \in \mathcal{H}$. But it is impossible because on $\mathcal{D}(A)$ the operators A and $I_{-2,0}$ coincide and therefore ω must be in $\mathcal{H}$. Hence $\psi \equiv 0$. So $\mathcal{D}$ and also $\mathcal{D}_{\lambda,\omega}$ are dense in $\mathcal{H}$.

Now we verify that the operator $A_{\lambda,\omega}$ which is defined on $\mathcal{D}_{\lambda,\omega}$ by (4) is symmetric:

$$(A_{\lambda,\omega}g_1, g_2) = (Af_1, g_2) = (Af_1, f_2) + \lambda^{-1}\,\overline{(Af_2, \eta)}\,(Af_1, \eta) =$$

$$= (f_1, Af_2) + \lambda^{-1}((Af_1, \eta)\eta, Af_2) = (g_1, Af_2) =$$

$$= (g_1, A_{\lambda,\omega}g_2), \; g_1, g_2 \in \mathcal{D}_{\lambda,\omega}$$

since $A \geqslant m > 0$ the range $R(A) = \mathcal{H}$. (4) implies that $R(A_{\lambda,\omega}) = \mathcal{H}$ also. It means that $A_{\lambda,\omega}$ is a self-adjoint operator since it is Hermitian and its range equals the whole space. Further from (5) it is obvious that on $\mathcal{D}$ the operators A and $A_{\lambda,\omega}$ coincide. It means that $A_{\lambda,\omega} \in \mathcal{A}_s(A)$. Introduce now the operator $\mathring{A}_0 = A \mid \mathcal{D} \equiv A_{\lambda,\omega} \mid \mathcal{D}$. From (3) it follows that $\dim N_0 = 1$ where $N_0 = M_0^{\perp}$, $M_0 := \mathring{A}\mathcal{D}$. So $A_{\lambda,\omega} \in \mathcal{A}_s^1(A)$. By the construction $A_{\lambda,\omega}^{-1}$ exists and is bounded. Finally the coincidence of the Friedrichs extension $(\mathring{A}_0)_F$ with A follows from condition $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$ (see Theorem 4). Hence we have proved that $A_{\lambda,\omega} \in \mathcal{A}_{s_1}^1(A)$. Conversely for given $A \in A_{s_1}^1(A)$ we introduce the Hermitian operator $A_0 = \widetilde{A} \mid \mathcal{D}$ where $\mathcal{D}$ is defined by (3). Further we consider the subspace $N_0 = \operatorname{Ker}(A_0)^* \equiv (A_0\mathcal{D})^{\perp}$. As an starting assumption $n \pm (A_0) = 1$, i. e. $\dim N_0 = 1$. Let $\eta \in N_0$ be a unit vector. Then we set ω equal $I_{-2,0}\,\eta$. We show that $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}$. Indeed if we assume that $\omega \in \mathcal{H}$ then

$$\langle \varphi, \omega \rangle = (\varphi, \omega) = 0, \ \varphi \in \mathcal{D}$$

since $\langle \varphi, \omega \rangle = \langle \varphi, I_{-2,0}\eta \rangle = (A\varphi, \eta)$ and $A\varphi \perp \eta$ for all $\varphi \in \mathcal{D}$. But $\mathcal{D}$ is dense in $\mathcal{H}$ and the equality $(\varphi, \omega) = 0$ implies that $\omega = 0$. That is a contradiction. Therefore $\omega \notin \mathcal{H}$. Moreover $\omega \notin \mathcal{H}_{+1}$. Indeed if $\omega \in \mathcal{H}_{+1}$ then $\mathcal{D}$ is not dense in $\mathcal{H}_+$ (see Theorem 4). In such case $A_\infty \neq A$. That also is a contradiction to the starting conditions. Now by the vector $\omega = I_{-2,0}\,\eta$ and by the number $\lambda \in \mathbb{R}^1$, $\lambda \neq 0$ we can define the operator $A_{\lambda,\omega}$ according to (4) and (5). It is obvious that all three operators $A_{\lambda,\omega}$, $\widetilde{A}$ and A coincide on $\mathcal{D}$. We have to show that choosing the number λ acording to (7) we ensure the equality $A_{\lambda,\omega} = \widetilde{A}$. For this aim consider in $\mathcal{H}$ a pair of vectors $v_\eta = \widetilde{A}^{-1}\eta$ and $u_\eta A^{-1}\eta$. It turns out that $v_\eta - u_\eta \in N_0$. In fact we have

