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Monic divisors of polynomial matrices over infinite field

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Abstract. The necessary and sufficient conditions of existence of monic divisors of a nonsingular polynomial matrix over an infinite field are established.

Introduction

The problem of existence of regular factor of a polynomial matrix over a field was one of the most investigated problems in the matrix theory in the middle and second half of last century. Above her worked on resolution of not one mathematical school. For its solution used methods that were based on the notion of Jordan chains [1, 2], mostly. However, the problem of existence of regular factor of a polynomial matrix was solved by P.S. Kazimirs'kii [3] methods, which were based on determinant matrix concepts and the values of the matrix on a system of roots of the polynomial. Belongs to the students of P.S. Kazimirs'kii, namely, V.M. Petrychkovych [4], in collaboration with whame he introduced the concept of semiscalar equivalence, and V.R. Zelisko [5],who first considered the group of matrices quasicommutative with diagonal matrix the significant contribution the problem in solving.

Various aspects of this problem are of interest for algebraist now. So in the Slusky [6] the number of monic divisors of a matrix polynomial of second order is estimated. The possibility of expansion the monic matrix in the product of monic linear factors depending on the location of the

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roots of the characteristic polynomial in the complex plane was explored by J. Maroulas, P. Psarrakos [7]. N.S. Dzhalyuk and V.M. Petrychkovych [8] investigate the monic divisors with prescribed canonical diagonal forms of polynomial matrices under conditions parallel to the respective factorization matrices to factorization of their canonical diagonal forms.

The necessary and sufficient conditions of existence of monic divisors of a nonsingular matrix polynomial, with certain restrictions on the canonical diagonal form divisor, over infinite field, were established in the paper [9]. In this article factorization of a nonsingular polynomial matrices is investigated. Namely, necessary and sufficient conditions of existence of a monic divisors of a nonsingular matrix polynomial over infinite field were established. Thus, the class of fields, for which this result is correct, is significantly wider than algebraically closed fields of characteristic zero, such fields can have a finite characteristic. This causes the novelty of the approach to solving this problem, based on the concepts of generating set and determinant matrix.

Main results

Let F be an infinite field , $A(x)$ be a nonsingular matrix of order n over $F[x]$, written in the form of matrix polynomial:

$$A(x) = A_k x^k + A_{k-1}x^{k-1} + \cdots + A_0.$$

The matrix $A(x)$ is called regular if $\det A_k \neq 0$ and monic if $A_k = E$ (the identity matrix of order n). We say that a matrix $A(x)$ is right-regularizable, if there exists an invertible matrix $U(x)$, such that

$$A(x)U(x) = Ex^r + D_{r-1}x^{r-1} + \cdots + D_0.$$

There exists such invertible matrices $P(x)$ and $Q(x)$, that

$$P(x)A(x)Q(x) = diag(\varepsilon_1(x), \ldots, \varepsilon_n(x)) = \Psi(x),$$

$\varepsilon_i(x)|\varepsilon_{i+1}(x)$, $i = 1, \ldots, n-1$. The matrix $\Psi(x)$ is called the canonical diagonal form (c.d.f.) or Smith normal form of the matrix $A(x)$. The matrices $P(x)$ and $Q(x)$ are called left and right transformation matrices of the matrix $A(x)$, respectively. Let $A(x) = B\,(x)\,C\,(x)$, where $B\,(x)$ has c.d.f. $\Phi(x) = diag\,(\varphi_1(x), \ldots, \varphi_n(x))$. According to [10] the matrix $\Phi(x)$ is the right and left divisors of matrix $\Psi(x)$. Moreover, if $B(x)$ is a monic matrix polynomial of degree r, then $\deg\det\Phi(x) = nr$.

Before formulating the main result we introduce some notions. Consider the lower unitriangular matrix:

$$V(\Psi,\Phi)=\left\|\begin{array}{ccccc} 1 & 0 & \dots & 0 & 0\\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)}k_{21} & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ \frac{\varphi_n}{(\varphi_n,\varepsilon_1)}k_{n1} & \frac{\varphi_n}{(\varphi_n,\varepsilon_2)}k_{n2} & \dots & \frac{\varphi_n}{(\varphi_n,\varepsilon_{n-1})}k_{n,n-1} & 1 \end{array}\right\|, \quad (1)$$

