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Derivations and relation modules for inverse semigroups

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Abstract. We define the derivation module for a homomorphism of inverse semigroups, generalizing a construction for groups due to Crowell. For a presentation map from a free inverse semigroup, we can then define its relation module as the kernel of a canonical map from the derivation module to the augmentation module. The constructions are analogues of the first steps in the Gruenberg resolution obtained from a group presentation. We give a new proof of the characterization of inverse monoids of cohomological dimension zero, and find a class of examples of inverse semigroups of cohomological dimension one.

1. Introduction

A cohomology theory for inverse semigroups was established in the fundamental work of Lausch [10] and Loganathan [14] but since then, there have been limited applications of homological algebra to inverse semigroups. However, important contributions have been made in closely related areas. These include an approach, due to Steinberg [18], to the study of homomorphisms of inverse semigroups based on derived categories, and developed in [18] in the wider context of morphisms of ordered groupoids; the topology of 2–complex models of inverse semigroup presentations and its application to amalgams, also due to Steinberg [19]; homotopy theory in the category of ordered groupoids [12], leading to an alternative proof

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of Steinberg's Factorization Theorem; and Funk's [4] study of the topos of representations of an inverse semigroup.

A key step in [14] was the reformulation of the module theory for an inverse semigroup S, due to Lausch [10], as the module theory of a left cancellative category $\mathfrak{L}(S)$. An S–module is then a functor $\mathfrak{L}(S) \to \mathbf{Ab}$ from $\mathfrak{L}(S)$ to the category of abelian groups. This enables the application of the (co)homology theory of categories to be applied directly to the study of inverse semigroups: for example, Loganathan [14, Theorem 4.5(i)] shows that free inverse monoids have cohomological dimension one. Recent work of Webb [22, 23] on the cohomology of categories has developed ideas from group representation theory, such as relation modules and the Schur multiplier, and formulated them for categories. Webb's results, when applied to Loganathan's category $\mathfrak{L}(S)$, give us an entrée to the ideas of augmentation and relation modules for inverse semigroups: the starting point in [23] is the Gruenberg resolution, whose construction we now recall.

Let $\mathcal{P} = \langle X : R \rangle$ be a group presentation of a group G. Gruenberg [6] (and see also [7, chapter 3]) gave a functorial construction from $\mathcal{P}$, of a free resolution of the trivial G–module $\mathbb{Z}$. This *Gruenberg resolution* has a number of interesting properties and we outline the construction, referring to [6, 7] for details.

Let $F = F(X)$ be the free group on X, let N be the normal closure of R in $F(X)$, and let $\theta : F \to G$ be the natural map with kernel N. Then θ induces a map $\theta : \mathbb{Z}F \to \mathbb{Z}G$ of integral group rings, and we let $\mathfrak{r}$ be its kernel. The augmentation ideal $\mathfrak{f}$ of F is the kernel of the augmentation map $\mathbb{Z}F \to \mathbb{Z}$ induced by mapping $w \mapsto 1$ for all $w \in F$: clearly $\mathfrak{r}$ and $\mathfrak{f}$ are two-sided ideals.

Theorem (Gruenberg, [6])**.** *The complex of $\mathbb{Z}G$–modules*

$$\ldots \to \mathfrak{r}^2/\mathfrak{r}^3 \to \mathfrak{f}\mathfrak{r}/\mathfrak{f}\mathfrak{r}^2 \to \mathfrak{r}/\mathfrak{r}^2 \to \mathfrak{f}/\mathfrak{f}\mathfrak{r} \to \mathbb{Z}G \to \mathbb{Z} \to 0$$

is a G–free resolution of $\mathbb{Z}$, and this construction gives a functor from the category of free presentations of G to the category of G–free resolutions of $\mathbb{Z}$.

Gruenberg [8] went on to show that his resolution gave rise to explicit formulae for the homology groups of G in terms of $\mathfrak{r}$ and $\mathfrak{f}$, generalising the Hopf formula for $H_2(G, \mathbb{Z})$.

The kernel of the map $\mathfrak{f}/\mathfrak{f}\mathfrak{r} \to \mathbb{Z}G$ is $\mathfrak{r}/\mathfrak{f}\mathfrak{r}$ and Gruenberg [6] shows that this is isomorphic to the *relation module* N^{ab}. (Indeed, more is true, as shown in [6]: the kernel of $\mathfrak{f}\mathfrak{r}^{n-1}/\mathfrak{f}\mathfrak{r}^n \to \mathfrak{r}^{n-1}/\mathfrak{r}^n$ is isomorphic to the tensor product $N^{ab} \otimes \cdots \otimes N^{ab}$ of n copies of the relation module, with the diagonal G–action.) Defining the relation module in this way permits the introduction of the concept in other algebraic settings where the operation of abelianisation has no obvious counterpart.

For a small category $\mathcal{C}$, the integral category algebra is the free abelian group $\mathbb{Z}\mathcal{C}$ having the morphisms of $\mathcal{C}$ as a basis. Multiplication is defined by the $\mathbb{Z}$–linear extension of the composition of morphisms in $\mathcal{C}$, with undefined compositions being set equal to zero. An inverse semigroup S therefore determines the category algebra $\mathbb{Z}\mathfrak{L}(S)$. However, Loganathan defines $\mathbb{Z}S$ as a $\mathfrak{L}(S)$–module determined by the free abelian groups on Green's $\mathcal{L}$–classes in S (see section 2 below). Beginning with Loganathan's definition leads to definitions of the augmentation ideal and relation modules differing from those that result from following the constructions of [23] for the category algebra $\mathbb{Z}\mathfrak{L}(S)$.

We shall begin with Loganathan's $\mathbb{Z}S$, and blend some of the ideas of [14] with Webb's approach to the cohomology of categories, so that we are able to define the relation module of an inverse semigroup presentation. The key ingredient is the construction of an $\mathfrak{L}(S)$–module D that corresponds to the module $\mathfrak{f}/\mathfrak{f}\mathfrak{r}$ in Gruenberg's resolution. Our construction generalises Crowell's *derived module* of a group homomorphism [3], and is based on the derived module of a morphism of groupoids as defined by Brown and Higgins in [2]. After some preliminaries on Loganathan's category $\mathfrak{L}(S)$ and $\mathfrak{L}(S)$–modules in section 2, we define the derivation module D_ϕ of an inverse semigroup homomorphism ϕ in section 3 and establish its adjoint relationship to the semidirect products of S with $\mathfrak{L}(S)$–modules. For subsequent applications, the most important result of section 3 is that if S is an inverse monoid presented by $\langle X : R \rangle$, and if $\theta : \mathrm{FIM}(X) \to S$ is the presentation map from the free inverse monoid on X to S, then the derivation module D_θ is a projective $\mathfrak{L}(S)$–module.

There is a canonical $\mathfrak{L}(S)$–map from D_θ to the augmentation module of S, and its kernel is defined to be the relation module of the presentation $\langle X : R \rangle$. We show that the set of relations R gives rise to a natural generating set for the relation module, and show that the relation module may also be interpreted as the first homology group of the Schützenberger graph of S with respect to the generating set X. This naturally leads to consideration of the class of inverse monoids whose Schützenberger graph is a forest: we call such an inverse monoid *arboreal*. The analogue of the Gruenberg resolution in this case shows that an arboreal inverse monoid has cohomological dimension one.

2. Modules and augmentation

Let S be an inverse semigroup, with semilattice of idempotents $E(S)$. Recall that the *natural partial order* on S may be defined as follows: for $s, t \in S$ we have $s \leqslant t$ if only if $s = et$ for some $e \in E(S)$. We have the following useful alternative characterisations of the natural partial order (see [11, Lemma 1.4.6] for example):

Lemma 2.1. *Let S be an inverse semigroup. Then the following are equivalent for $s, t \in S$:*

- $s \leqslant t$,
- $s = tf$ *for some* $f \in E(S)$,
- $s = ss^{-1}t$,
- $s = ts^{-1}s$.

