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Free normal dibands

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ABSTRACT. We construct a free normal diband, a free $(\ell n, n)$-diband, a free (n, rn)-diband and a free $(\ell n, rn)$-diband. We also describe the structure of free normal dibands and characterize some least congruences on these dibands.

1. Introduction and preliminaries

The notions of a dialgebra and a dimonoid were introduced by J.-L. Loday [1]. For further details and background see [1], [10].

J.-L. Loday constructed a free dimonoid [1]. Pirashvili [3] introduced the notion of a duplex and constructed a free duplex. Dimonoids in the sense of Loday [1] are examples of duplexes. In [6] a free commutative dimonoid was constructed. Free rectangular dimonoids (rectangular dibands) were construcred in [9].

In this paper the research which was started in [6] and [9] is continued. Here we construct a free normal diband, a free $(\ell n, n)$-diband, a free (n, rn)-diband and a free $(\ell n, rn)$-diband. It turns out that the operations of a dimonoid with left (right) normal bands coincide and it is a left (right) normal band. We also describe the structure of free normal dibands and, as a consequence, obtain the description of some least congruences on free normal dibands.

We refer to [6] and [9] for the terminology and notations.

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Recall that an idempotent semigroup S is called a normal band, if $axya = ayxa$ for all $a, x, y \in S$. It is well-known that a normal band satisfies any identity of the form

$$ax_1x_2...x_nb = ax_{1\pi}x_{2\pi}...x_{n\pi}b, \tag{1}$$

where π is a permutation of $\{1, 2, ..., n\}$.

A dimonoid $(D, \dashv, \vdash)$ will be called a normal diband, if both semigroups $(D, \dashv)$ and $(D, \vdash)$ are normal bands.

Lemma 1. *([11], Sect. 3.5, Lemma) Let $(D, \dashv, \vdash)$ be an arbitrary dimonoid, $x,\ a_i \in D,\ 1 \leq i \leq n,\ n \in N,\ n > 1$. Then*

(i) $(a_n \dashv ... \dashv a_i \dashv ... \dashv a_1) \vdash x = a_n \vdash ... \vdash a_i \vdash ... \vdash a_1 \vdash x;$

(ii) $x \dashv (a_1 \vdash ... \vdash a_i \vdash ... \vdash a_n) = x \dashv a_1 \dashv ... \dashv a_i \dashv ... \dashv a_n.$

Lemma 2. *Let $(D, \dashv, \vdash)$ be an idempotent dimonoid. Then $(D, \dashv)$ is a normal band if and only if $(D, \vdash)$ is a normal band.*

Proof. If $(D, \dashv)$ is a normal band, $a, x, y \in D$, then

$$a \dashv x \dashv y \dashv a = a \dashv y \dashv x \dashv a.$$

Multiplying both parts of the last equality on the right by a concerning the operation $\vdash$, we obtain

$$(a \dashv x \dashv y \dashv a) \vdash a = a \vdash x \vdash y \vdash a \vdash a = a \vdash x \vdash y \vdash a,$$

$$(a \dashv y \dashv x \dashv a) \vdash a = a \vdash y \vdash x \vdash a \vdash a = a \vdash y \vdash x \vdash a$$

according to Lemma 1 (i) and the idempotent property of the operation $\vdash$. So, $(D, \vdash)$ is a normal band.

Conversely, let $(D, \vdash)$ be a normal band. Then

$$a \vdash x \vdash y \vdash a = a \vdash y \vdash x \vdash a$$

for all $a, x, y \in D$. Multiplying both parts of the last equality on the left by a concerning the operation $\dashv$, we obtain

$$a \dashv (a \vdash x \vdash y \vdash a) = a \dashv a \dashv x \dashv y \dashv a = a \dashv x \dashv y \dashv a,$$

$$a \dashv (a \vdash y \vdash x \vdash a) = a \dashv a \dashv y \dashv x \dashv a = a \dashv y \dashv x \dashv a$$

according to Lemma 1 (ii) and the idempotent property of the operation $\dashv$. So, $(D, \dashv)$ is a normal band. □

For an arbitrary nonempty set X denote the set of all nonempty finite subsets of X by $B[X]$.

Let $(D,\dashv,\vdash)$ be an arbitrary dimonoid and D be a totally ordered set. For every $A=\{x_1,x_2,...,x_n\}\in B[D]$ assume

$$\overrightarrow{A}=x_1\vdash x_2\vdash ...\vdash x_n,$$

$$\overleftarrow{A}=x_1\dashv x_2\dashv ...\dashv x_n,$$

where $x_1<x_2<...<x_n$ in the total order.

Using the identity (1), the idempotent property of the operations of a normal diband and Lemma 1, we can prove the following lemma.

Lemma 3. *Let $(D,\dashv,\vdash)$ be a normal diband, D be a totally ordered set and $A,B,C\in B[D], C\subseteq B$, $a\in A$, $x,y\in D$. Then*

(i) $x\vdash a\vdash \overrightarrow{A}=x\vdash \overrightarrow{A}$;

(ii) $\overleftarrow{A}\dashv a\dashv x=\overleftarrow{A}\dashv x$;

(iii) $\overrightarrow{A}\vdash a\vdash x=\overrightarrow{A}\vdash x=\overleftarrow{A}\vdash x$;

(iv) $x\dashv a\dashv \overleftarrow{A}=x\dashv \overleftarrow{A}=x\dashv \overrightarrow{A}$;

(v) $x\vdash \overrightarrow{A\cup B}\vdash y=x\vdash \overrightarrow{A}\vdash \overrightarrow{B}\vdash y=x\vdash \overleftarrow{A\cup B}\vdash y$;

(vi) $x\dashv \overleftarrow{A\cup B}\dashv y=x\dashv \overleftarrow{A}\dashv \overleftarrow{B}\dashv y=x\dashv \overrightarrow{A\cup B}\dashv y$;

(vii) $x\vdash \overrightarrow{B}\vdash \overrightarrow{C}\vdash y=x\vdash \overrightarrow{C}\vdash \overrightarrow{B}\vdash y=x\vdash \overrightarrow{B}\vdash y$;

(viii) $x\dashv \overleftarrow{B}\dashv \overleftarrow{C}\dashv y=x\dashv \overleftarrow{C}\dashv \overleftarrow{B}\dashv y=x\dashv \overleftarrow{B}\dashv y$.