where

$$k_{ij}=\begin{cases} 0, & (\varphi_i,\varepsilon_j)=\varphi_j,\\ k_{ij0}+k_{ij1}x+\ldots+k_{ijh_{ij}}x^{h_{ij}}, & (\varphi_i,\varepsilon_j)\neq\varphi_j,\end{cases}$$

$$h_{ij}=\deg\frac{(\varphi_i,\varepsilon_j)}{\varphi_j}-1, i=2,3,\ldots,n, j=1,\ldots,n-1, i>j,$$

where k_{ijl}- parameters, $i=2,3,\ldots,n$, $j=1,\ldots,n-1$. Denote by $F(k)$ transcendental extension of field F by adjoining all parameters k_{ijl}. The main result of this paper is the following theorem:

Theorem 1. *Let F be an infinite field. In order that,*

$$A(x)=B(x)C(x),$$

where

$$B(x)=Ex^r+B_{r-1}x^{r-1}+\ \cdots\ +B_0,$$

$$B(x)\sim\Phi(x)=diag\left(\varphi_1(x),\ldots,\varphi_n(x)\right),\varphi_i(x)|\varphi_{i+1}(x),i=1,\ldots,n-1,$$

it is necessary and sufficient that, the matrix $\left(V(\Psi,\Phi)P(x)\right)^{-1}\Phi(x)$ be right-regularizable over $F(k)[x]$.

Note, that we can use one of the methods proposed in [3, 14], in order to verify that the matrix $\left(V(\Psi,\Phi)P(x)\right)^{-1}\Phi(x)$ is right-regularizable .

Before proving this theorem, we establish some auxiliary assertions. Consider the following sets of matrices:
$\mathbf{G}_\Phi=\{H(x)\in GL_n(F[x])\mid H(x)\Phi(x)=\Phi(x)H_1(x),\ for\ some\ H_1(x)\in GL_n(F[x])\}$,
$\mathbf{L}(\Psi,\Phi)=\{L(x)\in GL_n(F[x])\mid L(x)\Psi(x)=\Phi(x)L_1(x),\ for\ some$ $L_1(x)\in M_n(F[x])\}$, which, according to the results of [5, 11], consisting of a invertible matrices of the form

$$\left\|\begin{array}{ccccc} h_{11} & h_{12} & \dots & h_{1,n-1} & h_{1n}\\ \frac{\varphi_2}{\varphi_1}h_{21} & h_{22} & \dots & h_{2,n-1} & h_{2n}\\ \vdots & \vdots & \dots & \vdots & \vdots\\ \frac{\varphi_n}{\varphi_1}h_{n1} & \frac{\varphi_n}{\varphi_2}h_{n2} & \dots & \frac{\varphi_n}{\varphi_{n-1}}h_{n,n-1} & h_{nn}\end{array}\right\|, \quad (2)$$

$$\left\|\begin{array}{ccccc} l_{11} & l_{12} & \dots & l_{1,n-1} & l_{1n} \\ \frac{\varphi_2}{(\varphi_2,\varepsilon_1)} l_{21} & l_{22} & \dots & l_{2,n-1} & l_{2n} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \frac{\varphi_n}{(\varphi_n,\varepsilon_1)} l_{n1} & \frac{\varphi_n}{(\varphi_n,\varepsilon_2)} l_{n2} & \dots & \frac{\varphi_n}{(\varphi_n,\varepsilon_{n-1})} l_{n,n-1} & l_{nn} \end{array}\right\|, \qquad (3)$$

respectively. The set $\mathbf{G}_\Phi$ is a multiplicative group, and the set $\mathbf{L}(\Psi,\Phi)$ is be called generating set. Similarly, we introduce the set $\mathbf{G}_\Psi$. Denote by $\mathbf{G}^*_\Phi$ and $\mathbf{L}^*(\Psi,\Phi)$ the sets of matrices of the form (2) and (3) over $F(k)[x]$, respectively.

Denote by $\mathbf{V}(\Psi,\Phi)$ the set of lower unitriangular matrices, obtained from the matrix $V(\Psi,\Phi)$, when the all parameters k_{ijl} independently runs all possible values of the field F.