$$(v_\eta - u_\eta,\ M_0) = (\widetilde{A}^{-1}\eta - A^{-1}\eta, \quad M_0) = (\eta,\ (\widetilde{A}^{-1} - A^{-1})\,M_0)$$

where $M_0 = \widetilde{A}\mathcal{D} \equiv A\mathcal{D}$. Hence $v_\eta - u_\eta = c\eta$ where $c = \lambda^{-1}$ with λ^{-1} from (7). Using $\|\eta\| = 1$ we can write

$$v_\eta = u_\eta + \lambda^{-1}(Au_\eta,\ \eta)\,\eta$$

and now it is obious that $\eta = \widetilde{A}v_\eta = Au_\eta = A_{\lambda,\omega}v_\eta$. So the equality $\widetilde{A} = A_{\lambda,\omega}$ is established on the two subsets $\mathcal{D}$ and $\widetilde{N}_1 := \widetilde{A}^{-1}N_0$. By linearity it extends on the whole domain $\mathcal{D}(\widetilde{A}) = \mathcal{D}(A_{\lambda,\omega})$.

Finally we can define the singular perturbation $T_{\lambda,\omega}$ corresponding to $\widetilde{A}$ according to (2). Clearly the set $\Phi_0 := \operatorname{Ker} T_{\lambda,\omega} \equiv \mathcal{D}$ is dense in $\mathcal{H}_{+1}$. Therefore $T_{\lambda,\omega} \in \mathcal{T}_{s_1}^1(A)$. The theorem is proved.

T h e e x p l i c i t f o r m f o r t h e r e s o l v e n t o f $A_{\lambda,\omega}$. From (5) it follows that for each $g \in \mathcal{D}(A_{\lambda,\omega})$ the following representation is valid

$$g = f + \lambda^{-1}P_\eta Af, \quad f \in \mathcal{D}(A) \tag{5'}$$

where P_η is the projector in $\mathcal{H}$ on the vector $\eta \in \mathcal{N}_0$. Denote $h := A_{\lambda,\omega}\,g = Af$. Then from (5') we have

$$g = A_{\lambda,\omega}^{-1}h = A^{-1}h + \lambda^{-1}P_\eta h,$$

i. e.

$$A_{\lambda,\omega}^{-1} = A^{-1} + \lambda^{-1}P_\eta. \tag{8}$$

Now we show that an analogous formula is true for the resolvent of $A_{\lambda,\omega}$.

Denote

$$R_z := (A - z)^{-1}, \quad R_z^{\omega} := (A_{\lambda,\omega} - z)^{-1}$$

where $z \in p := p(A) \cap p(A_{\lambda,\omega})$ (p is the common set of the regular points for A and $A_{\lambda,\omega}$). Let A_0 be the common symmetric operator for A and $A_{\lambda,\omega}$, $\mathcal{D}(A_0) := \mathcal{D}$. Then for the deficiency subspace

$$N_z := ((A_0 - z)\mathcal{D})^{\perp}$$

there is the representation

$$N_z = \{c(I_{-2,0} - z)^{-1}\omega\}_{c \in \mathbb{C}}. \tag{9}$$

In fact using (1) we have

$$((A_0 - z)D, N_z) = 0 = (A_0\mathcal{D}, N_z) - z(\mathcal{D}, N_z) = \langle \mathcal{D}, I_{-2,0}N_z\rangle - z\langle \mathcal{D}, N_z\rangle.$$

Hence

$$I_{-2,0}N_z - zN_z = \{c\omega\}_{c \in \mathbb{C}}$$

and (9) is proved.