Loganathan's category $\mathfrak{L}(S)$ is now defined as follows. Its set of objects is $E(S)$, and the set of arrows is $\{(e,s) : e \in E(S), s \in S, e \geqslant ss^{-1}\}$. The identity arrow at $e \in E(S)$ is (e,e), and the arrow (e,s) has domain $\mathbf{d}(e,s) = e$ and range $\mathbf{r}(e,s) = s^{-1}s$. If $(e,s),(f,t) \in \mathfrak{L}(S)$ and $s^{-1}s = f$ then their composite is $(e,s)(f,t) = (e,st)$. It is now straightforward to check that $\mathfrak{L}(S)$ is a left-cancellative category.

An $\mathfrak{L}(S)$–module is then a functor $\mathcal{A} : \mathfrak{L}(S) \to \mathbf{Ab}$ to the category $\mathbf{Ab}$ of abelian groups. Modules for the category algebra $\mathbb{Z}\mathfrak{L}(S)$, as defined in [22], will be $\mathfrak{L}(S)$–modules in our sense, but the converse will be true if and only if $E(S)$ is finite. An $\mathfrak{L}(S)$–module $\mathcal{A}$ determines an abelian group A_e for each $e \in E(S)$ and a homomorphism $\alpha_{(e,s)} : A_e \to A_{s^{-1}s}$ for each arrow (e,s), such that $\alpha_{(e,e)}$ is the identity map on A_e, and such that $\alpha_{(e,s)}\alpha_{(f,t)} = \alpha_{(e,st)}$ whenever $s^{-1}s = f$. If $\mathcal{A}$ is an $\mathfrak{L}(S)$–module and $a \in A_e$, we shall write $a \triangleleft (e,s)$ for $a\alpha_{(e,s)} \in A_{s^{-1}s}$.

We may regard the poset $E(S)$ as a category in the usual way, with a unique morphism $(e,f) : e \to f$ whenever $e \leqslant f$ in $E(S)$. Then $E(S)^{op}$ is a subcategory of $\mathfrak{L}(S)$, and the restriction of an $\mathfrak{L}(S)$–module to $E(S)^{op}$ determines a presheaf of abelian groups $\mathcal{A}$ over $E(S)$. Hence an $\mathfrak{L}(S)$–module $\mathcal{A}$ may be considered as a presheaf of abelian groups over $E(S)$ with an S–action. Moreover, for any $\mathfrak{L}(S)$–module $\mathcal{A}$, the disjoint union $\mathcal{A}^{\sqcup} = \sqcup_{e \in E(S)} A_e$ is a commutative inverse semigroup, with the product $a * b$ of $a \in A_e$ and $b \in A_f$ defined by $a * b = (a \triangleleft (e,ef)) + (b \triangleleft (f,ef))$. The natural partial order on $\mathcal{A}^{\sqcup}$ then coincides with that induced by the action of $E(S)^{op}$: for $a \in A_e$ and $b \in A_f$,

$$a \leqslant b \iff e \leqslant f \text{ and } a = b \triangleleft (f,e).$$

Moreover, $\mathcal{A}^{\sqcup}$ also admits a semigroup action of S by endomorphisms, defined for $a \in A_e$ and $s \in S$ by $a \cdot s = a \triangleleft (e,es)$.

A homomorphism of $\mathfrak{L}(S)$–modules is just a natural transformation of functors. We call such a natural transformation an $\mathfrak{L}(S)$–map. Hence an $\mathfrak{L}(S)$–map $\mu : \mathcal{A} \to \mathcal{B}$ is determined by a family of abelian group homomorphisms $\mu_e : A_e \to B_e$ indexed by $E(S)$, such that the following square commutes:

$$\begin{tikzcd} A_e \arrow[r, "\mu_e"] \arrow[d, "\lhd(e,s)"'] & B_e \arrow[d, "\lhd(e,s)"] \\ A_{s^{-1}s} \arrow[r, "\mu_{s^{-1}s}"'] & B_{s^{-1}s} \end{tikzcd}$$

We denote the category of $\mathfrak{L}(S)$–modules and $\mathfrak{L}(S)$–maps by Mod_S. The *kernel* of an $\mathfrak{L}(S)$–map μ is the $\mathfrak{L}(S)$–module $\ker\mu$ defined by $(\ker\mu)_e = \ker(\mu_e : A_e \to B_e)$. Clearly $\ker\mu$ is then an $\mathfrak{L}(S)$–submodule of $\mathcal{A}$.

A fixed abelian group A determines a *trivial* $\mathfrak{L}(S)$–module $\underline{A}$, with $\underline{A}_e = A$ for all $e \in E(S)$, and $\alpha_{(e,s)} = \mathrm{id}$ for all (e,s). In particular, the group of integers determines the trivial module $\underline{\mathbb{Z}}$ (denoted by $\Delta\mathbb{Z}$ in [14]).

Given an $\mathfrak{L}(S)$–module $\mathcal{A}$ we can define the *semidirect product* $S \ltimes \mathcal{A}$, which will again be an inverse semigroup. The construction is a special case of both Billhardt's λ–semidirect product of inverse semigroups [1] and of Steinberg's semidirect product of ordered groupoids, and so as an instance of [18, proposition 3.3] we have the following result.

Proposition 2.2. *Let S be an inverse semigroup and $\mathcal{A}$ an $\mathfrak{L}(S)$–module. Define*

$$S \ltimes \mathcal{A} = \{(s,a) : s \in S, a \in A_{s^{-1}s}\}.$$

Then $S \ltimes \mathcal{A}$ is an inverse semigroup, with multiplication

$$\begin{aligned}(s,a)(t,b) &= (st, (a \lhd (s^{-1}s, s^{-1}st)) + (b \lhd (t^{-1}t, t^{-1}s^{-1}st))) \\ &= (st, (a \cdot t) * b).\end{aligned}$$

Clearly the projection $(s,a) \mapsto s$ is a homomorphism of inverse semigroups, and the construction $\mathcal{A} \to S \ltimes \mathcal{A}$ is a functor from the category of $\mathfrak{L}(S)$–modules to the slice category $(\mathcal{IS} \downarrow S)$ of inverse semigroups over S. In section 3 we shall construct a left adjoint, modifying a construction originally due to Crowell [3] and then generalised in [2]. This left adjoint will then give us the term corresponding to $\mathfrak{f}/\mathfrak{fr}$ in the Gruenberg resolution.

Now Loganathan [14] constructs the $\mathfrak{L}(S)$–module $\mathbb{Z}S$ as follows. For each idempotent $e \in E(S)$, let L_e be the $\mathcal{L}$–class of e and let $\mathbb{Z}S_e$ be the free abelian group with basis L_e. Now if $a \in L_e$ and $(e,s) \in \mathfrak{L}(S)$ we define $a \lhd (e,s) = as$. Since $e = a^{-1}a \geqslant ss^{-1}$, it follows that $(as)^{-1}(as) = s^{-1}a^{-1}as = s^{-1}s$, so that $as \in L_{s^{-1}s}$ and the basis transformation $a \mapsto as$ induces a homomorphism $\mathbb{Z}S_e \to \mathbb{Z}S_{s^{-1}s}$. Loganathan notes [14, Remark 4.2] that if S is an inverse monoid then $\mathbb{Z}S$ is a free $\mathfrak{L}(S)$–module. In this setting, a basis for a free $\mathfrak{L}(S)$–module is an $E(S)$–set, that is a family of disjoint sets X_e indexed by $e \in E(S)$.

