Abstract. Let $M$ be a right $R$-module and $\tau$ a preradical. We call $M$ $\tau$-$H$-supplemented if for every submodule $A$ of $M$ there exists a direct summand $D$ of $M$ such that $(A+D)/D \subseteq \tau(M/D)$ and $(A+D)/A \subseteq \tau(M/A)$. Let $\tau$ be a cohereditary preradical. Firstly, for a duo module $M = M_1 \oplus M_2$ we prove that $M$ is $\tau$-$H$-supplemented if and only if $M_1$ and $M_2$ are $\tau$-$H$-supplemented. Secondly, let $M = \bigoplus_{i=1}^{n} M_i$ be a $\tau$-supplemented module. Assume that $M_i$ is $\tau$-$M_j$-projective for all $j > i$. If each $M_i$ is $\tau$-$H$-supplemented, then $M$ is $\tau$-$H$-supplemented. We also investigate the relations between $\tau$-$H$-supplemented modules and $\tau$-(\oplus)-supplemented modules.

Introduction

Throughout this paper, $R$ denotes an associative ring with identity and modules are unital right $R$-modules. We use $N \leq M$ and $N \leq_d M$ to signify that $N$ is a submodule and a direct summand of $M$, respectively.

A functor $\tau$ from the category of the right $R$-modules Mod $- R$ to itself is called a preradical if it satisfies the following properties:

i) For any $R$-module $M$, $\tau(M)$ is a submodule of an $R$-module $M$,

ii) If $f : M' \to M$ is an $R$-module homomorphism, then $f(\tau(M')) \subseteq \tau(M)$ and $\tau(f)$ is the restriction of $f$ to $\tau(M')$.

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It is well known if \( K \) is a direct summand of \( M \), then \( \tau(K) = \tau(M) \cap K \) for a preradical \( \tau \). A preradical \( \tau \) is said to be cohereditary if, for every \( M \in \text{Mod} - R \) and every submodule \( N \) of \( M \), \( \tau(M/N) = (\tau(M) + N)/N \). We refer to [3] for details concerning radicals and preradicals. In this paper, \( \tau \) will be a preradical unless otherwise stated. Recall that a module \( M \) has the Summand Sum Property, (SSP) if the sum of any two direct summands of \( M \) is again a direct summand (see [4]).

Let \( M \) be a module. A submodule \( X \) of \( M \) is called fully invariant, if for every \( f \in \text{End}(M) \), \( f(X) \subseteq X \). The module \( M \) is called a duo module, if every submodule of \( M \) is fully invariant. The submodule \( A \) of \( M \) is called projection invariant in \( M \) if \( f(A) \subseteq A \), for any idempotent \( f \in \text{End}(M) \). A submodule \( K \) of \( M \) is called small in \( M \) (denoted by \( K \ll M \)) if \( N + K \neq M \) for any proper submodule \( N \) of \( M \).

Lifting modules were defined and studied by many authors. \( H \)-supplemented modules were introduced in [11] as a generalization of lifting modules. According to [11], a module \( M \) is called \( H \)-supplemented if for every submodule \( A \) of \( M \) there exists a direct summand \( D \) of \( M \) such that \( A + X = M \) if and only if \( D + X = M \) for any submodule \( X \) of \( M \). For more information about \( H \)-supplemented modules we refer the reader to [8], [10] and [11]. A module \( M \) is called \( \oplus \)-supplemented if for every submodule \( N \) of \( M \) there exists a direct summand \( D \) of \( M \) such that \( M = N + D \) and \( N \cap D \ll D \). According to [15], a module \( M \) is semiperfect if every factor module of \( M \) has a projective cover. By [15, 41.14 and 42.1], if \( P \) is projective, then \( P \) is semiperfect if and only if for every submodule \( K \) of \( P \) there exists a decomposition \( K = A \oplus B \) such that \( A \) is a direct summand of \( P \) and \( B \ll P \). By [5, Lemma 1.2] a projective module is \( \oplus \)-supplemented if and only if it is semiperfect.

In [2], for a radical \( \tau \), Al-Takhman, Lomp and Wisbauer defined and studied the concept of \( \tau \)-lifting, \( \tau \)-supplemented and \( \tau \)-semiperfect modules. Following [2], a module \( M \) is called \( \tau \)-lifting if every submodule \( N \) of \( M \) has a decomposition \( N = A \oplus B \) such that \( A \) is a direct summand of \( M \) and \( B \subseteq \tau(M) \) and they call \( M \) \( \tau \)-supplemented if for every submodule \( N \) of \( M \) there exists a submodule \( K \) of \( M \) such that \( N + K = M \) and \( N \cap K \subseteq \tau(K) \) (In this case \( K \) is called a \( \tau \)-supplement of \( N \) in \( M \)). They call a module \( M \) \( \tau \)-semiperfect if for every submodule \( N \) of \( M \), \( M/N \) has a projective \( \tau \)-cover. In this paper we define \( \tau \)-\( H \)-supplemented modules and investigate the general properties of such modules.

In Section 1 we will define \( \tau \)-\( H \)-supplemented modules and give an equivalent condition for such modules. Also we obtain some conditions which under the factor module of a \( \tau \)-\( H \)-supplemented module will be \( \tau \)-\( H \)-supplemented. Let \( M \) be a \( \tau \)-\( H \)-supplemented module for a cohereditary preradical \( \tau \). Then
(1) If $M$ is a distributive module, then $M/X$ is $\tau$-$H$-supplemented for every submodule $X$ of $M$.

(2) Let $N \leq M$ such that for each decomposition $M = M_1 \oplus M_2$ we have $N = (N \cap M_1) \oplus (N \cap M_2)$. Then $M/N$ is $\tau$-$H$-supplemented.

(3) Let $X$ be a projection invariant submodule of $M$. Then $M/X$ is $\tau$-$H$-supplemented. In particular, for every fully invariant submodule $A$ of $M$, $M/A$ is $\tau$-$H$-supplemented (Corollary 1).

In Section 2 we will study direct summands of $\tau$-$H$-supplemented modules. We show that, if $\tau$ is a cohereditary preradical, every direct summand of a $\tau$-$H$-supplemented module with SSP is $\tau$-$H$-supplemented (Theorem 2).