Consider the product of matrices $V(\Psi,\Phi)T(x) = U(x)$, where $T(x) = \|t_{ij}\|_1^n \in \mathbf{G}_\Psi$. Denote by $U_i(x)$ submatrices of the matrix $U(x)$ obtained by crossing out the first i rows and the first i columns of the matrix $U(x)$, $i = 1,\dots,n-1$. According to the Binee-Cauchy formula we have:

$$\det U_i(x) = \sum_j |V_{ij}(\Psi,\Phi)| \cdot |T_{ji}(x)| + \det T_i(x),$$

where $\sum_j |V_{ij}(\Psi,\Phi)| \cdot |T_{ji}(x)|$ are the sum of products of all possible maximal order $n-i$ minors of the matrix $V_{ij}(\Psi,\Phi)$, with exception of minor of the lower unitriangular matrix that is equal to one, the corresponding minors of the same order matrix $T_{ji}(x)$. $T_i(x)$ is the submatrices of the matrix $T(x)$ obtained by crossing out the first i rows and the first i columns of the matrix $T(x)$, $i = 1,\dots,n-1$.

Lemma 1. *A necessary and sufficient condition for*

$$\det U_i(\alpha) \equiv \sum_j |V_{ij}(\Psi,\Phi,\alpha)| \cdot |T_{ji}(\alpha)| + \det T_i(\alpha) \equiv 0,$$

is that each term of this sum is equal to zero.

Proof. Sufficiency is obviousl. *Necessity.* The minors $|V_{ij}(\Psi,\Phi,\alpha)|$ are the sum of products elements of field F and parameters k_{ijl},
$i = 2,3,\dots,n$, $j = 1,\dots,n-1$. Moreover the set of parameters that occurs in each of these minors is not repeated in any other minor. □

Lemma 2. *If* $\frac{\varphi_i}{\varphi_j}(\alpha) = 0$ *and* $\frac{\varepsilon_i}{\varepsilon_j}(\alpha) \neq 0$*, then* $k_{ij}(x) \not\equiv 0$*,* $i > j$*.*

Proof. Assume that $k_{ij}(x) \equiv 0$. Then $(\varphi_i,\varepsilon_j) = \varphi_j$. Thus

$$\frac{\varepsilon_i}{\varepsilon_j} = \frac{\varphi_i\varepsilon_i(\varphi_i,\varepsilon_j)}{(\varphi_i,\varepsilon_j)\varphi_i\varepsilon_j} = \frac{\varphi_i}{\varphi_j} \cdot \frac{(\varepsilon_i\varphi_i,\varepsilon_i\varepsilon_j)}{\varphi_i\varepsilon_j}.$$

Hence $\frac{\varepsilon_i}{\varepsilon_j}(\alpha) = 0$, a contradiction with the conditions of lemma. □

Lemma 3. *If* $\frac{\varepsilon_i}{\varepsilon_j}(\alpha) = 0$ *and* $\frac{\varepsilon_i}{\varepsilon_{j+1}}(\alpha) \neq 0$, $i > j$ *then* $\frac{\varepsilon_{j+1+t}}{\varepsilon_{j-l}}(\alpha) = 0$, $t = 0, 1, \dots, n-j-1$, $l = 0, 1, \dots, j-1$, $j+1+t > j-l$ *and* $\frac{\varepsilon_{s+1}}{\varepsilon_s}(\alpha) \neq 0$, $s = j+1, j+2, \dots, i-1$.

Proof. From the equality $\frac{\varepsilon_i}{\varepsilon_j} = \frac{\varepsilon_i}{\varepsilon_{j+1}} \cdot \frac{\varepsilon_{j+1}}{\varepsilon_j}$ it follows that $\frac{\varepsilon_{j+1}}{\varepsilon_j}(\alpha) = 0$. Since $\frac{\varepsilon_{j+1}}{\varepsilon_j} \Big| \frac{\varepsilon_{j+1+t}}{\varepsilon_{j-l}}$ then $\frac{\varepsilon_{j+1+t}}{\varepsilon_{j-l}}(\alpha) = 0$, $t = 0, 1, \dots, n-j-1$, $l = 0, 1, \dots, j-1$, $j+1+t > j-l$. Noting, that $\frac{\varepsilon_i}{\varepsilon_{j+1}} = \frac{\varepsilon_{j+2}}{\varepsilon_{j+1}} \cdot \frac{\varepsilon_{j+3}}{\varepsilon_{j+2}} \cdot \ldots \cdot \frac{\varepsilon_i}{\varepsilon_{i-1}}$ we conclude that $\frac{\varepsilon_{s+1}}{\varepsilon_s}(\alpha) \neq 0$, $s = j+1, j+2, \dots, i-1$. □

Similarly, we obtain the following result.