The augmentation map $\varepsilon_S : \mathbb{Z}S \to \underline{\mathbb{Z}}$ is the $\mathfrak{L}(S)$–map defined on the basis L_e of $\mathbb{Z}S_e$ by $s \mapsto 1 \in \underline{\mathbb{Z}}_e$. It is clear that this is an $\mathfrak{L}(S)$–map, and its kernel is the *augmentation module* $\mathfrak{s}$ of S.

Lemma 2.3. *For each $e \in E(S)$, the abelian group $\mathfrak{s}_e$ is freely generated by the elements $s - e$ with $e \neq s \in L_e$.*

Proof. Let $x = \sum_{i \in I} n_i s_i \in \mathfrak{s}_e$, so that each $s_i \in L_e$. Since $x\varepsilon_S = 0$, we have $\sum_{i \in I} n_i = 0$ and hence $\sum_{i \in I} n_i e = 0$. Then

$$x = \sum_{i \in I} n_i s_i - \sum_{i \in I} n_i e = \sum_{i \in I} n_i (s_i - e).$$

Hence the elements $s - e$ with $s \in L_e$ generate $\mathfrak{s}_e$, and since $\mathbb{Z}S_e$ is freely generated by L_e, it is clear that the elements $s - e$ with $e \neq s \in L_e$ are a basis. □

We remark that essentially the same definitions of $\mathbb{Z}G$ and its augmentation module were given for a groupoid G by Brown and Higgins [2]. In the next section we shall adapt another definition for groupoids from [2] in order to define the relation module of an inverse semigroup presentation.

3. The derivation module of a homomorphism

The second step in Gruenberg's resolution is the module $\mathfrak{f}/\mathfrak{fr}$, which is G–free on the cosets of the elements $1 - x, (x \in X)$. It is therefore isomorphic to the module $\mathfrak{f} \otimes_F \mathbb{Z}G$, which is Crowell's *derived module* of the homomorphism $\theta : F \to G$: indeed, Crowell gives the short exact sequence

$$0 \to N^{ab} \to \mathfrak{f} \otimes_F \mathbb{Z}G \to \mathfrak{g} \to 0$$

in [3]. We shall adapt a generalisation of Crowell's derived module due to Brown and Higgins [2] and apply it to a homomorphism of inverse semigroups. From a homomorphism $\phi : T \to S$ of inverse semigroups, we shall construct an $\mathfrak{L}(S)$–module D_ϕ, called the derivation module of ϕ, with a universal property detailed in Proposition 3.5, and with a canonical $\mathfrak{L}(S)$–map to the augmentation module of S.

Let $\mathcal{A}$ be an $\mathfrak{L}(S)$–module. We first define the notion of a $(\mathfrak{L}(T), \phi)$–derivation from $\mathfrak{L}(T)$ to $\mathcal{A}$, following the definition of a derivation to a module for a category algebra given in [22]. The properties of the composition of an $(\mathfrak{L}(T), \phi)$–derivation with the map $T \to \mathfrak{L}(T)$ that carries $t \mapsto (tt^{-1}, t)$ will then lead us to the formulation of the notion of a ϕ–derivation. Given a functor $\phi : \mathfrak{L}(T) \to \mathfrak{L}(S)$, we define an $(\mathfrak{L}(T), \phi)$–derivation to $\mathcal{A}$ to be a function $\zeta : \mathfrak{L}(T) \to \mathcal{A}^{\sqcup}$, defined on the arrows of $\mathfrak{L}(T)$, such that

- if (p, a) is an arrow of $\mathfrak{L}(T)$ then $(p, a)\zeta \in A_{(a^{-1}a)\phi}$,

- if $(p,a),(a^{-1}a,b)$ are composable arrows in $\mathfrak{L}(T)$, so that $a^{-1}a \geqslant bb^{-1}$, then
$$(p,ab)\zeta = ((p,a)(a^{-1}a,b))\zeta = (p,a)\zeta \lhd (a^{-1}a,b)\phi + (a^{-1}a,b)\zeta.$$

Now let ϕ be a homomorphism of inverse semigroups. Of course, ϕ induces a functor $\mathfrak{L}(T) \xrightarrow{\phi} \mathfrak{L}(S)$ mapping $(p,a) \mapsto (p\phi, a\phi)$. A *$\phi$–derivation* $\eta : T \to \mathcal{A}$ is a function $T \to \mathcal{A}^{\sqcup}$ such that

- if $a \in T$ then $a\eta \in A_{(a^{-1}a)\phi}$,
- if $a,b \in T$ and $a^{-1}a \geqslant bb^{-1}$, then
$$(ab)\eta = a\eta \lhd ((a^{-1}a)\phi, b\phi) + b\eta. \tag{3.1}$$

Example 3.1. If $\phi : T \to S$ is a homomorphism of inverse semigroups then the function $\eta : T \to \mathfrak{s}, t \mapsto t\phi - (t^{-1}t)\phi$ is a ϕ–derivation. If $t^{-1}t \geqslant uu^{-1}$ then
$$\begin{aligned}(tu)\eta &= (tu)\phi - (u^{-1}u)\phi \\ &= (t\phi)(u\phi) - u\phi + u\phi - (u^{-1}u)\phi \\ &= (t\phi - (t^{-1}t)\phi) \lhd ((t^{-1}t)\phi, u\phi) + u\phi - (u^{-1}u)\phi \\ &= (t\eta) \lhd ((t^{-1}t)\phi, u\phi) + u\eta.\end{aligned}$$

For each $e \in E(S)$ we define
$$\mathfrak{X}_e = \{(a,s) : a \in T, s \in L_e, (a^{-1}a)\phi \geqslant ss^{-1}\}.$$

We note that if $(a,s) \in \mathfrak{X}_e$ then $(a\phi)s \in L_e$. Now the group $D_{\phi,e}$ is defined as the (additive) abelian group generated by $\mathfrak{X}_e$ subject to all relations of the form
$$(ab,s) - (b,s) = (a,(b\phi)s) \tag{3.2}$$
where $a,b \in T$ with $a^{-1}a \geqslant bb^{-1}$ and $s \in S$. We denote the image of (a,s) in $D_{\phi,e}$ by $\langle a,s\rangle$. For subsequent use, we note the following consequences of the relations in $D_{\phi,e}$.

Lemma 3.2. (a) *If $(a,s) \in \mathfrak{X}_e$ then $\langle aa^{-1}, s\rangle = 0$ in $D_{\phi,e}$.*

(b) *If $a,b \in T$ with $b \geqslant a$, and $s \in S$ with $(a^{-1}a)\phi \geqslant ss^{-1}$, then $\langle a,s\rangle = \langle b,s\rangle$ in $D_{\phi,e}$.*

Proof. For (a) we have
$$\begin{aligned}0 = \langle a,s\rangle - \langle a,s\rangle &= \langle aa^{-1}a,s\rangle - \langle a,s\rangle \\ &= \langle a,(a^{-1}a)\phi s\rangle + \langle a^{-1}a,s\rangle - \langle a,s\rangle\end{aligned}$$

$$= \langle a, s\rangle + \langle a^{-1}a, s\rangle - \langle a, s\rangle$$
$$= \langle a^{-1}a, s\rangle.$$

For (b), we note that $a = ba^{-1}a$ with $b^{-1}b \geqslant a^{-1}a$, and therefore

$$\langle a, s\rangle = \langle ba^{-1}a, s\rangle = \langle b, (a^{-1}a)\phi s\rangle + \langle a^{-1}a, s\rangle = \langle b, s\rangle. \qquad \square$$

