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Generalized symmetric rings

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Abstract. In this paper, we introduce a class of rings which is a generalization of symmetric rings. Let R be a ring with identity. A ring R is called *central symmetric* if for any a, $b, c \in R$, $abc = 0$ implies bac belongs to the center of R. Since every symmetric ring is central symmetric, we study sufficient conditions for central symmetric rings to be symmetric. We prove that some results of symmetric rings can be extended to central symmetric rings for this general settings. We show that every central reduced ring is central symmetric, every central symmetric ring is central reversible, central semmicommutative, 2-primal, abelian and so directly finite. It is proven that the polynomial ring $R[x]$ is central symmetric if and only if the Laurent polynomial ring $R[x, x^{-1}]$ is central symmetric. Among others, it is shown that for a right principally projective ring R, R is central symmetric if and only if $R[x]/(x^n)$ is central Armendariz, where $n \geq 2$ is a natural number and (x^n) is the ideal generated by x^n.

1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. A ring is *reduced* if it has no nonzero nilpotent elements, similarly a ring R is called *central reduced* [2] if every nilpotent element of R is central. A weaker condition than "reduced" is defined by Lambek in [16]. A ring R is *symmetric* if for any a, $b, c \in R$, $abc = 0$ implies $acb = 0$ if and only if $abc = 0$ implies $bac = 0$. An equivalent condition

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on a ring is that whenever a product of any number of elements is zero, any permutation of the factors still yields product zero. A ring R is *right (left) principally quasi-Baer* [8] if the right (left) annihilator of a principal right (left) ideal of R is generated by an idempotent. Finally, a ring R is called *right (left) principally projective* if the right (left) annihilator of an element of R is generated by an idempotent [7]. Throughout this paper, $\mathbb{Z}$ denotes the ring of integers. We write $R[x]$ and $R[x, x^{-1}]$ for the polynomial ring and the Laurent polynomial ring, respectively.

2. Central symmetric rings

In this section we introduce a class of rings, called central symmetric rings, which is a generalization of symmetric rings. We investigate which properties of symmetric rings hold for the central case. Clearly, symmetric rings are central symmetric and central symmetric rings are central reversible. We supply an example to show that all central symmetric rings need not be symmetric. We provide another example to show that all central reversible rings may not be central symmetric. Therefore the class of central symmetric rings lies strictly between classes of symmetric rings and central reversible rings. We obtain sufficient conditions for central symmetric rings to be symmetric. It is shown that the class of central symmetric rings is closed under finite direct sums. We have an example which shows that the homomorphic image of a central symmetric ring is not central symmetric. Then we determine under what conditions a homomorphic image of a ring is central symmetric.

We now give our main definition.

Definition 2.1. A ring R is called *central symmetric* if for any $a, b, c \in R$, $abc = 0$ implies bac belongs to the center of R.

All commutative rings, reduced rings, central reduced rings and symmetric rings are central symmetric. One may suspect that central symmetric rings are symmetric. But the following example erases the possibility.

Example 2.2. Let x, y and z be indeterminates and consider the set

$$R = \{a_0 + a_1x + a_2y + a_3z \mid a_0, a_1, a_2, a_3 \in \mathbb{Z}\}$$

with componentwise addition and defining multiplication

$$(a_0 + a_1x + a_2y + a_3z)(b_0 + b_1x + b_2y + b_3z) =$$
$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_2b_0)y + (a_0b_3 + a_3b_0 + a_1b_2)z.$$

Then R is a ring with identity. From this multiplication all products are zero except that $xy = z$ and that 1 acts as an identity (see [21, Example 5.1]). Since $x^2 = y^2 = 0$, $xz = xxy = 0 = xyx = zx$ and $zy = xyy = 0 = yxy = yz$, z is central. Hence R is central symmetric. On the other hand, $yx = yx1 = 0$, but $xy1 = z$ and so R is not symmetric.

Recall that a ring R is *semiprime* if $aRa = 0$ implies $a = 0$ for $a \in R$. Our next aim is to find conditions under what a central symmetric ring is symmetric.

Proposition 2.3. *If R is a symmetric ring, then R is central symmetric. The converse holds if R satisfies any of the following conditions.*

(1) *R is a semiprime ring.*

(2) *R is a right (left) principally projective ring.*

(3) *R is a right (left) principally quasi-Baer ring.*

Proof. Clearly every symmetric ring is central symmetric. Conversely, assume that R is a central symmetric ring and $a, b, c \in R$ with $abc = 0$. It implies that bca is central due to $1 \in R$. Now consider the following cases.

(1) Let R be a semiprime ring. Since $(bca)x(bca) = 0$ for all $x \in R$, it follows that $bca = 0$. Therefore R is symmetric.