Note that the class of normal dibands is a subclass of the variety of all dimonoids which is closed under the taking of homomorphic images, subdimonoids and Cartesian products. Therefore it is a subvariety of the variety of all dimonoids. A dimonoid which is free in the variety of normal dibands will be called a free normal diband.

The necessary information about varieties of dimonoids can be found in [6].

Now we consider a free rectangular dimonoid [9].

Let $I_n=\{1,2,...,n\}$, $n>1$ and let $\{X_i\}_{i\in I_n}$ be a family of arbitrary nonempty sets X_i, $i\in I_n$. Define the operations $\dashv$ and $\vdash$ on $\prod_{i\in I_n}X_i$ by

$$(x_1,...,x_n)\dashv(y_1,...,y_n)=(x_1,...,x_{n-1},y_n),$$

$$(x_1,...,x_n)\vdash(y_1,...,y_n)=(x_1,y_2,...,y_n)$$

for all $(x_1,...,x_n),\ (y_1,...,y_n)\in\prod_{i\in I_n}X_i$.

Lemma 4. *([9], Lemma 4) For any $n>1$, $(\prod_{i\in I_n}X_i,\dashv,\vdash)$ is a rectangular dimonoid.*

Obviously, for any $n > 1$, $(\prod_{i \in I_n} X_i, \dashv, \vdash)$ is a normal diband. Let X be an arbitrary nonempty set and $X^3 = X \times X \times X$. We denote the dimonoid $(X^3, \dashv, \vdash)$ by $FRct(X)$.

Theorem 1. *([9], Theorem 1) $FRct(X)$ is a free rectangular dimonoid.*

If $f : D_1 \to D_2$ is a homomorphism of dimonoids, then the corresponding congruence on D_1 will be denoted by Δ_f.

2. Free normal dibands

In this section we construct a free normal diband.

Let $\{D_i\}_{i \in I}$ be a family of arbitrary dimonoids D_i, $i \in I$ and let $\overline{\prod}_{i \in I} D_i$ be a set of all functions $f : I \to \bigcup_{i \in I} D_i$ such that $if \in D_i$ for any $i \in I$. It easy to check that $\overline{\prod}_{i \in I} D_i$ with multiplications defined by

$$i(f \dashv g) = if \dashv ig, \;\; i(f \vdash g) = if \vdash ig,$$

where $i \in I$, $f, g \in \overline{\prod}_{i \in I} D_i$, is a dimonoid. It is called the Cartesian product of dimonoids D_i, $i \in I$. Observe that if I is finite, then the Cartesian product and the direct product coincide. The Cartesian product of a finite number of dimonoids $D_1, D_2, ..., D_n$ is denoted by $D_1 \times D_2 \times ... \times D_n$.

Let $FRct(X)$ be the free rectangular dimonoid (see Sect. 1), $B(X)$ be the semilattice of all nonempty finite subsets of X with respect to the operation of the set theoretical union and let

$$FND(X) = \{((x, y, z), A) \in FRct(X) \times B(X) \,|\, x, y, z \in A\}.$$

The main result of this section is the following.

Theorem 2. *$FND(X)$ is a free normal diband.*

Proof. Clearly, $FRct(X) \times B(X)$ is a dimonoid (see above). It is not difficult to see that $FND(X)$ is a subdimonoid of $FRct(X) \times B(X)$. It is clear that the operations $\dashv$ and $\vdash$ of $FND(X)$ are idempotent. For all $((x, y, z), A), ((a, b, c), B), ((s, c, t), C) \in FND(X)$ we have

$$((x, y, z), A) \dashv ((a, b, c), B) \dashv ((s, c, t), C) \dashv ((x, y, z), A) =$$

$$= ((x, y, c), A \cup B) \dashv ((s, c, t), C) \dashv ((x, y, z), A) =$$

$$= ((x, y, t), A \cup B \cup C) \dashv ((x, y, z), A) = ((x, y, z), A \cup B \cup C),$$

$$((x, y, z), A) \dashv ((s, c, t), C) \dashv ((a, b, c), B) \dashv ((x, y, z), A) =$$

$$= ((x, y, t), A \cup C) \dashv ((a, b, c), B) \dashv ((x, y, z), A) =$$

$$= ((x, y, c), A \cup C \cup B) \dashv ((x, y, z), A) = ((x, y, z), A \cup C \cup B).$$

Hence $FND(X)$ is a normal band concerning the operation $\dashv$. By Lemma 2 $FND(X)$ is a normal band concerning the operation $\vdash$. So, $FND(X)$ is a normal diband.

Let us show that $FND(X)$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary normal diband, T be a totally ordered set and let $\gamma : X \to T$ be an arbitrary map. For every $A = \{x_1, x_2, ..., x_n\} \in B[X]$ assume $A_\gamma = \{x_i\gamma \,|\, 1 \le i \le n\}$ and define a map

$$\mu : FND(X) \to (T, \dashv', \vdash') : ((x, y, z), A) \mapsto ((x, y, z), A)\mu,$$

assuming

$$((x, y, z), A)\mu = x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' z\gamma$$

for all $((x, y, z), A) \in FND(X)$.