Lemma 3.3. *Let $s \in S$ with $s^{-1}s = e$ and suppose that $\langle a, s\rangle \in D_{\phi,e}$. If $b \in T$ and $b\phi = s$ then $\langle ab, e\rangle \in D_{\phi,e}$ and*

$$\langle ab, e\rangle = \langle a, s\rangle + \langle b, e\rangle\,.$$

Proof. Set $f = (a^{-1}a)\phi$: then $f \geqslant ss^{-1}$ and so

$$((ab)\phi)^{-1}(ab)\phi = s^{-1}fs = s^{-1}s = e.$$

Therefore $\langle ab, e\rangle \in D_\phi$. Since $(a^{-1}ab)(a^{-1}ab)^{-1} = a^{-1}abb^{-1} \leqslant a^{-1}a$ we have

$$\begin{aligned}\langle ab, e\rangle &= \langle a(a^{-1}ab), e\rangle = \langle a, (a^{-1}ab)\phi\rangle + \langle a^{-1}ab, e\rangle \qquad \text{(by (3.2))}\\ &= \langle a, fs\rangle + \langle a^{-1}ab, e\rangle\\ &= \langle a, s\rangle + \langle b, e\rangle\end{aligned}$$

since $fs = s$ and, by Lemma 3.2(b), $\langle a^{-1}ab, e\rangle = \langle b, e\rangle$. $\square$

Lemma 3.4. *Suppose that T is generated as an inverse semigroup by a subset X. Then $D_{\phi,e}$ is generated, as an abelian group, by the subset*

$$\{\langle x, s\rangle : x \in X, (x^{-1}x)\phi \geqslant ss^{-1}, s^{-1}s = e\}.$$

Proof. Suppose that $t \in T$ with $t = ua$ for some $u \in T$ and $a \in X \cup X^{-1}$. Then $t = u(u^{-1}ua)$ and $u^{-1}u \geqslant u^{-1}uaa^{-1} = (u^{-1}ua)(u^{-1}ua)^{-1}$. The relations (3.2) then imply that, if $(t, s) \in \mathfrak{X}_e$, then

$$\langle t, s\rangle = \langle u, ((u^{-1}ua)\phi)s\rangle + \langle u^{-1}ua, s\rangle.$$

By part (b) of Lemma 3.2, we have $\langle u^{-1}ua, s\rangle = \langle a, s\rangle$ and so

$$\langle t, s\rangle = \langle u, ((u^{-1}ua)\phi)s\rangle + \langle a, s\rangle.$$

It now follows, by induction on the minimum length of a product of elements of $X \cup X^{-1}$ representing $t \in T$, that $D_{\phi,e}$ is generated as an abelian group by the subset $\{\langle x, s\rangle : x \in X \cup X^{-1}\}$. But, by part (a) of Lemma 3.2 and the relations (3.1), for any $(a, s) \in \mathfrak{X}_e$,

$$0 = \langle aa^{-1}, s\rangle = \langle a, (a^{-1}\phi)s\rangle + \langle a^{-1}, s\rangle$$

and so $\langle x^{-1}, s\rangle = -\langle x, (x^{-1}\phi)s\rangle$. $\square$

We may now define the *derivation module of* ϕ, denoted by D_ϕ. For $e \in E(S)$ we have $(D_\phi)_e = D_{\phi,e}$, and the action of (e,t) on a generator $\langle a,s\rangle$ of $D_{\phi,e}$ is given by $\langle a,s\rangle \lhd (e,t) = \langle a, st\rangle$. Since $s^{-1}s = e \geqslant tt^{-1}$, $(st)^{-1}(st) = t^{-1}s^{-1}st = t^{-1}t$, and so $\langle a, st\rangle \in D_{\phi,t^{-1}t}$. It is easy to check that D_ϕ is indeed an $\mathfrak{L}(S)$–module.

Proposition 3.5. *There exists a canonical* ϕ*–derivation* $\delta : T \to D_\phi$ *such that, given any* ϕ*–derivation* $\eta : T \to \mathcal{A}$ *to an* $\mathfrak{L}(S)$*–module* $\mathcal{A}$*, there is a unique* $\mathfrak{L}(S)$*–map* $\xi : D_\phi \to \mathcal{A}$ *such that* $\eta = \delta\xi$*.*

Proof. We define the map $\delta : T \to D_\phi$ by $t \mapsto \langle t, (t^{-1}t)\phi\rangle$. Then $t\delta \in D_{\phi,(t^{-1}t)\phi}$ and δ clearly satisfies the defining property for a ϕ–derivation given in (3.1).

Let $\mathcal{A}$ be an $\mathfrak{L}(S)$–module. Suppose that an $\mathfrak{L}(S)$–map $\xi : D_\phi \to \mathcal{A}$ satisfies $\eta = \delta\xi$, and consider $\langle a,s\rangle \in D_\phi^{\sqcup}$. Then $(a^{-1}a)\phi \geqslant ss^{-1}$ and

$$\begin{aligned}
\langle a,s\rangle\xi &= (\langle a, (a^{-1}a)\phi\rangle \lhd ((a^{-1}a)\phi, s))\xi \\
&= (\langle a, (a^{-1}a)\phi\rangle)\xi \lhd ((a^{-1}a)\phi, s) \\
&= (a\delta)\xi \lhd ((a^{-1}a)\phi, s) \\
&= a\eta \lhd ((a^{-1}a)\phi, s)\,,
\end{aligned}$$

so ξ is uniquely determined by η.

Given $\eta : T \to \mathcal{A}$, we may define a map $\xi : D_\phi \to \mathcal{A}$ by setting $\langle a,s\rangle\xi = a\eta \lhd ((a^{-1}a)\phi, s)$. Firstly, this will be well-defined on $D_\phi^{\sqcup}$ since, if $a, b \in T$ with $a^{-1}a \geqslant bb^{-1}$, then

$$\begin{aligned}
(\langle ab,s\rangle - \langle b,s\rangle)\xi &= ((ab)\eta \lhd ((b^{-1}b)\phi, s)) - (b\eta \lhd ((b^{-1}b)\phi, s)) \\
&= ((ab)\eta - b\eta) \lhd ((b^{-1}b)\phi, s) \\
&= (a\eta \lhd ((bb^{-1})\phi, b\phi)) \lhd ((b^{-1}b)\phi, s) \\
&= a\eta \lhd ((bb^{-1})\phi, (b\phi)s) \\
&= \langle a, (b\phi)s\rangle\xi\,,
\end{aligned}$$

and is an $\mathfrak{L}(S)$–map since

$$\begin{aligned}
(\langle a,s\rangle \lhd (e,x))\xi = \langle a, sx\rangle\xi &= a\eta \lhd ((a^{-1}a)\phi, sx) \\
&= a\eta \lhd ((a^{-1}a)\phi, s) \lhd (e,x) \\
&= \langle a,s\rangle\xi \lhd (e,x).
\end{aligned}$$ $\square$

Corollary 3.6. *Given any inverse semigroup homomorphism* $\phi : T \to S$*, there is an* $\mathfrak{L}(S)$*–map* $\partial_1 : D_\phi \to \mathbb{Z}S$ *whose image is the augmentation module* $\mathfrak{s}$*.*

Proof. The map ∂_1 is induced by the ϕ–derivation $t \mapsto t\phi - (t^{-1}t)\phi$ of Example 3.1, and so $\langle a, s\rangle\partial_1 = (a\phi - (a^{-1}a)\phi) \lhd s = (a\phi)s - s$. Lemma 2.3 then implies that its image is $\mathfrak{s}$. $\square$

Suppose that we have inverse semigroup homomorphisms $\phi : T \to S$ and $\psi : U \to S$ and that $\lambda : T \to U$ is a homomorphism of inverse semigroups such that $\phi = \lambda\psi$. Then λ is a morphism in the slice category $(\mathcal{IS} \downarrow S)$ and induces a mapping $\lambda_* : D_\phi \to D_\psi$ given by $\langle a, s\rangle \mapsto \langle a\lambda, s\rangle$. In this way the construction of the derivation module gives a functor $(\mathcal{IS} \downarrow S) \to \mathrm{Mod}_S$.