In Section 3 we will study direct sums of $\tau$-$H$-supplemented modules. Let $\tau$ be a cohereditary preradical. Let $M = M_1 \oplus M_2$ be a duo module. Then $M$ is $\tau$-$H$-supplemented if and only if $M_1$ and $M_2$ are $\tau$-$H$-supplemented (Theorem 4). Let $\tau$ be a cohereditary preradical. Let $M = \oplus_{i=1}^{n}M_i$ be a $\tau$-supplemented module. Assume that $M_i$ is $\tau$-$M_j$-projective for all $j > i$. If each $M_i$ is $\tau$-$H$-supplemented, then $M$ is $\tau$-$H$-supplemented (Corollary 4).

In Section 4 we will obtain the relations between $\tau$-$H$-supplemented modules and the other modules. Let $\tau$ be a cohereditary preradical. Let $M$ be a projective module such that every $\tau$-supplement submodule of $M$ is a direct summand. The following are equivalent: (Theorem 6)

(1) $M$ is $\tau$-supplemented;
(2) $M$ is $\tau$-lifting;
(3) $M$ is amply $\tau$-supplemented;
(4) $M$ is $\tau$-$H$-supplemented and $\tau(M)$ is QSL in $M$;
(5) $M$ is $\tau$-$\oplus$-supplemented.

1. Factor modules of $\tau$-$H$-supplemented modules

In this section we will define $\tau$-$H$-supplemented modules and give an equivalent condition for a module to be $\tau$-$H$-supplemented. Also we investigate some conditions for factor modules of a $\tau$-$H$-supplemented module to be $\tau$-$H$-supplemented.

Keskin Tütüncü, Nematollahi and Talebi give equivalent conditions for a module to be $H$-supplemented (see [8, Theorem 2.1]). Now we give the definition of a $\tau$-$H$-supplemented module based on their definition.

**Definition 1.** Let $M$ be a module. Then $M$ is $\tau$-$H$-supplemented in case for every $A \leq M$ there exists a direct summand $D$ of $M$ such that $(A + D)/A \subseteq \tau(M/A)$ and $(A + D)/D \subseteq \tau(M/D)$.
In this paper, \( \tau\)-\( H \)-supplement will mean that a direct summand \( D \) of \( M \) exists with the stated inclusions in Definition 1. The definition shows that every \( \tau \)-lifting module is \( \tau \)-\( H \)-supplemented.

Next we give an equivalent condition for a module to be \( \tau \)-\( H \)-supplemented.

**Proposition 1.** Let \( M \) be a module. Then \( M \) is \( \tau \)-\( H \)-supplemented if and only if for each \( A \leq M \) there exists a direct summand \( D \) of \( M \) and a submodule \( X \) of \( M \) such that \( A \subseteq X, D \subseteq X, X/A \subseteq \tau(M/A) \) and \( X/D \subseteq \tau(M/D) \).

**Proof.** (\( \Rightarrow \)) It is clear.

(\( \Leftarrow \)) Let \( A \leq M \). By assumption, there exist a direct summand \( D \) of \( M \) and \( X \leq M \) such that \( (A + D)/A \subseteq X/A \subseteq \tau(M/A) \) and \( (A + D)/D \subseteq X/D \subseteq \tau(M/D) \). Hence \( M \) is \( \tau \)-\( H \)-supplemented. \( \square \)

A factor module of a \( \tau \)-\( H \)-supplemented module need not be \( \tau \)-\( H \)-supplemented in general. Before giving a counter example to the fact that a factor module of a \( \tau \)-\( H \)-supplemented module need not be \( \tau \)-\( H \)-supplemented in case \( \tau = \text{Rad} \), we have to mention the following definitions:

A commutative ring \( R \) is a **valuation ring** if it satisfies one of the following three equivalent conditions:

1. For any two elements \( a \) and \( b \), either \( a \) divides \( b \) or \( b \) divides \( a \).
2. The ideals of \( R \) are linearly ordered by inclusion.
3. \( R \) is a local ring and every finitely generated ideal is principal.

A module \( M \) is called **finitely presented** if \( M \cong F/K \) for some finitely generated free module \( F \) and finitely generated submodule \( K \) of \( M \).

**Example 1.** Let \( R \) be a commutative local ring which is not a valuation ring and let \( n \geq 2 \). By [16, Theorem 2], there exists a finitely presented indecomposable module \( M = R^{(n)}/K \) which cannot be generated by fewer than \( n \) elements. By [5, Corollary 1.6], \( R^{(n)} \) is \( \bigoplus \)-supplemented and hence \( H \)-supplemented by [9, Proposition 2.1]. Being finitely generated, \( R^{(n)} \) is \( \text{Rad-H} \)-supplemented. Since \( M \) is not cyclic, it is not \( \bigoplus \)-supplemented, and hence not \( H \)-supplemented. Since \( M \) is finitely generated, it is not \( \text{Rad-H} \)-supplemented. (Note that since \( R/\text{Jac}R \) is semisimple, the preradical \( \text{Rad} \) is also cohereditary.)

In [8] and [10], the authors give some conditions for a factor module of an \( H \)-supplemented module to be \( H \)-supplemented. Now we give analogous of their conditions for a \( \tau \)-\( H \)-supplemented module.
Theorem 1. Let \( \tau \) be a cohereditary preradical. Let \( M \) be a \( \tau \)-H-supplemented module and \( X \leq M \). If for every direct summand \( K \) of \( M \), \( (X+K)/X \) is a direct summand of \( M/X \), then \( M/X \) is \( \tau \)-H-supplemented.

Proof. Let \( N/X \leq M/X \). Since \( M \) is \( \tau \)-H-supplemented, there exists a direct summand \( D \) of \( M \) such that \( (N+D)/N \subseteq \tau(M/N) \) and \( (N+D)/D \subseteq \tau(M/D) \). By assumption, \( (D+X)/X \) is a direct summand of \( M/X \). Since \( \tau \) is a cohereditary preradical, it is easy to check that 
\[
\frac{N/X+(D+X)/X}{N/X} \subseteq \tau\left(\frac{M/X}{N/X}\right) \quad \text{and} \quad \frac{N/X+(D+X)/X}{(D+X)/X} \subseteq \tau\left(\frac{M/X}{(D+X)/X}\right).
\]
Hence \( M/X \) is \( \tau \)-H-supplemented.