We show that μ is a homomorphism. We will use the axioms of a dimonoid, Lemma 3 and the idempotent property of the operations.

For arbitrary elements $((x, y, z), A)$, $((a, b, c), B) \in FND(X)$ we have

$$((x, y, z), A)\mu = x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' z\gamma,$$

$$((a, b, c), B)\mu = a\gamma \vdash' \overrightarrow{B_\gamma} \vdash' b\gamma \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma,$$

$$(((x, y, z), A) \dashv ((a, b, c), B))\mu = ((x, y, c), A \cup B)\mu =$$

$$= x\gamma \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma,$$

$$((x, y, z), A)\mu \dashv' ((a, b, c), B)\mu =$$

$$= (x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' z\gamma) \dashv' (a\gamma \vdash' \overrightarrow{B_\gamma} \vdash' b\gamma \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma) =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' z\gamma \dashv' a\gamma \dashv' \overrightarrow{B_\gamma} \dashv' b\gamma \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' z\gamma \dashv' a\gamma \dashv' \overleftarrow{B_\gamma} \dashv' b\gamma \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' \overleftarrow{B_\gamma} \dashv' b\gamma \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' \overleftarrow{B_\gamma} \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma} \dashv' \overleftarrow{B_\gamma} \dashv' c\gamma =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma =$$

$$= (x\gamma \vdash' \overrightarrow{A_\gamma}) \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$

$$= (x\gamma \vdash' \overrightarrow{A_\gamma}) \vdash' (y\gamma \dashv' \overrightarrow{(A \cup B)_\gamma} \dashv' c\gamma) \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$

$$= (x\gamma \vdash' \overrightarrow{A_\gamma}) \vdash' (y\gamma \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' c\gamma) \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$

$$= x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' c\gamma \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$
$$= x\gamma \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' c\gamma \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$
$$= x\gamma \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' (y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma) =$$
$$= x\gamma \vdash' \overrightarrow{(A \cup B)_\gamma} \vdash' y\gamma \dashv' \overleftarrow{(A \cup B)_\gamma} \dashv' c\gamma.$$

Thus,

$$(((x, y, z), A) \dashv ((a, b, c), B))\mu = ((x, y, z), A)\mu \dashv' ((a, b, c), B)\mu$$

for all $((x, y, z), A),\ ((a, b, c), B) \in FND(X)$. Analogously, we can prove that

$$(((x, y, z), A) \vdash ((a, b, c), B))\mu = ((x, y, z), A)\mu \vdash' ((a, b, c), B)\mu$$

for all $((x, y, z), A),\ ((a, b, c), B) \in FND(X)$. This completes the proof of Theorem 2. □

Obviously, the free normal diband $FND(X)$ generated by a finite set X is finite. Specifically, $|FND(X)| = \sum_{A \in B[X]} |A|^3$.

3. Dimonoids and (left, right) normal bands

In this section we show that the operations of a dimonoid $(D, \dashv, \vdash)$ with a left (respectively, right) normal band $(D, \vdash)$ (respectively, $(D, \dashv)$) coincide and construct a free $(\ell n, n)$-diband, a free (n, rn)-diband and a free $(\ell n, rn)$-diband.

Recall that an idempotent semigroup S is called a left normal band, if

$$axy = ayx \tag{2}$$

for all $a, x, y \in S$. If instead of (2) the identity

$$xya = yxa \tag{3}$$

holds, then S is a right normal band. It is well-known that a left normal band satisfies any identity of the form

$$ax_1x_2...x_n = ax_{1\pi}x_{2\pi}...x_{n\pi}, \tag{4}$$

where π is a permutation of $\{1, 2, ..., n\}$. Dually, a right normal band satisfies any identity of the form

$$x_1x_2...x_na = x_{1\pi}x_{2\pi}...x_{n\pi}a, \tag{5}$$

where π is a permutation of $\{1, 2, ..., n\}$.

Lemma 5. *The operations of a dimonoid* $(D, \dashv, \vdash)$ *coincide, if one of the following conditions holds:*

(i) $(D, \vdash)$ *is a left normal band;*

(ii) $(D, \dashv)$ *is a right normal band.*

Proof. (i) For all $x, y, z \in D$ we have

$$x \vdash (y \dashv z) = x \vdash (y \dashv z) \vdash (y \dashv z) =$$

$$= x \vdash (y \vdash z) \vdash (y \dashv z) = x \vdash (y \dashv z) \vdash (y \vdash z) =$$

$$= x \vdash (y \vdash z) \vdash (y \vdash z) = x \vdash (y \vdash z) =$$

$$= (x \vdash y) \vdash z = (x \vdash y) \dashv z$$

according to the idempotent property of the operation $\vdash$, the axioms of a dimonoid and the identity (2). Substituting $y = x$ in the last equality and using the idempotent property of the operation $\vdash$, we obtain $x \vdash z = x \dashv z$.

(ii) For all $x, y, z \in D$ we have

$$(x \vdash y) \dashv z = (x \vdash y) \dashv (x \vdash y) \dashv z =$$

$$= (x \vdash y) \dashv (x \dashv y) \dashv z = (x \dashv y) \dashv (x \vdash y) \dashv z =$$

$$= (x \dashv y) \dashv (x \dashv y) \dashv z = (x \dashv y) \dashv z =$$

$$= x \dashv (y \dashv z) = x \vdash (y \dashv z)$$

according to the idempotent property of the operation $\dashv$, the axioms of a dimonoid and the identity (3). Substituting $z = y$ in the last equality and using the idempotent property of the operation $\dashv$, we obtain $x \dashv y = x \vdash y$. □

From Lemma 5 (i) (respectively, Lemma 5 (ii)) it follows that a dimonoid $(D, \dashv, \vdash)$ with left (respectively, right) normal bands $(D, \dashv)$ and $(D, \vdash)$ is a left (respectively, right) normal band.