Proposition 3.7. *The derivation module functor $(T \xrightarrow{\phi} S) \mapsto D_\phi$ is left adjoint to the semidirect product functor $\mathcal{A} \mapsto S \ltimes \mathcal{A}$.*

Proof. The first part of the proof is the standard verification, adapted to our setting, of the correspondence between derivations and maps to semidirect products. The details are as follows. Let $\phi : T \to S$ be an inverse semigroup homomorphism and let $\alpha : T \to S \ltimes \mathcal{A}$ be a morphism in the slice category $(\mathcal{IS} \downarrow S)$, so that for all $t \in T$ we have $t\alpha = (t\phi, t\zeta)$ for some function $\zeta : T \to \mathcal{A}^{\sqcup}$. Since $(t\phi, t\zeta) \in S \ltimes \mathcal{A}$, it follows that $t\zeta \in A_{(t^{-1}t)\phi}$.

Now assume that $a, b \in T$ with $a^{-1}a \geqslant bb^{-1}$. Then

$$\begin{aligned}(a\alpha)(b\alpha) &= (a\phi, a\zeta)(b\phi, b\zeta)\\ &= (a\phi b\phi, a\zeta \lhd ((a^{-1}a)\phi, (a^{-1}ab)\phi)\\ &\qquad\qquad + b\zeta \lhd ((b^{-1}b)\phi, (b^{-1}a^{-1}ab)\phi))\end{aligned}$$

and, since $a^{-1}ab = b$,

$$\begin{aligned}&= (a\phi b\phi, a\zeta \lhd ((a^{-1}a)\phi, b\phi) + b\zeta \lhd ((b^{-1}b)\phi, (b^{-1}b)\phi))\\ &= (a\phi b\phi, a\zeta \lhd ((a^{-1}a)\phi, b\phi) + b\zeta).\end{aligned}$$

Now $(a\alpha)(b\alpha) = (ab)\alpha = ((ab)\phi, (ab)\zeta)$, and so if $a^{-1}a \geqslant bb^{-1}$, we have

$$(ab)\zeta = a\zeta \lhd ((a^{-1}a)\phi, b\phi) + b\zeta$$

and ζ is a ϕ-derivation.

By Proposition 3.5, ζ induces a unique $\mathfrak{L}(S)$–map $\widehat{\alpha} : D_\phi \to \mathcal{A}$ defined by

$$\widehat{\alpha} : \langle a, s\rangle \mapsto a\zeta \lhd ((a^{-1}a)\phi, s)\,. \tag{3.3}$$

Now an $\mathfrak{L}(S)$–map $\gamma : D_\phi \to \mathcal{A}$ induces an inverse semigroup homomorphism $\gamma^\dagger : T \to S \ltimes \mathcal{A}$ in the slice category $(\mathcal{IS} \downarrow S)$ given by

$$\gamma^\dagger : t \mapsto (t\phi, \langle t, (t^{-1}t)\phi\rangle\gamma). \tag{3.4}$$

We check that $\gamma^\dagger$ is indeed an inverse semigroup homomorphism as follows. For all $a, b \in T$ we have

$$(ab)\gamma^\dagger = ((ab)\phi, \langle ab, (b^{-1}a^{-1}ab)\phi\rangle\gamma).$$

Now we compute $(a\gamma^\dagger)(b\gamma^\dagger)$:

$$\begin{aligned}(a\gamma^\dagger)(b\gamma^\dagger) &= (a\phi, \langle a, (a^{-1}a)\phi\rangle\gamma)(b\phi, \langle b, (b^{-1}b)\phi\rangle\gamma) \\ &= (a\phi b\phi, \langle a, (a^{-1}a)\phi\rangle\gamma \lhd ((a^{-1}a)\phi, (a^{-1}ab)\phi) \\ &\qquad\qquad + \langle b, (b^{-1}b)\phi\rangle\gamma \lhd ((b^{-1}b)\phi, (b^{-1}a^{-1}ab)\phi)) \\ &= (a\phi b\phi, \langle a, (a^{-1}ab)\phi\rangle\gamma + \langle b, (b^{-1}a^{-1}ab)\phi\rangle\gamma) \\ &= (a\phi b\phi, (\langle a, (a^{-1}ab)\phi\rangle + \langle b, (b^{-1}a^{-1}ab)\phi\rangle)\gamma).\end{aligned}$$

Now it follows from part (b) of Lemma 3.2 that

$$\langle b, (b^{-1}a^{-1}ab)\phi\rangle = \langle a^{-1}ab, (b^{-1}a^{-1}ab)\phi\rangle$$

and then by Lemma 3.3 that

$$\langle a, (a^{-1}ab)\phi\rangle + \langle a^{-1}ab, (b^{-1}a^{-1}ab)\phi\rangle = \langle ab, (b^{-1}a^{-1}ab)\phi\rangle.$$

Hence

$$(a\gamma^\dagger)(b\gamma^\dagger) = (a\phi b\phi, \langle ab, (b^{-1}a^{-1}ab)\phi\rangle\gamma)$$

and so $\gamma^\dagger$ is an inverse semigroup homomorphism.

The correspondences $\alpha \mapsto \widehat{\alpha}$ (defined in (3.3)) and $\gamma \mapsto \gamma^\dagger$ (defined in (3.4)) are mutually inverse bijections, giving a natural isomorphism between the bifunctors $\mathrm{Mod}_S(D_-, -)$ and $(\mathcal{IS} \downarrow S)(-, S \ltimes -)$ confirming the adjunction given in the Proposition. $\square$

Theorem 3.8. *Let S be an inverse monoid generated by a set X, let $\mathrm{FIM}(X)$ be the free inverse monoid on X, and let $\theta : \mathrm{FIM}(X) \to S$ be the presentation map.*

(a) *$D_{\theta,e}$ is generated, as an abelian group, by the subset*

$$\{\langle x, s\rangle : x \in X, (x^{-1}x)\theta \geqslant ss^{-1}, s^{-1}s = e\}$$

and hence D_θ is generated as an $\mathfrak{L}(S)$–module by the set

$$\{\langle x, (x^{-1}x)\theta\rangle : x \in X\}.$$

(b) *D_θ is a projective $\mathfrak{L}(S)$–module.*

Proof. (a) This follows from Lemma 3.4.

(b) Let $\mu : \mathcal{A} \to \mathcal{B}$ be an epimorphism of $\mathfrak{L}(S)$–modules. For any $e \in E(S)$, the evaluation functor $\mathrm{eval}_e : \mathrm{Mod}_S \to \mathbf{Ab}$ given by $\mathcal{A} \mapsto A_e$ is exact, and hence $\mu : A_e \to B_{e\mu}$ is surjective.

Suppose that $\beta : D_\theta \to \mathcal{B}$ is an $\mathfrak{L}(S)$–map. By Proposition 3.7, β induces a inverse monoid homomorphism $\beta^\dagger : \mathrm{FIM}(X) \to S \ltimes \mathcal{B}$, and there is a surjection $\mu_* : S \ltimes \mathcal{A} \to S \ltimes \mathcal{B}$ of inverse monoids, given by $(s, a) \mapsto (s, a\mu)$. The freeness of $\mathrm{FIM}(X)$ then implies that we can lift $\beta^\dagger$ to $\alpha^\dagger : \mathrm{FIM}(X) \to S \ltimes \mathcal{A}$, with $\alpha^\dagger \mu_* = \beta^\dagger$. For $w \in \mathrm{FIM}(X)$, it follows that $w\alpha^\dagger = (w\theta, w\eta)$, where $\eta : \mathrm{FIM}(X) \to \mathcal{A}$ is a θ–derivation such that $w\eta\mu = \langle w, (w^{-1}w)\theta\rangle\beta$. By Proposition 3.5 η induces an $\mathfrak{L}(S)$–map $\alpha : D_\theta \to \mathcal{A}$ defined by

$$\alpha = \widehat{(\alpha^\dagger)} : \langle w, s\rangle \mapsto w\eta \lhd ((w^{-1}w)\theta, s),$$

and

$$\begin{aligned}
\langle w, s\rangle\alpha\mu &= (w\eta \lhd ((w^{-1}w)\theta, s))\mu \\
&= w\eta\mu \lhd ((w^{-1}w)\theta, s) \\
&= \langle w, (w^{-1}w)\theta\rangle\beta \lhd ((w^{-1}w)\theta, s) \\
&= (\langle w, (w^{-1}w)\theta\rangle \lhd ((w^{-1}w)\theta, s))\beta \\
&= \langle w, s\rangle\beta.
\end{aligned}$$