Let \( M \) be a module. Then \( M \) is called distributive if its lattice of submodules is a distributive lattice, equivalently for submodules \( K, L, N \) of \( M \), \( N+(K \cap L) = (N+K) \cap (N+L) \) or \( N \cap (K+L) = (N \cap K) + (N \cap L) \).

Corollary 1. Let \( M \) be a \( \tau \)-H-supplemented module for a cohereditary preradical \( \tau \).

1. If \( M \) is a distributive module, then \( M/X \) is \( \tau \)-H-supplemented for every submodule \( X \) of \( M \).
2. Let \( N \leq M \) such that for each decomposition \( M = M_1 \oplus M_2 \) we have \( N = (N \cap M_1) \oplus (N \cap M_2) \). Then \( M/N \) is \( \tau \)-H-supplemented.
3. Let \( X \) be a projection invariant submodule of \( M \). Then \( M/X \) is \( \tau \)-H-supplemented. In particular, for every fully invariant submodule \( A \) of \( M \), \( M/A \) is \( \tau \)-H-supplemented.

Proof. (1) Let \( D \) be a direct summand of \( M \). Then \( M = D \oplus D' \) for some submodule \( D' \) of \( M \). Now \( M/X = [(D+X)/X] + [(D'+X)/X] \) and \( X = X+(D \cap D') = (X+D) \cap (X+D') \). So \( M/X = [(D+X)/X] \oplus [(D'+X)/X] \). By Theorem 1, \( M/X \) is \( \tau \)-H-supplemented.

(2) Let \( L/N \leq M/N \). Since \( M \) is \( \tau \)-H-supplemented, there exists a direct summand \( D \) of \( M \) and a submodule \( X \) of \( M \) such that \( X/L \subseteq \tau(M/L) \) and \( X/D \subseteq \tau(M/D) \). Let \( M = D \oplus D' \). Then by hypothesis, \( N = (D \cap N) \oplus (D' \cap N) = (D+N) \cap (D'+N) \). So, \( (D+N)/N \oplus (D'+N)/N = M/N \). Now we have \( X/N \mid L/N \) and \( X/N \mid (D+N)/N \) and hence \( M/N \) is \( \tau \)-H-supplemented by Proposition 1.

(3) Clear by (2).

Proposition 2. Let \( M \) be a \( \tau \)-H-supplemented module for a cohereditary preradical \( \tau \) and \( N \leq M \). If for each idempotent \( e : M \to M \) there exists an idempotent \( f : M/N \to M/N \) such that \( \frac{(N+e(M))/N}{T/N} \subseteq \tau\left(\frac{M/N}{T/N}\right) \) where \( Imf = T/N \), then \( M/N \) is \( \tau \)-H-supplemented.

Proof. Let \( Y/N \leq M/N \). Since \( M \) is \( \tau \)-H-supplemented, there exists an idempotent \( e : M \to M \) and a submodule \( X \) of \( M \) such that \( X/e(M) \subseteq \)
$\tau(M/e(M))$ and $X/Y \subseteq \tau(M/Y)$ by Proposition 1. By hypothesis, there exists an idempotent $f : M/N \to M/N$ with $\text{Im} f = T/N$ such that $(N + e(M))/T \subseteq \tau(M/T)$. Now, $T/N$ is a direct summand of $M/N$ and $T/N \subseteq X/N$. Clearly $X/N \subseteq \tau(M/N)$ and $X/N \subseteq \tau(M/N)$.

Proposition 3. Let $\tau$ be a cohereditary preradical and $M_0$ a direct summand of a module $M$ such that for every decomposition $M = N \oplus K$ of $M$, there exist submodules $N'$ of $N$ and $K'$ of $K$ such that $M = M_0 \oplus N' \oplus K'$ with $\tau(K') = K'$. If $M$ is $\tau$-$H$-supplemented, then $M/M_0$ is $\tau$-$H$-supplemented.

Proof. Let $N/M_0 \leq M/M_0$. Since $M$ is $\tau$-$H$-supplemented, there exists a decomposition $M = K \oplus S$ such that $(N + K)/N \subseteq \tau(M/N)$ and $(N + K)/K \subseteq \tau(M/K)$. By hypothesis, $M = M_0 \oplus N' \oplus K'$ for $N' \leq K$ and $K' \leq S$ with $\tau(K') = K'$. Now it is easy to see that $(M_0 \oplus N')/M_0$ is a $\tau$-$H$-supplement of $N/M_0$ in $M/M_0$. $\square$

Let $M$ be an $R$-module and $\tau$ a preradical. By $P_\tau(M)$ we denote the sum of all submodules $N$ of $M$ with $\tau(N) = N$. The following Lemma will be very useful for us to prove Corollary 2.

Lemma 1. Let $\tau$ be any preradical and let $M$ be any module. Then

(1) $\tau(P_\tau(M)) = P_\tau(M)$.
(2) $P_\tau(M)$ is a fully invariant submodule of $M$.
(3) If $M = N \oplus K$, then $P_\tau(M) = P_\tau(N) \oplus P_\tau(K)$.

Corollary 2. Let $M$ be a $\tau$-$H$-supplemented module for a cohereditary preradical $\tau$. If $P_\tau(M)$ is a direct summand of $M$, then $P_\tau(M)$ and $M/P_\tau(M)$ are $\tau$-$H$-supplemented.

Proof. By Corollary 1(3) and Lemma 1(2), $M/P_\tau(M)$ is $\tau$-$H$-supplemented. Let $L$ be a submodule of $M$ such that $M = P_\tau(M) \oplus L$. Let $M = N \oplus K$. Now, by Lemma 1(3), $M = P_\tau(N) \oplus P_\tau(K) \oplus L$. Therefore $M/L \cong P_\tau(M)$ is $\tau$-$H$-supplemented by Proposition 3 and Lemma 1(1). $\square$

2. Direct summands of $\tau$-$H$-supplemented modules

In this section we will consider direct summands of $\tau$-$H$-supplemented modules. We investigate some conditions for direct summands of a $\tau$-$H$-supplemented module to be $\tau$-$H$-supplemented. We call a module $M$ completely $\tau$-$H$-supplemented provided every direct summand of $M$ is $\tau$-$H$-supplemented. The following Theorem is an analogue of [10, Theorem 2.7].
Theorem 2. (1) Every $\tau$-lifting module is completely $\tau$-$H$-supplemented.