Consider the semigroups $X_{\ell z}$, X_{rz}, X_{rb} and the dimonoids $X_{\ell z,rz}$, $X_{rb,rz}$, $X_{\ell z,rb}$ which were defined in [9]. It is easy to see that $X_{\ell z}$, X_{rz}, X_{rb} are normal bands and $X_{\ell z,rz}$, $X_{rb,rz}$, $X_{\ell z,rb}$ are normal dibands.

Let

$$B_{rb}(X) = \{((x, y), A) \in X_{rb} \times B(X) \,|\, x, y \in A\},$$

$$B_{\ell z}(X) = \{(x, A) \in X_{\ell z} \times B(X) \,|\, x \in A\},$$

$$B_{rz}(X) = \{(x, A) \in X_{rz} \times B(X) \,|\, x \in A\},$$

$$B_{\ell z,rb}(X) = \{((x, y), A) \in X_{\ell z,rb} \times B(X) \,|\, x, y \in A\},$$

$$B_{rb,rz}(X) = \{((x, y), A) \in X_{rb,rz} \times B(X) \,|\, x, y \in A\},$$

$$B_{\ell z,rz}(X) = \{(x, A) \in X_{\ell z,rz} \times B(X) \,|\, x \in A\}.$$

It is clear that $B_{rb}(X)$, $B_{\ell z}(X)$, $B_{rz}(X)$ are subsemigroups of $X_{rb} \times B(X)$, $X_{\ell z} \times B(X)$, $X_{rz} \times B(X)$ respectively, and $B_{\ell z,rb}(X)$, $B_{rb,rz}(X)$, $B_{\ell z,rz}(X)$ are subdimonoids of $X_{\ell z,rb} \times B(X)$, $X_{rb,rz} \times B(X)$, $X_{\ell z,rz} \times B(X)$ respectively. By [2] $B_{rb}(X)$, $B_{\ell z}(X)$ and $B_{rz}(X)$ are the free normal band, the free left normal band and the free right normal band respectively.

A dimonoid $(D, \dashv, \vdash)$ will be called a $(\ell n, n)$-diband, if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a normal band. A dimonoid $(D, \dashv, \vdash)$ will be called a (n, rn)-diband, if $(D, \dashv)$ is a normal band and $(D, \vdash)$ is a right normal band. A dimonoid $(D, \dashv, \vdash)$ will be called a $(\ell n, rn)$-diband, if $(D, \dashv)$ is a left normal band and $(D, \vdash)$ is a right normal band.

Note that every left (right) normal band is normal and the class of $(\ell n, n)$-dibands ((n, rn)-dibands, $(\ell n, rn)$-dibands) is a subvariety of the variety of all normal dibands. A dimonoid which is free in the variety of $(\ell n, n)$-dibands (respectively, (n, rn)-dibands, $(\ell n, rn)$-dibands) will be called a free $(\ell n, n)$-diband (respectively, free (n, rn)-diband, free $(\ell n, rn)$-diband).

For the proofs of the following three lemmas we will use the notations from Sect. 1 and from the proof of Theorem 2.

Lemma 6. *$B_{\ell z,rb}(X)$ is a free $(\ell n, n)$-diband.*

Proof. Clearly, $B_{\ell z,rb}(X)$ is a $(\ell n, n)$-diband. Let us show that $B_{\ell z,rb}(X)$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary $(\ell n, n)$-diband, T be a totally ordered set and let $\gamma : X \to T$ be an arbitrary map. Define the map

$$\phi_{\ell n,n} : B_{\ell z,rb}(X) \to (T, \dashv', \vdash') :$$

$$((x, y), A) \mapsto ((x, y), A)\phi_{\ell n,n} = x\gamma \vdash' \overrightarrow{A_\gamma} \vdash' y\gamma \dashv' \overleftarrow{A_\gamma}.$$

Similarly to the proof of Theorem 2, we can show that $\phi_{\ell n,n}$ is a homomorphism. For this, we also use (4). □

Lemma 7. *$B_{rb,rz}(X)$ is a free (n, rn)-diband.*

Proof. Obviously, $B_{rb,rz}(X)$ is a (n, rn)-diband. Show that $B_{rb,rz}(X)$ is free.

Let $(T, \dashv', \vdash')$ be an arbitrary (n, rn)-diband, T be a totally ordered set and let $\gamma : X \to T$ be an arbitrary map. Define the map

$$\phi_{n,rn} : B_{rb,rz}(X) \to (T, \dashv', \vdash') :$$

$$((x, y), A) \mapsto ((x, y), A)\phi_{n,rn} = \overrightarrow{A_\gamma} \vdash' x\gamma \dashv' \overleftarrow{A_\gamma} \dashv' y\gamma.$$

Analysis similar to that in the proof of Theorem 2 shows that $\phi_{n,rn}$ is a homomorphism. Our proof also uses (5). □

Lemma 8. *$B_{\ell z,rz}(X)$ is a free $(\ell n, rn)$-diband.*

Proof. It is evident that $B_{\ell z,rz}(X)$ is a $(\ell n, rn)$-diband. Let $(T, \dashv', \vdash')$ be an arbitrary $(\ell n, rn)$-diband, T be a totally ordered set and let $\gamma : X \to T$ be an arbitrary map. Define the map

$$\phi_{\ell n,rn} : B_{\ell z,rz}(X) \to (T, \dashv', \vdash') :$$

$$(x, A) \mapsto (x, A)\phi_{\ell n,rn} = \overrightarrow{A_\gamma} \vdash' x\gamma \dashv' \overleftarrow{A_\gamma}.$$

Similarly to the proof of Theorem 2, the fact that $\phi_{\ell n,rn}$ is a homomorphism can be proved. To do this, also use (4) and (5). □

4. Decompositions of $FND(X)$

In this section we describe the structure of free normal dibands and characterize some least congruences on these dibands.