We know [2] thet the difference

$$R_z^{\omega} - R_z := B_z : N_z \to N_z.$$

Since it follows from (9) that dim $N_{\xi} = 1$, $\xi \in \rho$ the operator B_z may be written as

$$B_z\varphi = \lambda_z^{-1}(\varphi, \eta_z)\eta\bar{z}, \tag{10}$$

where $\eta_z\,(\eta_{\bar{z}})$is a unit vector in $N_z\,(N_{\bar{z}})$. Note that $\eta \equiv \eta_0$.

T h e o r e m 2. *If* (8) *is valid for the operator* $A_{\lambda,\omega}$ *defined by* (4), (5), *then for its resolvent the representation true*:

$$R_z^{\omega} = R_z + \lambda^{-1}|\eta_z><\eta_{\bar{z}}| \tag{11}$$

where number

$$\lambda_z^{-1} = (\lambda - za_{0,\bar{z}})^{-1}, \quad a_{0,\bar{z}} := (\eta, \eta_{\bar{z}}) \tag{12}$$

and a dyadic operator $|\psi><\psi|$ *acts as follows* : $\varphi \to (\varphi, \psi)\psi$.

P r o o f. The validity of formula (11) follows from he general theory of self-adjoint extentions of symmetric operators. We have to prove (12). For this aim we use the Hilbert resolvent identity:

$$R_{\mu} - R_{\nu} = (\mu - \nu)R_{\mu}R_{\nu}.$$

We write

$$R_z^{\omega} - R^{\omega} = zR_z^{\omega}R^{\omega} \quad (R^{\omega} \equiv R_{z=0}^{\omega} \equiv A_{\lambda,\omega}^{-1})$$

and substitute instead R^{ω} and R_z^{ω} by their representations given by (8) and (11). After some simplification we have

$$\lambda_z^{-1}B_z((A - z)R - z\lambda^{-1}P_{\eta}) = \lambda^{-1}AR_zP_{\eta}.$$

Further we act on the vector η an take the product with vector n_z. Carrying out the calculation we obtain

$$\lambda_z^{-1}(1 - z\lambda^{-1}a_{0\bar{z}}) = \lambda^{-1}$$

and finally (12).

E x a m p l e. Let $\mathcal{H} = L^2([1, \infty); dx)$ and A be the multiplicator on a variable x. Set $\omega = 1 \in \mathcal{H}_{-2} \setminus \mathcal{H}$. Then $\eta = A^{-1}\omega = \dfrac{1}{x}$. The norm $\|\omega\|_2 = \|\eta\| = \int_1^{\infty} x^{-2}dx = 1$. Note that if $f \in \mathcal{D}(A)$ then $\int_1^{\infty} f(x)\,dx < \infty$ because it is equal to $Af, \eta)$. It is convenient to write $A_{\lambda,\omega} \equiv A_{\lambda}$, for arbitrary $\lambda \in \mathbb{R}^{-1}$. The domain of A_{λ} has the form

$$\left\{g \in \mathcal{H} \mid g = f + \lambda^{-1}\left(\int_1^{\infty} f(x)\,dx\right)x^{-1}, \quad f \in \mathcal{D}(A)\right\}.$$

Moreover

$$A_\lambda g = Af.$$

The eigenvector of the operator A_λ is explicitly given by

$$\psi_\alpha(x) = \lambda^{-1}(x-\alpha)^{-1} \equiv \lambda^{-1}\eta_\alpha$$

and it corresponds to the eigenvalue $\alpha = 1 - \exp(-\lambda)$. In fact let $\psi_\alpha(x) = \varphi_\alpha(x) + \lambda^{-1}(\varphi_\alpha, \eta)\cdot\eta$ under the condition $\langle\varphi_\alpha, 1\rangle = 1$. Then for φ_α we have: $\varphi_\alpha = \alpha\lambda^{-1}(x-\alpha)^{-1}x^{-1}$. And for $\psi_\alpha(x)$ we obtain

$$\psi_\alpha(x) = \alpha\lambda^{-1}(x-\alpha)^{-1}x^{-1} + \lambda^{-1}x^{-1} = \lambda^{-1}(\alpha(x-a)^{-1}+1)x^{-1} = \lambda^{-1}(x-\alpha)^{-1}.$$