Lemma 4. *If* $\frac{\varepsilon_i}{\varepsilon_j}(\alpha) = 0$ *and* $\frac{\varepsilon_{i-1}}{\varepsilon_j}(\alpha) \neq 0$, $i > j$, *then* $\frac{\varepsilon_{i+d}}{\varepsilon_{i-1-p}}(\alpha) = 0$, $d = 0, 1, \dots, n-i$, $p = 0, 1, \dots, i-2$, $i+d > i-1-p$ *and* $\frac{\varepsilon_{k+1}}{\varepsilon_k}(\alpha) \neq 0$, $k = j, j+1, j+2, \dots, i-2$.

Lemma 5. $\left(\frac{\varphi_{i+1}}{\varphi_i}(x), \det U_i(x)\right) = 1$ *for* $i = 1, \dots, n-1$.

Proof. Suppose, that for some $i = r$, $\left(\frac{\varphi_{r+1}}{\varphi_r}(x), \det U_r(x)\right) = \delta(x) \neq const.$ Let F' be a polynomial decomposition field $\frac{\varphi_{r+1}}{\varphi_r}(x)$ and $\delta(\alpha) = 0$, $\alpha \in F'$. Then $\frac{\varphi_{r+1}}{\varphi_r}(\alpha) = 0$ and

$$\det U_r(\alpha) = \sum_j |V_{rj}(\Psi, \Phi, \alpha)| \cdot |T_{jr}(\alpha)| + \det T_r(\alpha) \equiv 0. \tag{4}$$

Let j be the first index for which $\frac{\varepsilon_{r+1}}{\varepsilon_j}(\alpha) = 0$ and $\frac{\varepsilon_{r+1}}{\varepsilon_{j+1}}(\alpha) \neq 0$, $0 \leq j < r+1$, where $\varepsilon_0(x) = 1$. According to lemma 3 $\frac{\varepsilon_{j+1+t}}{\varepsilon_{j-l}}(\alpha) = 0$, $t = 0, 1, \dots, n-j-1$, $l = 0, 1, \dots, j-1$, $j+1+t > j-l$ and $\frac{\varepsilon_{s+1}}{\varepsilon_s}(\alpha) \neq 0$, $s = j+1, j+2, \dots, r$.

Let m be the first index for which $\frac{\varepsilon_m}{\varepsilon_r}(\alpha) = 0$ and $\frac{\varepsilon_{m-1}}{\varepsilon_r}(\alpha) \neq 0$, $r+1 < m \leq n$. According to lemma 4 $\frac{\varepsilon_{m+d}}{\varepsilon_{m-1-p}}(\alpha) = 0$, $d = 0, 1, \dots, n-m$, $p = 0, 1, \dots, m-2$, $m+d > m-1-p$ and $\frac{\varepsilon_{k+1}}{\varepsilon_k}(\alpha) \neq 0$, $k = r, r+1, \dots, m-2$.

Equality $\frac{\varepsilon_f}{\varepsilon_g} = \frac{\varepsilon_{g+1}}{\varepsilon_g} \cdot \frac{\varepsilon_{g+2}}{\varepsilon_{g+1}} \cdots \frac{\varepsilon_f}{\varepsilon_{f-1}}$ implies, that $\frac{\varepsilon_f}{\varepsilon_g}(\alpha) \neq 0$ for $f \geq g$, where $f = r+1, r+2, \dots, m-1$, $g = j+1, j+2, \dots, r$. From, that $\frac{\varphi_f}{(\varphi_f, \varepsilon_g)} \Big| \frac{\varepsilon_f}{\varepsilon_g}$, follows that $\frac{\varphi_f}{(\varphi_f, \varepsilon_g)}(\alpha) \neq 0$, $f = r+1, r+2, \dots, m-1$, $g = j+1, j+2, \dots, r$. Since $\frac{\varphi_{r+1}}{\varphi_r} \Big| \frac{\varphi_{r+1+q}}{\varphi_{r-h}}$, then $\frac{\varphi_{r+1+q}}{\varphi_{r-h}}(\alpha) = 0$, $q = 0, 1, \dots, n-r-1$, $h = 0, 1, \dots, r-1$, $r+1+q > r-h$. According to lemma 2 $k_{fg}(x) \not\equiv 0$, $f = r+1, r+2, \dots, m-1$, $g = j+1, j+2, \dots, r$. That is, the elements $\frac{\varphi_f}{(\varphi_f, \varepsilon_g)}(\alpha) k_{fg}(x)$, $f = r+1, r+2, \dots, m-1$, $g = j+1, j+2, \dots, r$ are nonzero.