Let

$$B_{(i,j,k)}(X) = \{A \in B(X) \,|\, i, j, k \in A\},$$

$$B^{(i)}_{rb}(X) = \{((x, y), A) \in B_{rb}(X) \,|\, i \in A\},$$

$$B^{(i,j)}_{\ell z}(X) = \{(x, A) \in B_{\ell z}(X) \,|\, i, j \in A\},$$

$$B^{(i,j)}_{rz}(X) = \{(x, A) \in B_{rz}(X) \,|\, i, j \in A\},$$

$$B^{(i)}_{\ell z,rb}(X) = \{((x, y), A) \in B_{\ell z,rb}(X) \,|\, i \in A\},$$

$$B^{(i)}_{rb,rz}(X) = \{((x, y), A) \in B_{rb,rz}(X) \,|\, i \in A\},$$

$$B^{(i,j)}_{\ell z,rz}(X) = \{(x, A) \in B_{\ell z,rz}(X) \,|\, i, j \in A\}$$

for all $i, j, k \in X$. It is evident that $B_{(i,j,k)}(X)$, $B^{(i)}_{rb}(X)$, $B^{(i,j)}_{\ell z}(X)$, $B^{(i,j)}_{rz}(X)$ are subsemigroups of $B(X)$, $B_{rb}(X)$, $B_{\ell z}(X)$, $B_{rz}(X)$ respectively, and $B^{(i)}_{\ell z,rb}(X)$, $B^{(i)}_{rb,rz}(X)$, $B^{(i,j)}_{\ell z,rz}(X)$ are subdimonoids of $B_{\ell z,rb}(X)$, $B_{rb,rz}(X)$, $B_{\ell z,rz}(X)$ respectively.

For all $i, j, k \in X$ put

$$M_{(i,j,k)} = \{((x, y, z), A) \in FND(X) \,|\, (x, y, z) = (i, j, k)\},$$

$$M_{(i,j)} = \{((x, y, z), A) \in FND(X) \,|\, (x, y) = (i, j)\},$$

$$M_{(i,j]} = \{((x, y, z), A) \in FND(X) \,|\, (y, z) = (i, j)\},$$

$$M_{[i,j]} = \{((x, y, z), A) \in FND(X) \,|\, (x, z) = (i, j)\},$$

$$M_{(i)} = \{((x, y, z), A) \in FND(X) \,|\, x = i\},$$

$$M_{(i]} = \{((x,y,z),A) \in FND(X) \,|\, y = i\},$$
$$M_{[i]} = \{((x,y,z),A) \in FND(X) \,|\, z = i\};$$

for all $i, j \in X$, $Y \in B(X)$ such that $i, j \in Y$ put

$$M^Y_{(i,j)} = \{((x,y,z),A) \in FND(X) \,|\, ((x,y),A) = ((i,j),Y)\},$$
$$M^Y_{(i,j]} = \{((x,y,z),A) \in FND(X) \,|\, ((y,z),A) = ((i,j),Y)\},$$
$$M^Y_{[i,j]} = \{((x,y,z),A) \in FND(X) \,|\, ((x,z),A) = ((i,j),Y)\};$$

for all $i \in X$, $Y \in B(X)$ such that $i \in Y$ put

$$M^Y_{(i)} = \{((x,y,z),A) \in FND(X) \,|\, (x,A) = (i,Y)\},$$
$$M^Y_{(i]} = \{((x,y,z),A) \in FND(X) \,|\, (y,A) = (i,Y)\},$$
$$M^Y_{[i]} = \{((x,y,z),A) \in FND(X) \,|\, (z,A) = (i,Y)\};$$

for all $Y \in B(X)$ put

$$M^Y = \{((x,y,z),A) \in FND(X) \,|\, A = Y\}.$$

The notion of a diband of subdimonoids was introduced in [4] and investigated in [5] (see also [9]).

Subsequently, we will deal with diband decompositions and band decompositions of free normal dibands.

The following structure theorem gives decompositions of free normal dibands into dibands of subsemigroups.

Theorem 3. *Let $FND(X)$ be the free normal diband. Then*

(i) *$FND(X)$ is a rectangular diband $FRct(X)$ of subsemigroups $M_{(i,j,k)}$, $(i,j,k) \in FRct(X)$ such that $M_{(i,j,k)} \cong B_{(i,j,k)}(X)$ for every $(i,j,k) \in FRct(X)$;*

(ii) *$FND(X)$ is a diband $X_{\ell z,rb}$ of subsemigroups $M_{(i,j)}$, $(i,j) \in X_{\ell z,rb}$ such that $M_{(i,j)} \cong B^{(i,j)}_{rz}(X)$ for every $(i,j) \in X_{\ell z,rb}$;*

(iii) *$FND(X)$ is a diband $X_{rb,rz}$ of subsemigroups $M_{(i,j]}$, $(i,j) \in X_{rb,rz}$ such that $M_{(i,j]} \cong B^{(i,j)}_{\ell z}(X)$ for every $(i,j) \in X_{rb,rz}$;*

(iv) *$FND(X)$ is a left and right diband $X_{\ell z,rz}$ of subsemigroups $M_{(i]}$, $i \in X_{\ell z,rz}$ such that $M_{(i]} \cong B^{(i)}_{rb}(X)$ for every $i \in X_{\ell z,rz}$;*