This shows that D_θ is projective. □

Proposition 3.7 is based on the adjunction result for groupoid actions given in [2], but may also be seen as a special case of a result of Nico [16] for the semidirect product of categories. This work is elaborated by Steinberg and Tilson in [20], and by Steinberg in [18] for the category of ordered groupoids. The results of [18] are of particular relevance since the category of inverse semigroups is isomorphic to the subcategory of *inductive* groupoids in the category of ordered groupoids. Steinberg constructs a left adjoint $\mathrm{Der}(\phi)$ to the semidirect product, where the latter is regarded as a functor from the category of ordered groupoid actions to the category of ordered functors: from a pair (G, H) of ordered groupoids with H acting on G, the semidirect product $H \ltimes G$ is an ordered groupoid with a projection map $H \ltimes G \to H$. We have restricted attention to $\mathfrak{L}(S)$–modules, and so have replaced the category of ordered groupoid actions by Mod_S, and the category of ordered functors by the slice category $(\mathcal{IS} \downarrow S)$. Steinberg's adjunction then tells us that morphisms $T \to S \ltimes \mathcal{A}$ in $(\mathcal{IS} \downarrow S)$ correspond bijectively to a certain subclass of morphisms $\mathrm{Der}(\phi) \to \mathcal{A}$ in the category of ordered groupoid actions. Following through the details of Steinberg's construction, it is readily seen that the subclass of morphisms are those that arise from ϕ–derivations $T \to \mathcal{A}$.

4. Presentations and relation modules

Let S be an inverse semigroup given by a presentation $\mathcal{P} = \mathrm{Inv}[X : R]$, and let $\theta : \mathrm{FIS}(X) \to S$ be the associated presentation map. Corollary 3.1 gives an $\mathfrak{L}(S)$–map $\partial_1 : D_\theta \to \mathfrak{s}$: its kernel $\mathcal{M}$ is an $\mathfrak{L}(S)$–module called the *relation module* of $\mathcal{P}$. Our first task is to find a convenient generating set for the relation module: we approach this in stages in our next result.

Theorem 4.1. (a) *The relation module M_e at $e \in E(S)$ is generated, as an abelian group, by all elements of the form $\langle u, s\rangle - \langle v, s\rangle$, where $u\theta = v\theta$ and $s^{-1}s = e$.*

(b) *A smaller generating set for M_e, as an abelian group, is given by the set of all elements of the form $\langle pr_1q, s\rangle - \langle pr_2q, s\rangle$, where $p, q \in \mathrm{FIS}(X)$, $(r_1, r_2) \in R$ and $s^{-1}s = e$.*

(c) *The relation module $\mathcal{M}$ is generated as an $\mathfrak{L}(S)$–module by the set of all elements of the form $\langle r_1, e\rangle - \langle r_2, e\rangle$ where $(r_1, r_2) \in R$, $e \in E(S)$, and $(r_1^{-1}r_2)\theta = e$.*

Proof. (a) Let P_e be the subgroup of M_e generated by the set of elements specified in part (a) of the theorem. Now suppose that $\alpha = \sum_{i\in I} \varepsilon_i\langle u_i, s_i\rangle$ is an element of the relation module $\mathcal{M}_e$ at e (where $\varepsilon_i = \pm 1$), so that

$$\alpha\partial_1 = \sum_{i\in I} \varepsilon_i((u_i\theta)s_i - s_i) = 0. \tag{4.1}$$

Since $\mathbb{Z}S$ is (additively) free on the elements of the $\mathcal{L}$–class L_e, the terms in the sum in (4.1) must cancel in pairs. We fix such a pairing of cancelling terms, and for each s_i occurring in the given expression for α we choose $w_{s_i} \in \mathrm{FIS}(X)$ with $w_{s_i}\theta = s_i$.

Suppose that in our pairing of cancelling terms, $(u_i\theta)s_i$ is paired with s_i. Then $(u_i\theta)s_i = s_i$ and, by Lemma 3.2, $\langle u_i, s_i\rangle = \langle u_iw_{s_i}w_{s_i}^{-1}, s_i\rangle$ and $\langle u_iw_{s_i}w_{s_i}^{-1}, s_i\rangle = 0$. It follows that

$$\langle u_i, s_i\rangle = \langle u_iw_{s_i}w_{s_i}^{-1}, s_i\rangle - \langle u_iw_{s_i}w_{s_i}^{-1}, s_i\rangle \in P_e.$$

Now suppose that $(u_i\theta)s_i$ is paired with $(u_j\theta)s_j$: that is $(u_i\theta)s_i = (u_j\theta)s_j$ and $-\varepsilon_i = \varepsilon_j$. Then by Lemma 3.3,

$$\langle u_iw_{s_i}, e\rangle = \langle u_i, s_i\rangle + \langle w_{s_i}, e\rangle \text{ and } \langle u_jw_{s_j}, e\rangle = \langle u_j, s_j\rangle + \langle w_{s_j}, e\rangle,$$

with $\langle u_iw_{s_i}, e\rangle - \langle u_jw_{s_j}, e\rangle \in P_e$. Therefore, we may write

$$\varepsilon_i\langle u_i, s_i\rangle + \varepsilon_j\langle u_j, s_j\rangle = \varepsilon_i(\langle u_iw_{s_i}, e\rangle - \langle u_jw_{s_j}, e\rangle) - \varepsilon_i\langle w_{s_i}, e\rangle - \varepsilon_j\langle w_{s_j}, e\rangle,$$

and modulo P_e, we see that $\varepsilon_i\langle u_i, s_i\rangle + \varepsilon_j\langle u_j, s_j\rangle$ is equal to $-\varepsilon_i\langle w_{s_i}, e\rangle - \varepsilon_j\langle w_{s_j}, e\rangle$.

Finally, if $(u_i)\theta s_i$ is paired with s_j, then $\langle u_i w_{s_i}, e\rangle - \langle w_{s_j}, e\rangle \in P_e$ and, again by Lemma 3.3,

$$\langle u_i, s_i\rangle = (\langle u_i w_{s_i}, e\rangle - \langle w_{s_j}, e\rangle) + \langle w_{s_j}, e\rangle - \langle w_{s_i}, e\rangle.$$

These considerations show that modulo P_e, the element $\alpha \in \mathcal{M}_e$ is equal to a sum $\beta = \sum_k m_k\langle w_{s_k}, e\rangle$ with $m_k \in \mathbb{Z}$ and the s_k distinct. But then $\beta\partial_1 = \sum_k m_k(s_k - e) = 0$ is a sum of $\mathbb{Z}$–independent elements of $\mathbb{Z}S$, and so each $m_k = 0$. Therefore $\beta = 0$, and we conclude that α is in P_e.