(2) Let $M$ be a $\tau$-$H$-supplemented module for a cohereditary preradical $\tau$. If $M$ has the SSP, then $M$ is completely $\tau$-$H$-supplemented.

Proof. (1) It is clear since by [2, 2.10] every direct summand of a $\tau$-lifting module is again $\tau$-lifting.

(2) Assume that $M$ is $\tau$-$H$-supplemented and $M$ has the SSP. Let $N$ be a direct summand of $M$. We will show that $N$ is $\tau$-$H$-supplemented. Let $M = N \oplus N'$ for some submodule $N'$ of $M$. Suppose that $A$ is a direct summand of $M$. Since $M$ has the SSP, $A + N'$ is a direct summand of $M$. Let $M = (A + N') \oplus B$ for some $B \leq M$. Then $M/N' = (A + N')/N' \oplus (B + N')/N'$. Hence by Theorem 1, $M/N'$ is $\tau$-$H$-supplemented and so $N$ is $\tau$-$H$-supplemented.

Proposition 4. Let $M$ be a duo module. Then $M$ has the SSP.

Proof. See [10, Page 969].

Corollary 3. Let $\tau$ be a cohereditary preradical. Let $M$ be a $\tau$-$H$-supplemented duo module. Then $M$ is completely $\tau$-$H$-supplemented.

The following is an example for Theorem 2(2) in case $\tau = \text{Rad}$.

Example 2. Let $F$ be a field and $R$ the upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Since $R/JacR$ is semisimple, the preradical $\text{Rad}$ is cohereditary. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, let $M = A \oplus (R/B)$. Then $M$ is $H$-supplemented by [6, Lemma 3]. Also $M$ has the SSP. Therefore $M$ is a completely $\tau$-$H$-supplemented module by Theorem 2(2).

3. Direct sums of $\tau$-$H$-supplemented modules

The following example shows that any (finite) direct sum of $\tau$-$H$-supplemented modules need not be $\tau$-$H$-supplemented for $\tau = \text{Rad}$. We will show that under some conditions it will be true.

Example 3. Let $R$ be a commutative local ring and $M$ a finitely generated $R$-module. Assume $M \cong \bigoplus_{i=1}^{n} R/I_i$. Since every $I_i$ is fully invariant in $R$, every $R/I_i$ is $\tau$-$H$-supplemented by Corollary 1(3). By [11, Lemma A.4], $M$ is $\tau$-$H$-supplemented if $I_1 \leq I_2 \leq \ldots \leq I_n$. If we don’t have the condition $I_1 \leq I_2 \leq \ldots \leq I_n$, $M$ is not $\tau$-$H$-supplemented. (Note that since $M$ is finitely generated, $M$ is $H$-supplemented if and only if it is $\tau$-$H$-supplemented.)
We call a module $M$ $\tau$-semilocal provided that $M/\tau(M)$ is semisimple. Clearly $\tau$-supplemented modules are $\tau$-semilocal.

**Lemma 2.** Let $M$ be a $\tau$-$H$-supplemented module for a cohereditary preradical $\tau$. Then $M$ is $\tau$-semilocal.

**Proof.** Let $N/\tau(M) \leq M/\tau(M)$. Since $M$ is $\tau$-$H$-supplemented, there exists a direct summand $D$ of $M$ such that $(N + D)/N \subseteq \tau(M/N)$ and $(N + D)/D \subseteq \tau(M/D)$. Since $D \leq_d M$, $M = D \oplus D'$ for some submodule $D'$ of $M$. Then $M = D' + N$. It follows that $M/\tau(M) = N/\tau(M) + (D' + \tau(M))/\tau(M)$. Since $N \cap D' \subseteq \tau(D')$, $M/\tau(M) = N/\tau(M) \oplus (D' + \tau(M))/\tau(M)$. Hence $M/\tau(M)$ is semisimple.

**Proposition 5.** Let $M$ be a module. Then the following are equivalent for a cohereditary preradical $\tau$:

1. $M$ is $\tau$-$H$-supplemented;
2. $M$ is $\tau$-semilocal and each submodule (direct summand) of $M/\tau(M)$ lifts to a direct summand of $M$.

**Proof.** (1) ⇒ (2) By Lemma 2, we only prove the last statement. Let $N/\tau(M) \leq M/\tau(M)$. Since $M$ is $\tau$-$H$-supplemented, there exists $D \leq_d M$ such that $(N + D)/N \subseteq \tau(M/N)$ and $(N + D)/D \subseteq \tau(M/D)$. Then $D \subseteq N$. Hence $N/\tau(M) = (D + \tau(M))/\tau(M)$. This means $N/\tau(M)$ lifts to $D$.

(2) ⇒ (1) Let $N \leq M$. Then by assumption, $(N + \tau(M))/\tau(M) = \overline{N}$ is a direct summand of $M/\tau(M) = \overline{M}$. Then by assumption $\overline{N} = \overline{L}$ such that $M = L \oplus K$. The rest is easy by taking $L$ as a $\tau$-$H$-supplement of $N$ in $M$.

**Theorem 3.** Let $\tau$ be a cohereditary preradical. Let $M = \bigoplus_{i \in I} H_i$ be a direct sum of $\tau$-$H$-supplemented modules $H_i$ ($i \in I$). Assume that each direct summand of $M/\tau(M)$ lifts to a direct summand of $M$. Then $M$ is $\tau$-$H$-supplemented.

**Proof.** Clearly $M/\tau(M)$ is semisimple by Lemma 2. Now $M$ is $\tau$-$H$-supplemented by Proposition 5.

**Theorem 4.** Let $\tau$ be a cohereditary preradical. Let $M = M_1 \oplus M_2$ be a duo module. Then $M$ is $\tau$-$H$-supplemented if and only if $M_1$ and $M_2$ are $\tau$-$H$-supplemented.