Thus, all possible minors of maximal order , which contained the last columns and minors building on the last rows and on the last columns of the matrix are non equal to zero. Taking into account lemma 1, from identity (4), follows that all possible minors of maximal order containing the last rows and minors building on the last columns and on the last rows of the matrix are equal to zero. Hence the matrix has the form:

$$T(\alpha) = \left\| \begin{array}{cccc} * & * & * & * \\ \mathbf{0}_{(r-j)\times j} & S'_{(r-j)\times(r-j)} & H'_{(r-j)\times(m-r-1)} & T'_{(r-j)\times(n-m+1)} \\ \mathbf{0}_{(m-r-1)\times j} & S''_{(m-r-1)\times(r-j)} & H''_{(m-r-1)\times(m-r-1)} & T''_{(m-r-1)\times(n-m+1)} \\ \mathbf{0}_{(n-m+1)\times j} & \mathbf{0}_{(n-m+1)\times(r-j)} & \mathbf{0}_{(n-m+1)\times(m-r-1)} & T'''_{(n-m+1)\times(n-m+1)} \end{array} \right\|,$$

where $\mathbf{0}_{i\times j}$ is a zero $i \times j$ matrix. Note that when $j = 0$, that the matrix $\mathbf{0}_{i\times j}$ is empty. Decomposing the matrix $T(\alpha)$ on the last $(n-j)$ rows, we obtain that $\det T(\alpha) = 0$, which contradict to the invertibility of matrix $T(\alpha)$. □

Proof of Theorem 1

Proof. Sufficiency. Consider the matrix $A(x)$, as a matrix over $F(k)[x]$. Let the matrix $(V(\Psi,\Phi)P(x))^{-1}\Phi(x) = B(x)$ be right-regularizable over $F(k)[x]$. That is, there exist such matrix $U(x) \in GL_n(F(k)[x])$, that

$$B(x)U(x) = Ex^r + B_{r-1}x^{r-1} + \ldots + B_1x + B_0 = D(x).$$

According to proposition 1 of [12], the set

$$(\mathbf{L}^*(\Psi,\Phi)P(x))^{-1}\Phi(x)GL_n(F(k)[x])$$

is the set of all left divisors of matrix $A(x)$ with c.d.f. $\Phi(x)$ over $F(k)[x]$. Since $V(\Psi,\Phi) \in \mathbf{L}^*(\Psi,\Phi)$, $D(x)$ is a left divisor of the matrix $A(x)$: $A(x) = D(x)C(x)$. Basing on lemma 4 of [14] the matrix $B(x)$ is right-regularizable over $F(k)[x]$, if and only if,

$$\det M_B = f(k_{n10},\ldots,k_{n,n-1,h_{n,n-1}},x) \not\equiv 0,$$

where M_B is an corresponding matrix to matrix polynomial $B(x)$. According to lemma 5 of [14] coefficients of monic matrix polynomials $B(x)U(x)$, have the form $B_k = \sum_{i=0}^{k} T_iM_{(r-k)+i,r} = \sum_{i=0}^{k} \frac{1}{\det M_B}T_iM_{ij}$, $k = 0,1,\ldots,r-1$, $i,j = 0,1,\ldots,n$, where T_i are coefficients of the matrix polynomial $B(x)$, $M_{(r-k)+i,r}$ are corresponding blocks of a matrix M_B^{-1}. In the infinite field exist such elements $p_{n10},p_{n20},\ldots,p_{n,n-1,h_{n,n-1}},p_{nn}$, that $f(p_{n10},\ldots,p_{nn}) \neq 0$. Then the matrix $\overline{D}(x)$, obtained from a matrix $D(x)$ of variables replacing $k_{n10},\ldots,k_{n,n-1,h_{n,n-1}},x$ by the relevant elements $p_{n10},\ldots,p_{n,n-1,h_{n,n-1}},p_{nn}$ of the field F will be a monic divisors of the matrix $A(x)$.