(b) Let Q_e be the subgroup of M_e generated by the set of elements specified in part (b) of the theorem. Let $\langle u, s\rangle - \langle v, s\rangle$, as in part (a), be a generator of P_e. Now S is presented by $\mathrm{Inv}[X : R]$ and so, since $u\theta = v\theta$, there exists a finite sequence of elements $w_0, w_1, \ldots, w_m \in \mathrm{FIM}(X)$ with $w_0 = u$, $w_m = v$ and such that for each i, $0 \leqslant i < m$, we may write $w_i = p_i r_i q_i$ and $w_{i+1} = p_i r_i' q_i$ with $p_i, q_i \in \mathrm{FIM}(X)$ and with either (r_i, r_i') or (r_i', r_i) in R. We show by induction on m that the element $\langle u, s\rangle - \langle v, s\rangle$ of D_θ is in the subgroup Q_e. If $m = 1$ then $u = p_0 r_0 q_0$ and $v = p_0 r_0' q_0$ and so

$$\langle u, s\rangle - \langle v, s\rangle = \langle p_0 r_0 q_0, s\rangle - \langle p_0 r_0' q_0, s\rangle \in Q_e.$$

If $m > 1$ we have $u = p_0 r_0 q_0$ and $w_1 = p_0 r_0' q_0$

$$\langle u, s\rangle - \langle v, s\rangle = \langle p_0 r_0 q_0, s\rangle - \langle p_0 r_0' q_0, s\rangle + \langle w_1, s\rangle - \langle v, s\rangle.$$

Now we have $w_1\theta = v\theta$ and the elements w_1, v are linked by the sequence $w_1, \ldots, w_m \in \mathrm{FIM}(X)$, and by induction $\langle w_1, s\rangle - \langle v, s\rangle \in Q_e$. It follows that $\langle u, s\rangle - \langle v, s\rangle \in Q_e$.

(c) We show that a generator $\langle pr_1q, s\rangle - \langle pr_2q, s\rangle$ is an $\mathfrak{L}(S)$–translate of the element $\langle r_1, f\rangle - \langle r_2, f\rangle$, where $f = (r_1^{-1}r_2)\theta$. Now by Lemma 3.3,

$$\langle pr_1q, s\rangle = \langle p(p^{-1}p)r_1q, s\rangle = \langle p, (p^{-1}pr_1q)\theta s\rangle + \langle p^{-1}pr_1q, s\rangle$$

and similarly

$$\langle pr_2q, s\rangle = \langle p, (p^{-1}pr_2q)\theta s\rangle + \langle p^{-1}pr_2q, s\rangle\,.$$

Now $(p^{-1}pr_1q)\theta = (p^{-1}pr_2q)\theta$, and so

$$\langle pr_1q, s\rangle - \langle pr_2q, s\rangle = \langle p^{-1}pr_1q, s\rangle - \langle p^{-1}pr_2q, s\rangle = \langle r_1q, s\rangle - \langle r_2q, s\rangle$$

by part (b) of Lemma 3.2. We see that $\langle pr_1q, s\rangle - \langle pr_2q, s\rangle$ does not depend on p. In the same manner,

$$\langle r_1q, s\rangle = \langle r_1(r_1^{-1}r_1q), s\rangle = \langle r_1, (r_1^{-1}r_1q)\theta s\rangle + \langle r_1^{-1}r_1q, s\rangle$$

$$= \langle r_1, (r_1^{-1}r_1q)\theta s\rangle + \langle q, s\rangle$$

since $\langle r_1^{-1}r_1q, s\rangle = \langle q, s\rangle$ by part (b) of Lemma 3.2. It follows that

$$\langle r_1q, s\rangle - \langle r_2q, s\rangle = \langle r_1, (r_1^{-1}r_1q)\theta s\rangle - \langle r_2, (r_2^{-1}r_2q)\theta s\rangle = \langle r_1, s'\rangle - \langle r_2, s'\rangle$$

where $s' = (r_1^{-1}r_1q)\theta s = (r_2^{-1}r_2q)\theta s$. Then finally we have

$$\langle r_1, s'\rangle - \langle r_2, s'\rangle = (\langle r_1, e\rangle - \langle r_2, e\rangle) \lhd (e, s'). \qquad \square$$

Corollary 4.2. *Let S be an inverse monoid presented by* $\mathrm{Inv}[X : R]$ *with presentation map $\theta : \mathrm{FIM}(X) \to S$. Then there is an exact sequence of $\mathfrak{L}(S)$–modules*

$$\mathcal{F}(R) \to D_\theta \to \mathbb{Z}S \to \underline{\mathbb{Z}} \to \underline{0}$$

where $\mathcal{F}(R)$ is the free $\mathfrak{L}(S)$–module on the $E(S)$–set $X = \{X_e : e \in E(S)\}$, with $X_e = \{(r_1, r_2) \in R : (r_1^{-1}r_2)\theta = e\}$.

4.1. The Schützenberger representation

Suppose that S is an inverse monoid presented by $\mathrm{Inv}[X : R]$ with presentation map $\theta : \mathrm{FIM}(X) \to S$. The (left) Schützenberger graph of S with respect to X, denoted by $\mathrm{Sch}^{\mathcal{L}}(S, X)$, has vertex set S and, for each $(x, s) \in X \times S$ with $(x^{-1}x)\theta \geqslant ss^{-1}$, has a directed edge labelled by (x, s) with initial vertex s and terminal vertex $(x\theta)s$. Equivalently, there is a directed edge from s to xs labelled by (x, s) whenever $s\mathcal{L}(x\theta)s$ in S. The connected components of $\mathrm{Sch}^{\mathcal{L}}(S, X)$ are the full subgraphs spanned by the $\mathcal{L}$–classes of S, and the connected component containing $e \in E(S)$ is denoted by $\mathrm{Sch}^{\mathcal{L}}(S, X, e)$.

The usual convention for Schützenberger graphs [19, 21] is to have a directed edge (s, x) from s to sx whenever $s\mathcal{R}sx$. To get a natural right $\mathfrak{L}(S)$–module, we have to use the left Schützenberger graphs.

Suppose that $s^{-1}s = e$ and that $(e, t) \in \mathfrak{L}(S)$. Then by defining

$$s \lhd (e, t) = st \text{ and } (a, s) \lhd (e, t) = (a, st)$$

we get a graph map $\mathrm{Sch}^{\mathcal{L}}(S, X, e) \to \mathrm{Sch}^{\mathcal{L}}(S, X, t^{-1}t)$ and in this way we obtain a functor from $\mathfrak{L}(S)$ to the category of directed graphs. The cellular chain complex $C^{\mathcal{L}}(S, X)$ of $\mathrm{Sch}^{\mathcal{L}}(S, X)$ is then a complex of $\mathfrak{L}(S)$–modules, with $C_0^{\mathcal{L}}(S, X) = \mathbb{Z}S$ and with the group $C_1^{\mathcal{L}}(S, X)_e$ free abelian on the set

$$\{(x, s) : x \in X, s \in S, (x^{-1}x)\theta \geqslant ss^{-1}, s^{-1}s = e\}.$$

The boundary map $C_1^{\mathcal{L}}(S, X)_e \to C_0^{\mathcal{L}}(S, X)_e$ maps $(x, s) \mapsto (x\theta)s - s$.

Theorem 4.3. *Suppose that S is an inverse monoid presented by* $\mathrm{Inv}[X :$

R] with presentation map $\theta : \mathrm{FIM}(X) \to S$. Then the derivation module D_θ is isomorphic to $C_1^{\mathcal{L}}(S,X)$ and the relation module $\mathcal{M}$ is isomorphic to the first homology group of the Schützenberger graph $\mathrm{Sch}^{\mathcal{L}}(S,X)$.