**Proof.** Note that for $A \leq M$, we can write $A = (A \cap M_1) \oplus (A \cap M_2)$. ($\Rightarrow$) Assume that $M$ is $\tau$-$H$-supplemented. Since $M_1$ and $M_2$ are fully invariant submodules of $M$, $M_1$ and $M_2$ are $\tau$-$H$-supplemented by Corollary 1(3).
Suppose that \( M_1 \) and \( M_2 \) are \( \tau\)-\( H \)-supplemented. Let \( A \leq M \). Then 
\[
A = (A \cap M_1) \oplus (A \cap M_2).
\]
By assumption, there exist direct summands \( D_1 \) of \( M_1 \) and \( D_2 \) of \( M_2 \) such that 
\[
((A \cap M_1) + D_1)/(A \cap M_1) \subseteq \tau(M_1/(A \cap M_1)),
\]
\[
((A \cap M_1) + D_1)/D_1 \subseteq \tau(M_1/D_1)
\]
and 
\[
((A \cap M_2) + D_2)/(A \cap M_2) \subseteq \tau(M_2/(A \cap M_2)),
\]
\[
((A \cap M_2) + D_2)/D_2 \subseteq \tau(M_2/D_2).
\]
It is not hard to see that 
\[
(A + (D_1 + D_2))/A \subseteq \tau(M/A)
\]
and 
\[
(A + (D_1 + D_2))/(D_1 + D_2) \subseteq \tau(M/(D_1 + D_2)).
\]
Namely, \( D_1 \oplus D_2 \) is a \( \tau\)-\( H \)-supplement of \( A \) in \( M \). Hence \( M \) is \( \tau\)-\( H \)-supplemented.

**Definition 2.** Let \( M \) and \( N \) be two modules. Let \( \tau \) be a preradical. Then \( N \) is called \( \tau\)-\( M \)-projective if, for any \( K \leq M \) and any homomorphism 
\[
f : N \longrightarrow M/K
\]
there exists a homomorphism \( h : N \longrightarrow M \) such that 
\[
\text{Im}(f - \pi h) \subseteq \tau(M/K),
\]
where \( \pi : M \longrightarrow M/K \) is the natural epimorphism.

**Lemma 3.** Let \( M = M_1 \oplus M_2 \). Consider the following conditions:

1. \( M_1 \) is \( \tau\)-\( M_2 \)-projective;

2. For every \( K \leq M \) with \( K + M_2 = M \), there exists \( M_3 \leq M \) such that 
\[
M = M_2 \oplus M_3
\]
and \( (K + M_3)/K \subseteq \tau(M/K) \).

Then (1) \( \Rightarrow \) (2).

**Proof.** Let \( K \leq M \) and \( M = K + M_2 \). Consider the epimorphism \( \pi : M_2 \longrightarrow M/K \) with \( m_2 \mapsto m_2 + K(m_2 \in M_2) \) and the homomorphism 
\[
h : M_1 \longrightarrow M/K \text{ with } m_1 \mapsto m_1 + K(m_1 \in M_1).
\]
Since \( M_1 \) is \( \tau\)-\( M_2 \)-projective, there exist a homomorphism \( \overline{h} : M_1 \longrightarrow M_2 \) and a submodule \( X \) of \( M \) with \( K \subseteq X \) such that 
\[
\text{Im}(h - \pi \overline{h}) = X/K \subseteq \tau(M/K).
\]
Let \( M_3 = \{a - \overline{h}(a) \mid a \in M_1\} \). Clearly \( M = M_2 \oplus M_3 \). Since \( K + M_3 \subseteq X \), 
\[
(K + M_3)/K \subseteq X/K.
\]
Hence \( (K + M_3)/K \subseteq \tau(M/K) \).

**Lemma 4.** Let \( A \) and \( \{M_i\}_{i=1}^n \) be modules. If each \( M_i \) is \( \tau\)-\( A \)-projective, 
for \( i = 1, 2, \ldots n \), then \( \bigoplus_{i=1}^n M_i \) is \( \tau\)-\( A \)-projective.

**Proof.** The proof is straightforward.

**Theorem 5.** Let \( \tau \) be a cohereditary preradical. Let \( M = M_1 \oplus M_2 \) 
be a \( \tau\)-supplemented module. Assume \( M_1 \) is \( \tau\)-\( M_2 \)-projective (or \( M_2 \) is \( \tau\)-\( M_1 \)-projective). If \( M_1 \) and \( M_2 \) are \( \tau\)-\( H \)-supplemented, then \( M \) is \( \tau\)-\( H \)-supplemented.

**Proof.** Let \( Y \leq M \).

**Case 1:** Let \( M = Y + M_2 \). Then by Lemma 3, there exists \( M_3 \leq M \) such that 
\[
M = M_3 \oplus M_2 \text{ and } (Y + M_3)/Y \subseteq \tau(M/Y),
\]
Since \( M/M_3 \) is \( \tau\)-\( H \)-supplemented, there exist \( X/M_3 \leq M/M_3 \) and a direct summand \( D/M_3 \).
of \( M/M_3 \) such that \( \frac{X/M_3}{Y + M_3/M_3} \subseteq \tau(\frac{M/M_3}{Y + M_3/M_3}) \) and \( \frac{X/M_3}{D/M_3} \subseteq \tau(\frac{M/M_3}{D/M_3}) \) by Proposition 1. Clearly, \( D \) is a direct summand of \( M \). It is easy to check that \( X/D \subseteq \tau(M/D) \) and \( X/Y \subseteq \tau(M/Y) \). Therefore \( M \) is \( \tau-H \)-supplemented by Proposition 1.

**Case 2:** Let \( Y + M_2 \neq M \). Since \( M \) is \( \tau \)-supplemented, \( M/\tau(M) \) is semisimple. Then there exists a submodule \( K \) of \( M \) containing \( \tau(M) \) such that \( M/\tau(M) = K/\tau(M) \oplus (Y + M_2 + \tau(M))/\tau(M) \). So \( M = (K + Y) + M_2 \) and \( \tau(M) = K \cap (Y + M_2 + \tau(M)) = \tau(M) + (K \cap (Y + M_2)) \) and hence \( K \cap (Y + M_2) \subseteq \tau(M) \). By Lemma 3, there exists \( M_4 \leq M \) such that \( M = M_2 \oplus M_4 \) and \( (K + Y + M_4)/(K + Y) \subseteq \tau(M/(K + Y)) \). This implies that \( K + Y + M_4 \subseteq \tau(M) + K + Y = K + Y \). Now \( M/M_2 \) and \( M/M_4 \) are \( \tau-H \)-supplemented. Therefore there exist submodules \( X_1/M_2 \) of \( M/M_2 \) and \( X_2/M_4 \) of \( M/M_4 \) and direct summands \( D_1/M_2 \) of \( M/M_2 \) and \( D_2/M_4 \) of \( M/M_4 \) such that \( \frac{X_1/M_2}{Y + M_2/M_2} \subseteq \tau(\frac{M/M_2}{Y + M_2/M_2}) \), \( \frac{X_2/M_4}{D_1/M_2} \subseteq \tau(\frac{M/M_4}{D_1/M_2}) \), \( \frac{X_2/M_4}{Y + K + M_4/M_4} \subseteq \tau(\frac{M/M_4}{Y + K + M_4/M_4}) \) and \( \frac{X_2/M_4}{D_2/M_4} \subseteq \tau(\frac{M/M_4}{D_2/M_4}) \). Clearly, \( D_1 \cap D_2 \) is a direct summand of \( M \). Let \( M = (D_1 \cap D_2) \oplus L \) for some submodule \( L \) of \( M \). Then by [7, Lemma 1.2], \( M = D_2 \oplus (D_1 \cap L) \). Note that we have that \( X_1 \subseteq \tau(M) + D_1 \), \( X_1 \subseteq \tau(M) + M_2 + Y \), \( X_2 \subseteq \tau(M) + D_2 \) and \( X_2 \subseteq \tau(M) + Y + K + M_4 = K + Y \). Now,