(v) *$FND(X)$ is a diband $B_{\ell z,rb}(X)$ of subsemigroups $M^Y_{(i,j)}$, $((i,j),Y) \in B_{\ell z,rb}(X)$ such that $M^Y_{(i,j)} \cong Y_{rz}$ for every $((i,j),Y) \in B_{\ell z,rb}(X)$;*

(vi) *$FND(X)$ is a diband $B_{rb,rz}(X)$ of subsemigroups $M^Y_{(i,j]}$, $((i,j),Y) \in B_{rb,rz}(X)$ such that $M^Y_{(i,j]} \cong Y_{\ell z}$ for every $((i,j),Y) \in B_{rb,rz}(X)$;*

(vii) *$FND(X)$ is a diband $B_{\ell z,rz}(X)$ of subsemigroups $M^Y_{(i]}$, $(i,Y) \in B_{\ell z,rz}(X)$ such that $M^Y_{(i]} \cong Y_{rb}$ for every $(i,Y) \in B_{\ell z,rz}(X)$.*

Proof. (i) By Theorem 2 the map

$$\mu_{FRct} : FND(X) \to FRct(X) :$$

$$((x,y,z),A) \mapsto ((x,y,z),A)\mu_{FRct} = (x,y,z)$$

is a homomorphism. It is clear that $M_{(i,j,k)}, (i,j,k) \in FRct(X)$ is a class of $\Delta_{\mu_{FRct}}$ which is a subdimonoid of $FND(X)$. If $((x,y,z),A)$, $((a,b,c),B) \in M_{(i,j,k)}$, then $x = a = i$, $y = b = j$, $z = c = k$ and

$$((x,y,z),A) \dashv ((a,b,c),B) = ((x,y,c),A \cup B) = ((i,j,k),A \cup B),$$

$$((x,y,z),A) \vdash ((a,b,c),B) = ((x,b,c),A \cup B) = ((i,j,k),A \cup B).$$

Hence the operations of $M_{(i,j,k)}$ coincide and so, it is a semigroup. It is not difficult to show that for every $(i,j,k) \in FRct(X)$ the map

$$M_{(i,j,k)} \to B_{(i,j,k)}(X) : ((i,j,k),A) \mapsto A$$

is an isomorphism.

(ii) By Theorem 2 the map

$$\mu_{\ell z,rb} : FND(X) \to X_{\ell z,rb} : ((x,y,z),A) \mapsto ((x,y,z),A)\mu_{\ell z,rb} = (x,y)$$

is a homomorphism. It is evident that $M_{(i,j)}$, $(i,j) \in X_{\ell z,rb}$ is a class of $\Delta_{\mu_{\ell z,rb}}$ which is a subdimonoid of $FND(X)$. If $((x,y,z),A)$, $((a,b,c),B) \in M_{(i,j)}$, then $x = a = i$, $y = b = j$. Similarly to (i), the operations of $M_{(i,j)}$ coincide and so, it is a semigroup. It is easy to check that for every $(i,j) \in X_{\ell z,rb}$ the map

$$M_{(i,j)} \to B^{(i,j)}_{rz}(X) : ((i,j,z),A) \mapsto (z,A)$$

is an isomorphism.

(iii) By Theorem 2 the map

$$\mu_{rb,rz} : FND(X) \to X_{rb,rz} : ((x,y,z),A) \mapsto ((x,y,z),A)\mu_{rb,rz} = (y,z)$$

is a homomorphism. Similarly to (ii), $M_{(i,j]}$, $(i,j) \in X_{rb,rz}$ is a class of $\Delta_{\mu_{rb,rz}}$ which is a semigroup isomorphic to $B^{(i,j)}_{\ell z}(X)$.

(iv) By Theorem 2 the map

$$\mu_{\ell z,rz} : FND(X) \to X_{\ell z,rz} : ((x,y,z),A) \mapsto ((x,y,z),A)\mu_{\ell z,rz} = y$$

is a homomorphism. Then $M_{(i]}$, $i \in X_{\ell z,rz}$ is a class of $\Delta_{\mu_{\ell z,rz}}$ which is a subdimonoid of $FND(X)$. If $((x,y,z),A)$, $((a,b,c),B) \in M_{(i]}$, then

$y = b = i$. Similarly to (i), the operations of $M_{(i]}$ coincide and so, it is a semigroup. It is easily seen that for every $i \in X_{\ell z,rz}$ the map

$$M_{(i]} \to B^{(i)}_{rb}(X) : ((x,i,z),A) \mapsto ((x,z),A)$$

is an isomorphism.

(v) By Theorem 2 the map

$$\mu^*_{\ell z,rb} : FND(X) \to B_{\ell z,rb}(X) :$$

$$((x,y,z),A) \mapsto ((x,y,z),A)\mu^*_{\ell z,rb} = ((x,y),A)$$

is a homomorphism. Then $M^Y_{(i,j)}$, $((i,j),Y) \in B_{\ell z,rb}(X)$ is a class of $\Delta_{\mu^*_{\ell z,rb}}$ which is a subdimonoid of $FND(X)$. If $((x,y,z),A), ((a,b,c),B) \in M^Y_{(i,j)}$, then $x = a = i$, $y = b = j$, $A = B = Y$. Similarly to (i), the operations of $M^Y_{(i,j)}$ coincide and so, it is a semigroup. It is immediate to check that for every $((i,j),Y) \in B_{\ell z,rb}(X)$ the map

$$M^Y_{(i,j)} \to Y_{rz} : ((i,j,z),Y) \mapsto z$$

is an isomorphism.