Let us to find now a number α. From the condition $\langle\varphi_\alpha, 1\rangle = 1$ after calculation we have

$$1 = \alpha\lambda^{-1}\int_1^\infty (x-\alpha)^{-1}dx = -\lambda^{-1}\ln(1-\alpha).$$

So

$$\exp(-\lambda) = 1-\alpha \text{ and } \alpha = 1-\exp(-\lambda).$$

Therefore for arbitrary $\lambda \in \mathbb{R}^1$ the operator A_λ has the eigenvector ψ_α with eigenvalue $\alpha = 1 - \exp(-\lambda) < 1$.

The additional point spectrum of $A_{\lambda,\omega}$. Consider the question on additional point spectrum of the operator $A_{\lambda,\omega}$. With regard to (4), (5) the eigenvalue problem for $A_{\lambda,\omega}$ has a form:

$$A_{\lambda,\omega}\psi_\alpha = \alpha\psi_\alpha = \alpha(\varphi_\alpha + \lambda^{-1}(A\varphi_\alpha, \eta)\eta = A\varphi_\alpha$$

where

$$\psi_\alpha = \varphi_\alpha + \lambda^{-1}(A\varphi_\alpha, \eta)\eta \in \mathcal{D}_{\lambda,\omega}\alpha \in \mathbb{R}^1.$$

We can take the length of the vector φ_α such that

$$(A\varphi_\alpha, \eta) = 1. \tag{13}$$

Then we have

$$A_{\lambda,\omega}\psi_\alpha = \alpha\psi_\alpha = A\varphi_\alpha = \alpha\varphi_\alpha + \alpha\lambda^{-1}\eta.$$

It gives

$$\varphi_\alpha = \alpha\lambda^{-1}(A-\alpha)^{-1}\eta \equiv \alpha\lambda^{-1}R_\alpha\eta$$

and

$$\psi_\alpha = \alpha\lambda^{-1}R_\alpha\eta + \lambda^{-1}\eta \equiv \lambda^{-1}(\alpha R_\alpha + 1)\eta.$$

Using the operator identity $A(A-\alpha)^{-1} = \alpha(A-\alpha)^{-1} + 1$ we have finally

$$\psi_\alpha = \lambda^{-1}AR_\alpha\eta = \lambda^{-1}R_\alpha\omega = \lambda^{-1}\eta_\alpha. \tag{14}$$

The quality (13) contains the condition for the value α. Rewrite (13) using (1) in the following form

$$1 = (A\varphi_\alpha, \eta) = \alpha\lambda^{-1}(AR_\alpha\eta, \eta) = \alpha\lambda^{-1}\langle R_\alpha\eta, \omega\rangle.$$

So

$$\lambda = \alpha(\eta, \eta_\alpha) = \alpha a_{0,\alpha}. \tag{15}$$

Thus we have theorem 3.

Theorem 3. *Let $\lambda \in \mathbb{R}^1$, $\omega \in \mathcal{H}_{-2} \setminus \mathcal{H}_{-1}$. Then the essential spectrum of A and $A_{\lambda,\omega}$ coincide*

$$\sigma_{\text{ess}}(A_{\lambda,\omega}) = \sigma_{\text{ess}}(A).$$

For each real $\alpha \in \rho(A)$ the vector ψ_α given by (14) *is an eigenvector for the operator $A_{\lambda,\omega}$ where λ satisfies* (15), *i. e.*

$$\sigma_\rho(A_{\lambda,\omega}) = \alpha$$

if λ and α are connected by (15).