\[
X_1 \cap X_2 \subseteq (\tau(M) + M_2 + Y) \cap (Y + K) = (\tau(M) + Y) + (M_2 \cap (Y + K)) \subseteq \tau(M) + Y + [K \cap (Y + M_2)] + [Y \cap (K + M_2)] = \tau(M) + Y
\]

and

\[
X_1 \cap X_2 \subseteq (\tau(M) + D_1) \cap (\tau(M) + D_2) = (\tau(D_2) + D_1) \cap (\tau(D_1 \cap L) + D_2) = \tau(D_2) + [(D_2 + \tau(D_1 \cap L)) \cap D_1] = \tau(D_2) + \tau(D_1 \cap L) + (D_1 \cap D_2) \subseteq \tau(M) + (D_1 \cap D_2).
\]

Therefore \( (X_1 \cap X_2)/Y \subseteq \tau(M/Y) \) and \( (X_1 \cap X_2)/(D_1 \cap D_2) \subseteq \tau(M/(D_1 \cap D_2)) \). Thus \( M \) is \( \tau-H \)-supplemented by Proposition 1.

**Corollary 4.** Let \( \tau \) be a cohereditary preradical. Let \( M = \oplus_{i=1}^n M_i \) be a \( \tau \)-supplemented module. Assume that \( M_i \) is \( \tau-M_j \)-projective for all \( j > i \). If each \( M_i \) is \( \tau-H \)-supplemented, then \( M \) is \( \tau-H \)-supplemented.

**Proof.** By Lemma 4 and Theorem 5.

\[ \square \]


4. Relations between $\tau$-$H$-supplemented modules and the others

A module $M$ is called $\tau$-$\oplus$-supplemented if for every $A \leq M$, there exists a $B \leq_d M$ such that $A + B = M$ and $A \cap B \subseteq \tau(B)$. Clearly every $\tau$-lifting module is $\tau$-$\oplus$-supplemented and every $\tau$-$\oplus$-supplemented module is $\tau$-supplemented.

Next we will show that under some conditions every $\tau$-$\oplus$-supplemented module is $\tau$-$H$-supplemented.

**Proposition 6.** Let $\tau$ be any preradical. Assume $M$ is $\tau$-$\oplus$-supplemented such that whenever $M = M_1 \oplus M_2$ then $M_1$ and $M_2$ are relatively projective. Then $M$ is $\tau$-$H$-supplemented.

**Proof.** Let $N \leq M$. Since $M$ is $\tau$-$\oplus$-supplemented, there exists a decomposition $M = M_1 \oplus M_2$ such that $M = N + M_2$ and $N \cap M_2 \subseteq \tau(M_2)$ for submodules $M_1, M_2$ of $M$. By hypothesis, $M_1$ is $M_2$-projective. By [11, Lemma 4.47], we obtain $M = A \oplus M_2$ for some submodule $A$ of $M$ such that $A \leq N$. Then $N = A \oplus (M_2 \cap N)$. It is easy to see that $(N + A)/A \subseteq \tau(M/A)$ and $(N + A)/N \subseteq \tau(M/N)$. Thus $M$ is $\tau$-$H$-supplemented.

**Corollary 5.** Let $\tau$ be any preradical. Let $M$ be a $\tau$-$\oplus$-supplemented module. If $M$ is projective, then $M$ is $\tau$-$H$-supplemented.

Let $e = e^2 \in R$. Then $e$ is called a left (right) semicentral idempotent if $xe = exe$ ($ex = exe$), for all $x \in R$. The set of all left (right) semicentral idempotents is denoted by $S_l(R)$ ($S_r(R)$). A ring $R$ is called Abelian if every idempotent is central.

**Proposition 7.** Let $\tau$ be a preradical and $M$ an $R$-module such that $\text{End}(M)$ is Abelian and $X \leq M$ implies $X = \sum_{i \in I} h_i(M)$ where $h_i \in \text{End}(M)$. If $M$ is $\tau$-$\oplus$-supplemented, then $M$ is $\tau$-$H$-supplemented and satisfies the $(D_3)$-condition.

**Proof.** Let $X \leq M$, $X = \sum_{i \in I} h_i(M)$ with $h_i \in \text{End}(M)$. Since $M$ is $\tau$-$\oplus$-supplemented, there exists a direct summand $eM$ such that $X + eM = M$ and $(X \cap eM) \subseteq \tau(eM)$ for some $e^2 = e \in \text{End}(M)$. Since $\text{End}(M)$ is Abelian, $(1-e)X = (1-e)M = (1-e)\sum_{i \in I} h_i(M) = \sum_{i \in I} h_i(1-e)(M) \subseteq X$. Therefore $X = (1-e)M \oplus (X \cap eM)$. Then $(1-e)M$ is a $\tau$-$H$-supplement of $X$. If $eM + fM = M$ for $e^2 = e$, $f^2 = f \in \text{End}(M)$, then $eM \cap fM = efM$ with $(ef)^2 = ef$. So $M$ satisfies the $(D_3)$-condition.

Recall that for a commutative ring $R$, an $R$-module $M$ is said to be a multiplication module if for each $X \leq M$, $X = MA$ for some ideal $A$ of $R$. 