Proof. By Corollary 3.6 we have a commutative square of $\mathfrak{L}(S)$–maps

$$\begin{tikzcd} C_1^{\mathcal{L}}(S,X) \arrow[r] \arrow[d, two heads, "\kappa"'] & C_0^{\mathcal{L}}(S,X) \arrow[d, "\cong"] \\ D_\theta \arrow[r, "\partial_1"] & \mathbb{Z}S \end{tikzcd}$$

in which the left-hand map κ is the $\mathfrak{L}(S)$–map induced (as a homomorphism of abelian groups) by $(x,s) \mapsto \langle x,s\rangle$, and κ restricts to a surjection $H_1(\mathrm{Sch}^{\mathcal{L}}(S,X)) \to \mathcal{M}$. Now the function $X \to S \ltimes C_1^{\mathcal{L}}(S,X)$ that maps $x \mapsto (x\theta,(x,(x^{-1}x)\theta)$ induces an inverse monoid homomorphism $\mathrm{FIM}(X) \to S \ltimes C_1^{\mathcal{L}}(S,X)$ and by Proposition 3.7 there is then an $\mathfrak{L}(S)$–map $D_\theta \to C_1^{\mathcal{L}}(S,X)$ mapping $\langle x,(x^{-1}x)\theta\rangle \mapsto (x,(x^{-1}x)\theta)$ giving an inverse to κ. □

4.2. Cohomological dimension

The classification of all small categories of cohomological dimension zero was completed by Laudal [9], following a conjecture of Oberst [17]. Laudal's result was then used by Leech [13] to classify the inverse monoids of cohomological dimension zero: the classification is the expected generalisation of the result that a group has cohomological dimension zero if and only if it is trivial. We present another proof here that makes direct use of the fact that, if S has cohomological dimension zero, then $\underline{\mathbb{Z}}$ is a projective $\mathfrak{L}(S)$–module.

Theorem 4.4 (Leech [13]). *An inverse monoid S has cohomological dimension zero if and only if it is a semilattice.*

Proof. If S is a semilattice then $\mathbb{Z}S = \underline{\mathbb{Z}}$ and so $\mathbb{Z}$ is a projective $\mathfrak{L}(S)$–module, and hence $\mathrm{cd}(S) = 0$.

Conversely, if $\mathrm{cd}(S) = 0$ then $\underline{\mathbb{Z}}$ has a projective resolution $P_0 \to \mathbb{Z}$ and hence $\underline{\mathbb{Z}}$ is a projective $\mathfrak{L}(S)$–module. In particular, the augmentation map $\varepsilon : \mathbb{Z}S \to \underline{\mathbb{Z}}$ splits as an $\mathfrak{L}(S)$–map, and so there is an $\mathfrak{L}(S)$–map $\sigma : \underline{\mathbb{Z}} \to \mathbb{Z}S$ such that $\sigma\varepsilon = \mathrm{id}$. Let 1_e denote the copy of $1 \in \mathbb{Z}$ in $\underline{\mathbb{Z}}_e$ (that is, in the copy of $\mathbb{Z}$ indexed by $e \in E(S)$) and let $\widehat{1} = 1_1$. We set $\widehat{1}\sigma = w \in \mathbb{Z}S_1$. Then $w\varepsilon = \widehat{1}$ and w determines σ since

$$1_e\sigma = (\widehat{1} \lhd (1,e))\sigma = w \lhd (1,e) = we \in \mathbb{Z}L_e^S\,.$$

Now for any $s \in S$, we have

$$ws = w \lhd (1,s) = (\widehat{1}\sigma) \lhd (1,s) = (\widehat{1} \lhd (1,s))\sigma = 1_{s^{-1}s}\sigma = ws^{-1}s \quad (4.2)$$

in the free abelian group $\mathbb{Z}S_{s^{-1}s}$.

We may write $w = \sum_{t_i^{-1}t_i=1} m_{t_i} t_i$ (with $m_{t_i} \in \mathbb{Z}$ and at most finitely many non-zero). Suppose that $s \in S$ with $ss^{-1} = 1$. Then for each t_i with $m_{t_i} \neq 0$, there exists a t_j with $m_{t_j} \neq 0$ such that $t_i s = t_j s^{-1} s$. But then

$$t_i t_i^{-1} = (t_i s)(t_i s)^{-1} = (t_j s^{-1} s)(t_j s^{-1} s)^{-1} = t_j s^{-1} s t_j^{-1} \leqslant t_j t_j^{-1} \,.$$

Since the collection of such t_i is finite, there must exist t_k occurring in w with $t_k t_k^{-1} = t_k s^{-1} s t_k^{-1}$, or equivalently with $t_k = t_k s^{-1} s$. Then

$$1 = t_k^{-1} t_k = s^{-1} s t_k^{-1} t_k = s^{-1} s \,.$$

We see that $ss^{-1} = 1 = s^{-1}s$, and so Green's Ł–class at $1 \in E(S)$ coincides with the $\mathcal{H}$–class H_1, and is a group. In particular, $\mathbb{Z}S_1$ is the integral group ring $\mathbb{Z}H_1$. Now for all $h \in H_1$ we have, by (4.2),

$$\sum_{t_i^{-1}t_i=1} m_{t_i} t_i h = \sum_{t_i^{-1}t_i=1} m_{t_i} t_i \,.$$

Taking $h = t_i^{-1} t_j$ in the group H_1, we see that $m_{t_i} = m_{t_j}$, and since $\sum_i m_{t_i} = 1$, it follows that $|H_1| = 1$. In particular $w = 1$. From (4.2) we see, for any $s \in S$, that $s = s^{-1}s$ and so S is a semilattice. □

We note the contrast between this result and the theory of monoids of cohomological dimension zero. Laudal [9, Lemma A] shows that if a monoid M has (left) monoid cohomological dimension equal to zero then M has a right zero. See [5] for further results on the cohomological dimension of monoids.

Arboreal inverse monoids

An *arboreal* inverse monoid S is an inverse monoid given by a presentation of the form

$$S = \mathrm{Inv}[X : e_i = f_i \,, (i \in I)]$$

where I is some indexing set and e_i, f_i are idempotents in $\mathrm{FIM}(X)$. Therefore e_i, f_i are *Dyck words* in $(X \cup X^{-1})^*$, that is words whose freely reduced form is equal to 1. Free inverse monoids and free groups are arboreal inverse monoids, as is the bicyclic monoid B, presented by $\mathrm{Inv}[x : xx^{-1} = 1]$.

Arboreal inverse monoids are the main concern of [15], wherein it is shown that arboreal inverse monoids with generating set X are precisely the E–unitary quotients of $\mathrm{FIM}(X)$ with maximum group image $F(X)$, and are also characterised as the inverse monoids each of whose Schützenberger graphs is a tree. This latter characterization has motivated the use of the adjective 'arboreal'. A principal result of [15] is that finitely presented arboreal inverse monoids have decidable word problem. The following result may be readily deduced from the work of Steinberg [19] and also follows from Funk's topos-theoretic approach to inverse semigroup representations [4].

Theorem 4.5. *An arboreal inverse monoid S has cohomological dimension equal to* 1.

Proof. Let S be presented by $S = \mathrm{Inv}[X : e_i = f_i\,, (i \in I)]$, and let $\theta : \mathrm{FIM}(X) \to S$ be the presentation map. By part (c) of Theorem 4.1, the relation module M_e at $e \in E(S)$ is generated by $\mathfrak{L}(S)$–translates of elements of the form $\langle e_i, e_i\theta\rangle - \langle f_i, f_i\theta\rangle$ (where $e_i\theta = f_i\theta \in S$). By Lemma 3.2, all such elements are equal to 0. It follows that M_e is trivial, and hence that $\partial_1 : D_\theta \to \mathfrak{s}$ is injective. By [14, Remark 4.2], for a monoid S the $\mathfrak{L}(S)$–module $\mathbb{Z}S$ is free, and by Corollary 3.8 we know that D_θ is projective. Therefore the short exact sequence

$$\underline{0} \to D_\theta \xrightarrow{\partial_1} \mathbb{Z}S \xrightarrow{\varepsilon_S} \underline{\mathbb{Z}} \to \underline{0}$$

is a $\mathfrak{L}(S)$–projective resolution of $\underline{\mathbb{Z}}$. □

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