(vi) By Theorem 2 the map

$$\mu^*_{rb,rz} : FND(X) \to B_{rb,rz}(X) :$$

$$((x,y,z),A) \mapsto ((x,y,z),A)\mu^*_{rb,rz} = ((y,z),A)$$

is a homomorphism. Similarly to (v), $M^Y_{(i,j]}$, $((i,j),Y) \in B_{rb,rz}(X)$ is a class of $\Delta_{\mu^*_{rb,rz}}$ which is a semigroup isomorphic to $Y_{\ell z}$.

(vii) By Theorem 2 the map

$$\mu^*_{\ell z,rz} : FND(X) \to B_{\ell z,rz}(X) :$$

$$((x,y,z),A) \mapsto ((x,y,z),A)\mu^*_{\ell z,rz} = (y,A)$$

is a homomorphism. Similarly to (iv), $M^Y_{(i]}$, $(i,Y) \in B_{\ell z,rz}(X)$ is a class of $\Delta_{\mu^*_{\ell z,rz}}$ which is a semigroup isomorphic to Y_{rb}. □

If ρ is a congruence on a dimonoid $(D, \dashv, \vdash)$ such that $(D, \dashv, \vdash)/\rho$ is a $(\ell n, n)$-diband (respectively, (n, rn)-diband, $(\ell n, rn)$-diband), then we say that ρ is a $(\ell n, n)$-congruence (respectively, (n, rn)-congruence, $(\ell n, rn)$-congruence).

Using the terminology of [9], from Theorem 3 we obtain

Corollary 1. *Let $FND(X)$ be the free normal diband. Then*

(i) $\Delta_{\mu_{FRct}}$ *is the least rectangular diband congruence on* $FND(X)$*;*

(ii) $\Delta_{\mu_{\ell z,rb}}$ *is the least* $(\ell z, rb)$*-congruence on* $FND(X)$*;*

(iii) $\Delta_{\mu_{rb,rz}}$ *is the least* (rb, rz)*-congruence on* $FND(X)$*;*

(iv) $\Delta_{\mu_{\ell z,rz}}$ *is the least left zero and right zero congruence on* $FND(X)$*;*

(v) $\Delta_{\mu^*_{\ell z,rb}}$ *is the least* $(\ell n, n)$*-congruence on* $FND(X)$*;*

(vi) $\Delta_{\mu^*_{rb,rz}}$ *is the least* (n, rn)*-congruence on* $FND(X)$*;*

(vii) $\Delta_{\mu^*_{\ell z,rz}}$ *is the least* $(\ell n, rn)$*-congruence on* $FND(X)$*.*

Proof. (i) By Theorem 1 $FRct(X)$ is the free rectangular dimonoid. According to Theorem 3 (i) we obtain (i).

(ii) By Lemma 7 from [9] $X_{\ell z,rb}$ is the free $(\ell z, rb)$-dimonoid. According to Theorem 3 (ii) we obtain (ii).

The proof of (iii) is similar.

(iv) By Lemma 5 from [9] $X_{\ell z,rz}$ is the free left zero and right zero dimonoid. According to Theorem 3 (iv) we obtain (iv).

(v) By Lemma 6 $B_{\ell z,rb}(X)$ is the free $(\ell n, n)$-diband. According to Theorem 3 (v) we obtain (v).

The proof of (vi) is similar.

(vii) By Lemma 8 $B_{\ell z,rz}(X)$ is the free $(\ell n, rn)$-diband. According to Theorem 3 (vii) we obtain (vii). □

The following structure theorem gives decompositions of free normal dibands into bands of subdimonoids.

Theorem 4. *Let $FND(X)$ be the free normal diband. Then*

(i) *$FND(X)$ is a rectangular band X_{rb} of subdimonoids $M_{[i,j]}$, $(i, j) \in X_{rb}$ such that $M_{[i,j]} \cong B^{(i,j)}_{\ell z,rz}(X)$ for every $(i, j) \in X_{rb}$;*

(ii) *$FND(X)$ is a left band $X_{\ell z}$ of subdimonoids $M_{(i)}$, $i \in X_{\ell z}$ such that $M_{(i)} \cong B^{(i)}_{rb,rz}(X)$ for every $i \in X_{\ell z}$;*

(iii) *$FND(X)$ is a right band X_{rz} of subdimonoids $M_{[i]}$, $i \in X_{rz}$ such that $M_{[i]} \cong B^{(i)}_{\ell z,rb}(X)$ for every $i \in X_{rz}$;*

(iv) *$FND(X)$ is a normal band $B_{rb}(X)$ of subdimonoids $M^Y_{[i,j]}$, $((i, j), Y) \in B_{rb}(X)$ such that $M^Y_{[i,j]} \cong Y_{\ell z,rz}$ for every $((i, j), Y) \in B_{rb}(X)$;*

(v) *$FND(X)$ is a left normal band $B_{\ell z}(X)$ of subdimonoids $M^Y_{(i)}$, $(i, Y) \in B_{\ell z}(X)$ such that $M^Y_{(i)} \cong Y_{rb,rz}$ for every $(i, Y) \in B_{\ell z}(X)$;*

(vi) *$FND(X)$ is a right normal band $B_{rz}(X)$ of subdimonoids $M^Y_{[i]}$, $(i, Y) \in B_{rz}(X)$ such that $M^Y_{[i]} \cong Y_{\ell z,rb}$ for every $(i, Y) \in B_{rz}(X)$;*

(vii) *$FND(X)$ is a semilattice $B(X)$ of subdimonoids M^Y, $Y \in B(X)$ such that $M^Y \cong FRct(Y)$ for every $Y \in B(X)$.*

Proof. (i) By Theorem 2 the map

$$\mu_{rb} : FND(X) \to X_{rb} : ((x, y, z), A) \mapsto ((x, y, z), A)\mu_{rb} = (x, z)$$

is a homomorphism. It is clear that $M_{[i,j]}$, $(i, j) \in X_{rb}$ is a class of $\Delta_{\mu_{rb}}$ which is a subdimonoid of $FND(X)$. It can be shown that for every $(i, j) \in X_{rb}$ the map

$$M_{[i,j]} \to B^{(i,j)}_{\ell z,rz}(X) : ((i, y, j), A) \mapsto (y, A)$$

is an isomorphism.