**H-supplemented modules**
Corollary 6. Let $\tau$ be a preradical and $M$ a $\tau$$\oplus$-supplemented module. If $M$ satisfies one of the following conditions, then $M$ is $\tau$-$H$-supplemented.

(1) $M$ is a multiplication module and $R$ is commutative.
(2) $M$ is cyclic and $R$ is commutative.

Proof. (1) Assume $M$ is a multiplication module. Let $X \leq M$. Then $X = MA$ for some ideal $A$ of $R$. For each $a \in A$, define $h_a : M \to M$ by $h_a(m) = ma$ for all $m \in M$. Then $h_a$ is an $R$-homomorphism and $X = MA = \sum_{a \in A} h_a(M)$. Since every multiplication module is a duo module, thus if $e^2 = e \in S = \text{End}(M)$, then $e, 1-e \in S_1(S)$. Therefore $e$ is central. So $\text{End}(M)$ is Abelian. By Proposition 7, $M$ is $\tau$-$H$-supplemented.

(2) Clear by (1) since every cyclic module over a commutative ring is a multiplication module.

Now we investigate the relations between $\tau$-$H$-supplemented modules and the others. A module $M$ is called amply $\tau$-supplemented if for any submodules $K$ and $V$ of $M$ such that $M = K + V$, there is a submodule $U$ of $V$ such that $K + U = M$ and $K \cap U \subseteq \tau(U)$.

Lemma 5. Let $\tau$ be any preradical and let $M$ be a projective module. The following are equivalent:

(1) $M$ is $\tau$-supplemented;
(2) $M$ is amply $\tau$-supplemented.

Proof. Clearly an amply $\tau$-supplemented module is $\tau$-supplemented. For the converse: Let $M = U + V$ be a $\tau$-supplement of $U$ in $M$. For an $f \in \text{End}(M)$ with $\text{Im}(f) \subseteq V$ and $\text{Im}(I - f) \subseteq U$ we have $f(U) \subseteq U$, $M = U + f(X)$ and $f(U \cap X) = U \cap f(X)$ (from $u = f(x)$ we derive $x - u = (I - f)(x) \in U$ and $x \in U$). Since $U \cap X \subseteq \tau(X)$, we also have $U \cap f(X) \subseteq \tau(f(X))$, i.e. $f(X)$ is a $\tau$-supplement of $U$ with $f(X) \subseteq V$. Hence $M$ is amply $\tau$-supplemented.

Let $M$ be any module. A submodule $U$ of $M$ is called quasi strongly lifting (QSL) in $M$ if whenever $(A + U)/U$ is a direct summand of $M/U$, there exists a direct summand $P$ of $M$ such that $P \leq A$ and $P + U = A + U$ (see [1]).

Lemma 6. Let $\tau$ be a cohereditary preradical and let $M$ be any module. The following are equivalent:

(1) $M$ is $\tau$-lifting;
(2) $M$ is $\tau$-$H$-supplemented and $\tau(M)$ is QSL in $M$.

Proof. By Lemma 2 and [1, Lemma 3.5 and Proposition 3.6].
Lemma 7. Let $\tau$ be any preradical and let $M$ be a projective module such that every $\tau$-supplement submodule of $M$ is a direct summand of $M$. The following are equivalent:
(1) $M$ is $\tau$-supplemented;
(2) $M$ is amply $\tau$-supplemented;
(3) $M$ is $\tau$-lifting;
(4) $M$ is $\tau$-$\oplus$-supplemented.

Proof. (1) $\Leftrightarrow$ (2) By Lemma 5.
(1) $\Rightarrow$ (3) By [1, Lemma 3.2].
(3) $\Rightarrow$ (1) and (1) $\Leftrightarrow$ (4) are clear by definitions and the assumption that every $\tau$-supplement submodule of $M$ is a direct summand of $M$. \hfill $\Box$

Now we have the following Theorem:

Theorem 6. Let $\tau$ be a cohereditary preradical. Let $M$ be a projective module such that every $\tau$-supplement submodule of $M$ is a direct summand. The following are equivalent:
(1) $M$ is $\tau$-supplemented;
(2) $M$ is $\tau$-lifting;
(3) $M$ is amply $\tau$-supplemented;
(4) $M$ is $\tau$-$H$-supplemented and $\tau(M)$ is QSL in $M$;
(5) $M$ is $\tau$-$\oplus$-supplemented.

As we see in Example 3 a finite direct sum of $\tau$-$H$-supplemented modules need not be $\tau$-$H$-supplemented. We will show that a finite direct sum of $\tau$-$\oplus$-supplemented modules is $\tau$-$\oplus$-supplemented.

Lemma 8. Let $N, L \leq M$ such that $N + L$ has a $\tau$-supplement $H$ in $M$ and $N \cap (H + L)$ has a $\tau$-supplement $G$ in $N$. Then $H + G$ is a $\tau$-supplement of $L$ in $M$.

Proof. Let $H$ be a $\tau$-supplement of $N + L$ in $M$ and $G$ be a $\tau$-supplement of $N \cap (H + L)$ in $N$. Then $M = (N + L) + H$ such that $(N + L) \cap H \subseteq \tau(H)$ and $N = [N \cap (H + L)] + G$ such that $(H + L) \cap G \subseteq \tau(G)$. Since $(H + G) \cap L \subseteq [(G + L) \cap H] + [(H + L) \cap G] \subseteq \tau(H) + \tau(G) \subseteq \tau(H + G)$, $H + G$ is a $\tau$-supplement of $L$ in $M$. \hfill $\Box$

Theorem 7. For a ring $R$, any finite direct sum of $\tau$-$\oplus$-supplemented $R$-modules is $\tau$-$\oplus$-supplemented.

Proof. Let $M = M_1 \oplus \ldots \oplus M_n$ and $M_i$ be a $\tau$-$\oplus$-supplemented module for each $1 \leq i \leq n$. To prove that $M$ is $\tau$-$\oplus$-supplemented it is sufficient to assume $n = 2$. 