Necessity. Let $A(x) = B(x)C(x)$, where $B(x)$ is a monic divisor of the matrix $A(x)$ with c.d.f. $\Phi(x)$. According to proposition 1 of [12], the set $(\mathbf{L}(\Psi,\Phi)P(x))^{-1}\,\Phi(x)GL_n(F[x])$ is the set of all left divisors of matrix $A(x)$. Therefore, the matrix $B(x)$ can be written as $B(x) = (L(x)P(x))^{-1}\,\Phi(x)K(x)$, where $L(x) \in \mathbf{L}(\Psi,\Phi)$, $K(x) \in GL_n(F[x])$. Based on theorem 2 of [9] $L(x) = H(x)V_0(x)S(x)$, where $H(x) \in \mathbf{G}_\Phi$, $V_0(x) \in \mathbf{V}(\Psi,\Phi)$, $S(x) \in \mathbf{G}_\Psi$. Then

$$B(x) = (L(x)P(x))^{-1}\,\Phi(x)K(x) = (H(x)V_0(x)S(x)P(x))^{-1}\,\Phi(x)K(x).$$

According to [5, 11], the set of all left transforming matrices of the matrix $A(x)$ has the form $\mathbf{P}_A = \mathbf{G}_\Psi P$. Thus, $S(x)P(x) = P_0(x)$ is a left transforming matrix of the matrix $A(x)$. Then

$$B(x) = (H(x)V_0(x)P_0(x))^{-1}\,\Phi(x)K(x) =$$

$$= P_0^{-1}(x)V_0^{-1}(x)H^{-1}(x)\Phi(x)K(x).$$

Since $H^{-1}(x)\Phi(x) = \Phi(x)H_1(x)$, where $H_1(x) \in GL_n(F[x])$ then

$$B(x) = P_0^{-1}(x)V_0^{-1}(x)\Phi(x)H_1(x)K(x) = (V_0(x)P_0(x))^{-1}\,\Phi(x)K_1(x).$$

The matrix $B(x)$ is the monic and therefore the matrix $(V_0(x)P_0(x))^{-1}\,\Phi(x)$ is right-regularizable. By replacing in the matrix $V_0(x)$ coefficients of polynomials by corresponding parameters k_{ijl}, we obtain that the matrix $(V(\Psi,\Phi)P_0(x))^{-1}\,\Phi(x)$ is right-regularizable over $F(k)[x]$.

We show that regardless of the choice of transforming matrix $P_1(x)$ the matrix $D(x) = (V(\Psi,\Phi)P_1(x))^{-1}\Phi(x)$ is right-regularizable also. Since $P_1(x) = N(x)P_0(x)$, where $N(x) \in \mathbf{G}_\Psi$, then

$$D(x) = (V(\Psi,\Phi)P_1(x))^{-1}\,\Phi(x) = (V(\Psi,\Phi)N(x)P_0(x))^{-1}\,\Phi(x) =$$

$$= ((V(\Psi,\Phi)N(x))\,P_0(x))^{-1}\,\Phi(x).$$

Taking into account lemma 3 of [12], and lemma 5, there exist a matrix $T(x) \in \mathbf{G}^*_\Phi$, such that $T(x)V(\Psi,\Phi)N(x)$ is lower unitriangular matrix over $F(k)[x]$. According to lemma 3 of [13] there exists such matrix $T_1(x) \in \mathbf{G}^*_\Phi$, that

$$T_1(x)T(x)V(\Psi,\Phi)N(x) = V_1(\Psi,\Phi) \in \mathbf{V}(\Psi,\Phi)$$

Then

$$D(x) = (T_1(x)T(x)\,(V(\Psi,\Phi)N(x))\,P_0(x))^{-1}\,T_1(x)T(x)\Phi(x) =$$

$$= ((T_1(x)T(x)V(\Psi,\Phi)N(x))\,P_0(x))^{-1}\,\Phi(x)\left(\tilde{T}(x)\tilde{T}_1(x)\right) =$$

$$= (V_1(\Psi, \Phi)P_0(x))^{-1}\,\Phi(x)T_2(x).$$

Denoting parameters k_{ijl} in k'_{ijl} in the matrix $V_1(\Psi, \Phi)$ we get, that the matrix $(V(\Psi, \Phi)P_1(x))^{-1}\,\Phi(x)$ is right-regularizable over $F(k)[x]$. $\square$

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