Description of $\mathcal{S}^1_s$. Let $A = A^* \geqslant m > 0$ be a unbounded operator in $\mathcal{H}$. Introduce the continuous scale (A-scale) of he Hilbert spaces

$$\{\mathcal{H}_l\}_{l \in \mathbb{R}^1} \|\cdot\|_l = \| A^{l/2} \cdot \|.$$

Note that $\mathcal{H}_l$, $l \geqslant 0$ coincides with $\mathcal{D}(A^{1/2})$. The mapping

$$A^{\frac{l-d}{2}} : \mathcal{H}_l \to \mathcal{H}_d,\ l,\ d \in \mathbb{R}^1$$

is isometrical, densely defined and has a dense range. So its closure is the unitary operator which we denote by

$$I_{d,l} : \mathcal{H}_l \to \mathcal{H}_d,\ I_{l,d} = I^{-1}_{d,l}.$$

The triple

$$\mathcal{H}_{-l} \supset \mathcal{H} \supset \mathcal{H}_l,\ l < 0$$

is a rigged Hibert space. It means that $\mathcal{H}_{-l}$ is dual to $\mathcal{H}_l$ with respect to $\mathcal{H}$. The last property preserves under shifts along A-scale. The triple

$$\mathcal{H}_{-l+d} \supset \mathcal{H}_d \supset \mathcal{H}_{l+d},\ l > 0,\ d \in \mathbb{R}^1$$

is also the rigged Hilbert space.

Let $l > 0$ be fixed. Denote $\mathcal{H}_\pm \equiv \mathcal{H}_{\pm l}$ and $\mathcal{H}_{+,+} \equiv \mathcal{H}_{2l}$. Ler $\mathcal{D}_+$ be some closed subspace in $\mathcal{H}_+$. Under what condition is $\mathcal{D}_+$ dense in $\mathcal{H}$? Denote $N_+ := \mathcal{D}_+^\perp$ in $\mathcal{H}_+$, $N := I_{0,+}N_+$, $N_- := I_{-,+}N_+$.

Theorem 4. *The subspace $\mathcal{D}_+ \subset \mathcal{H}_+$ is dense in $\mathcal{H}$ iff one of the following equivalent conditions:*

a) $N_- \cap \mathcal{H} = \{0\}$;

b) $N \cap \mathcal{H}_+ = \{0\}$;

c) $N_+ \cap \mathcal{H}_{+,+} = \{0\}$

is fulfilled. In particular if $\dim N = 1$ and a vector $\eta \in N$ then $\mathcal{D}_+$ is dense in $\mathcal{H}$ iff:

a) $\omega \in \mathcal{H}_- \setminus \mathcal{H}$, $\omega := I_{-,0}\eta$;

b) $\eta \in \mathcal{H} \setminus \mathcal{H}_+$;

c) $\eta_+ \in \mathcal{H}_+ \setminus \mathcal{H}_{+,+}$, $\eta_+ := I_{+,0}\eta$.

Proof. We only prove a). Lef ψ be in $\mathcal{H}$ and $\psi \perp \mathcal{D}_+$, i. e. $(\psi, \mathcal{D}_+) = 0$. Then using (1) we have

$$0 = (\psi \mathcal{D}_+) = \langle \psi, \mathcal{D}_+ \rangle = (I_{+,-}\psi, \mathcal{D}_+)_+.$$

It means that $I_{+,-}\psi \in N_+$ and therefore $\psi \in N_-$. It shows that condition a) is equivalent to the property $\mathcal{D}_+$ be a dense set in $\mathcal{H}$. The next statements are proved by the same way with using the invariance A-scale under the shifts.

Let a vector $\eta_+ \in \mathcal{H}_l$, $l \geqslant 2$, $\eta := A\eta_+$ and $\omega := I_{-2,+2}\eta_+ \equiv I_{-2,0}\eta$ and let $T_{\lambda,\omega}$ ($\lambda \in \mathbb{R}^1$, $\lambda \neq 0$) be an operator of the form (2).

Proposition 3.

$$T_{\lambda,\omega} \in \mathcal{S}^1_s \Leftrightarrow \eta \in \mathcal{H} \setminus \mathcal{D}(A).$$

Proof follows from Theorem 4.

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