(ii) By Theorem 2 the map

$$\mu_{\ell z} : FND(X) \to X_{\ell z} : ((x, y, z), A) \mapsto ((x, y, z), A)\mu_{\ell z} = x$$

is a homomorphism. It is evident that $M_{(i)}$, $i \in X_{\ell z}$ is a class of $\Delta_{\mu_{\ell z}}$ which is a subdimonoid of $FND(X)$. It is easy to check that for every $i \in X_{\ell z}$ the map

$$M_{(i)} \to B^{(i)}_{rb,rz}(X) : ((i, y, z), A) \mapsto ((y, z), A)$$

is an isomorphism.

(iii) By Theorem 2 the map

$$\mu_{rz} : FND(X) \to X_{rz} : ((x, y, z), A) \mapsto ((x, y, z), A)\mu_{rz} = z$$

is a homomorphism. Similarly to (ii), $M_{[i]}$, $i \in X_{rz}$ is a class of $\Delta_{\mu_{rz}}$ which is a dimonoid isomorphic to $B^{(i)}_{\ell z,rb}(X)$.

(iv) By Theorem 2 the map

$$\mu^*_{rb} : FND(X) \to B_{rb}(X) : ((x, y, z), A) \mapsto ((x, y, z), A)\mu^*_{rb} = ((x, z), A)$$

is a homomorphism. Similarly to (i), $M^Y_{[i,j]}$, $((i, j), Y) \in B_{rb}(X)$ is a class of $\Delta_{\mu^*_{rb}}$ which is a dimonoid isomorphic to $Y_{\ell z,rz}$.

(v) By Theorem 2 the map

$$\mu^*_{\ell z} : FND(X) \to B_{\ell z}(X) : ((x, y, z), A) \mapsto ((x, y, z), A)\mu^*_{\ell z} = (x, A)$$

is a homomorphism. It is clear that $M^Y_{(i)}$, $(i, Y) \in B_{\ell z}(X)$ is a class of $\Delta_{\mu^*_{\ell z}}$ which is a subdimonoid of $FND(X)$. It can be shown that for every $(i, Y) \in B_{\ell z}(X)$ the map

$$M^Y_{(i)} \to Y_{rb,rz} : ((i, y, z), Y) \mapsto (y, z)$$

is an isomorphism.

(vi) By Theorem 2 the map

$$\mu^*_{rz} : FND(X) \to B_{rz}(X) : ((x,y,z),A) \mapsto ((x,y,z),A)\mu^*_{rz} = (z,A)$$

is a homomorphism. Similarly to (v), $M^Y_{[i]}$, $(i,Y) \in B_{rz}(X)$ is a class of $\Delta_{\mu^*_{rz}}$ which is a dimonoid isomorphic to $Y_{\ell z,rb}$.

(vii) By Theorem 2 the map

$$\mu^* : FND(X) \to B(X) : ((x,y,z),A) \mapsto ((x,y,z),A)\mu^* = A$$

is a homomorphism. Clearly, M^Y, $Y \in B(X)$ is a class of Δ_{μ^*} which is a subdimonoid of $FND(X)$. One can show that for every $Y \in B(X)$ the map

$$M^Y \to FRct(Y) : ((x,y,z),Y) \mapsto (x,y,z)$$

is an isomorphism. □

If ρ is a congruence on a dimonoid $(D,\dashv,\vdash)$ such that the operations of $(D,\dashv,\vdash)/\rho$ coincide and it is a (left, right) normal band, then we say that ρ is a (left, right) normal band congruence.

Using the terminology of [9], from Theorem 4 we obtain

Corollary 2. *Let $FND(X)$ be the free normal diband. Then*

(i) $\Delta_{\mu_{rb}}$ *is the least rectangular band congruence on $FND(X)$;*

(ii) $\Delta_{\mu_{\ell z}}$ *is the least left zero congruence on $FND(X)$;*

(iii) $\Delta_{\mu_{rz}}$ *is the least right zero congruence on $FND(X)$;*

(iv) $\Delta_{\mu^*_{rb}}$ *is the least normal band congruence on $FND(X)$;*

(v) $\Delta_{\mu^*_{\ell z}}$ *is the least left normal band congruence on $FND(X)$;*

(vi) $\Delta_{\mu^*_{rz}}$ *is the least right normal band congruence on $FND(X)$;*

(vii) Δ_{μ^*} *is the least semilattice congruence on $FND(X)$.*

Proof. (i) X_{rb} is the free rectangular band (see Sect. 3 of [9]). By Theorem 4 (i) we obtain (i).

(ii) It is well-known that $X_{\ell z}$ is the free left zero semigroup. By Theorem 4 (ii) we obtain (ii).

The proof of (iii) is similar.

(iv) $B_{rb}(X)$ is the free normal band (see Sect. 3). By Theorem 4 (iv) we obtain (iv).

(v) $B_{\ell z}(X)$ is the free left normal band (see Sect. 3). By Theorem 4 (v) we obtain (v).

The proof of (vi) is similar.

(vii) It is well-known that $B(X)$ is the free semilattice. By Theorem 4 (vii) we obtain (vii). □

Note that the least congruences on dimonoids and the corresponding decompositions of these dimonoids were also described in [4] and [6–9].

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