Let $L \leq M$. Then $M = M_1 + M_2 + L$ so that $M_1 + M_2 + L$ has a $\tau$-supplement $0$ in $M$. Let $H$ be a $\tau$-supplement of $M_2 \cap (M_1 + L)$ in $M_2$ such that $H \leq_d M_2$. By Lemma 8, $H$ is a $\tau$-supplement of $M_1 + L$ in $M$. Let $K$ be a $\tau$-supplement of $M_1 \cap (L + H)$ in $M_1$ such that $K \leq_d M_1$. Again by applying Lemma 8, we get that $H + K$ is a $\tau$-supplement of $L$ in $M$. Since $H \leq_d M_2$ and $K \leq_d M_1$, it follows that $H + K = H \oplus K \leq_d M$. Thus $M = M_1 \oplus M_2$ is $\tau$-supplemented.

Note that by the same proof as the proof of Theorem 7, any finite sum of $\tau$-supplemented modules is $\tau$-supplemented.

**Theorem 8.** Let $\tau$ be a cohereditary preradical. Let $R$ be a $\tau$-supplemented ring (i.e. $R_R$ is $\tau$-supplemented) such that every finite direct sum of the copies of $R$ is distributive. Then the following are equivalent:

1. $R$ is $\tau$-H-supplemented;
2. Every finitely generated free $R$-module is $\tau$-H-supplemented;
3. Every finitely generated projective $R$-module is $\tau$-H-supplemented;
4. If $F$ is a finitely generated free $R$-module and $N$ a fully invariant submodule, then $F/N$ is $\tau$-H-supplemented.

**Proof.** (1) $\Rightarrow$ (3) Let $M$ be a finitely generated projective $R$-module. Then $M$ is isomorphic to a direct summand of a finitely generated free module $F$. By Corollary 4, $F$ is $\tau$-H-supplemented. Thus $M$ is $\tau$-H-supplemented by Corollary 1(1).

(3) $\Rightarrow$ (2) $\Rightarrow$ (1) and (4) $\Rightarrow$ (1) are clear.

(2) $\Rightarrow$ (4) By (2), $F$ is $\tau$-H-supplemented. The result follows from Corollary 1(3). □

We next consider the preradical $\overline{Z}$.

Let $M$ be a module and $\mathcal{S}$ denote the class of all small modules. Talebi and Vanaja defined $\overline{Z}(M)$ in [13] as follows:

$\overline{Z}(M) = \bigcap \{\ker g \mid g \in \text{Hom}(M, L), L \in \mathcal{S}\}$. The module $M$ is called cosingular (non-cosingular) if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Clearly every non-cosingular module is $\overline{Z}$-H-supplemented. Also if $R$ is a non-cosingular ring, then every $R$-module is $\overline{Z}$-H-supplemented by [13, Proposition 2.5 and Corollary 2.6].

Let $M$ be a module and $\tau_M$ a preradical on $\sigma[M]$. In [12], the authors call a module $N \in \sigma[M]$ $\tau_M$-semiperfect if it satisfies one of the following conditions (see [12, Proposition 2.1 and Definition 2.2]):

1. For every submodule $K$ of $N$ there exists a decomposition $K = A \oplus B$ such that $A$ is a projective direct summand of $N$ in $\sigma[M]$ and $B \subseteq \tau_M(N)$;
(2) For every submodule $K$ of $N$, there exists a decomposition $N = A \oplus B$ such that $A$ is projective in $\sigma[M]$, $A \leq K$ and $K \cap B \subseteq \tau_M(N)$.

If $\sigma[M] = \text{Mod} - R$, then they call $N$ $\tau$-semiperfect.

By the above definition, every $\tau$-semiperfect module is $\tau$-lifting and hence $\tau$-$H$-supplemented. Also if $M$ is projective we have the following:

$\tau$-semiperfect $\iff$ $\tau$-lifting $\iff$ $\tau$-$\oplus$-supplemented $\Rightarrow$ $\tau$-$H$-supplemented

In [12, Theorem 2.23], the authors showed that their $\tau$-semiperfect module definition agrees with the definition of $\tau$-semiperfect module in the sense of [2] for a projective module and for the preradical $\text{Soc}$. In [14], Tribak and Keskin Tütüncü studied $\mathbb{Z}$-lifting modules and $\mathbb{Z}$-semiperfect modules in the sense of [12]. They also investigate some conditions for the preradical $\mathbb{Z}$ for two definitions of $\tau$-semiperfect modules to be agreed (see [14, Proposition 5.8 and Proposition 5.11]).

A $\tau$-$H$-supplemented module need not be $H$-supplemented. Of course if $\tau(M) \ll M$ and $\tau$ is cohereditary, then every $\tau$-$H$-supplemented module is $H$-supplemented.

\textbf{Example 4.} Let $K$ be a field and let $R = \prod_{n \geq 1} K_n$ with $K_n = K$. By [14, Example 4.1(1)] $R$ is not semiperfect. Since $R$ is projective, $R$ is not $\oplus$-supplemented by [5, Lemma 1.2]. Hence $R$ is not $H$-supplemented. Again by [14, Example 4.1(1)], the module $R$ is $\mathbb{Z}$-semiperfect in the sense of [12] and so it is $\mathbb{Z}$-$H$-supplemented.

If $R$ is a DVR (Discrete Valuation Ring), then the $R$-module $R$ is semiperfect and hence $H$-supplemented.

Now we give an equivalent condition for a module to be $\mathbb{Z}$-$\oplus$-supplemented module under some assumptions.

\textbf{Proposition 8.} Let $R$ be a commutative ring, $P$ a projective module with $\text{Rad}(P) \ll P$ and assume $P$ to have finite hollow dimension. Then the following are equivalent:

1. $P$ is $\mathbb{Z}$-$\oplus$-supplemented;
2. $P = P_1 \oplus P_2 \oplus P_3$ with $P_1$ $\oplus$-supplemented and $\text{Rad}(P_1) = \mathbb{Z}(P_1)$, $P_2$ semisimple and $\mathbb{Z}(P_3) = P_3$.

\textbf{Proof.} (1) $\Rightarrow$ (2) See the proof of [14, Corollary 4.3] and [5, Lemma 2.1].

(2) $\Rightarrow$ (1) By [14, Corollary 4.3] all $P_1$, $P_2$ and $P_3$ are $\mathbb{Z}$-semiperfect in the sense of [12] and hence $\mathbb{Z}$-$\oplus$-supplemented. By Theorem 7, $P$ is $\mathbb{Z}$-$\oplus$-supplemented. $\square$

\textbf{References}

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