(2) Let R be a right principally projective ring. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Thus $ae = 0$. Since $bc \in r_R(a) = eR$, we have $bc = ebc$. It follows that $bca = ebca = bcae = 0$. Therefore R is symmetric. A similar proof may be given for left principally projective rings.

(3) Let R be a right principally quasi-Baer ring. Then there exists an idempotent $e \in R$ such that $r_R(aR) = eR$. Hence $ae = 0$. On the other hand, $axbca = abcax = 0$ for all $x \in R$. This implies that $bca \in r_R(aR) = eR$, and so $bca = ebca = bcae = 0$. Thus R is symmetric. In a similar way every left principally quasi-Baer central symmetric ring is symmetric. $\square$

The following is an apparent consequence of Proposition 2.3.

Corollary 2.4. *If R is a central symmetric ring, then the following conditions are equivalent.*

(1) *R is a right principally projective ring.*

(2) *R is a left principally projective ring.*

(3) *R is a right principally quasi-Baer ring.*

(4) *R is a left principally quasi-Baer ring.*

Proof. (1) $\Rightarrow$ (2) Let R be a right principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $r_R(a) = eR$. Also by Proposition 2.3, R is symmetric. Let $b \in l_R(a)$. Being R symmetric,

$ab = 0$ and so $b = eb$. Since e is central, we have $b = be$. Hence $l_R(a) \subseteq Re$. Obviously, $Re \subseteq l_R(a)$. Thus R is a left principally projective ring.

(2) $\Rightarrow$ (3) Let R be a left principally projective ring and $a \in R$. Then there exists an idempotent $e \in R$ such that $l_R(a) = Re$. Since e is central, we have $ea = ae = 0$. Let $b \in r_R(aR)$. Then $axb = 0$ for all $x \in R$. Hence $bax = 0$, and $ba = 0$ due to $1 \in R$. Thus $b = be = eb$, and so $r_R(aR) \subseteq eR$. Hence $eR = r_R(aR)$ since $eR \subseteq r_R(aR)$ holds also. Therefore R is a right principally quasi-Baer ring.

(3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) are similar to the proofs of preceding cases. $\square$

It is folklore that all reduced rings are symmetric. In the central case, we have the following result.

Lemma 2.5. *If R is a central reduced ring, then it is central symmetric.*

Proof. Let $abc = 0$ for some a, b, $c \in R$. Then $(cab)^2 = cabcab = 0$ so cab is central. On the other hand, $(bca)^2 = bcabca = 0$ so bca is central. For any $r \in R$, $(arbc)^2 = arbcarbc = abcar^2bc = 0$ since bca is central. So $arbc$ is central. For any $r \in R$, $(abrc)^3 = abrcabrcabrc = abcabrcabrrc = 0$ since cab is central, and $(bcra)^2 = bcrabcra = 0$ implies $bcra$ is central. $(carb)^2 = carbcarb = cabcarrb = 0$. So $carb$ is central. For any r, $s \in R$, $(arbsc)^2 = arbscarbsc = arbcarbs^2c = abcar^2bs^2c = 0$ since $carb$ and then bca are central. Hence $arbsc$ is central. $(bac)^4 = b(acbac)b(acbac) = 0$ since $acbac$ is central and $(acbac)^2 = 0$. Hence bac is central. $\square$

In the following we prove when a central symmetric ring is reduced.

Theorem 2.6. *Let R be a central symmetric ring. Then we have*

(1) *If R is a semiprime ring, then R is reduced.*

(2) *If R is a right (left) principally projective ring, then R is reduced.*

Proof. (1) Let $a \in R$ with $a^2 = 0$. For any $r \in R$, $ra^2 = 0$. By hypothesis ara is central, $(ara)^2 = 0$ and so $aRa = 0$. Since R is semiprime, $a = 0$.
(2) Let R be a right principally projective ring and $a \in R$ with $a^2 = 0$. There exists $e^2 = e \in R$ such that $r_R(a) = eR$. Then $a \in r_R(a)$ and $ae = 0$ and $a = ea$. By Proposition 2.3, e is central and so $a = ea = ae = 0$. This proof is left-right symmetric since idempotents are central. $\square$

According to Cohn [9], a ring R is said to be *reversible* if for any a, $b \in R$, $ab = 0$ implies $ba = 0$. Similarly, a ring R is called *central reversible* if for any a, $b \in R$, $ab = 0$ implies ba is central in R. It is well known that every symmetric ring is reversible and the converse holds for semiprime rings. In this direction we have the following.

Lemma 2.7. *Let R be a central symmetric ring. Then R is central reversible. The converse statement holds if R is a semiprime ring.*

Proof. Let a, $b \in R$ with $ab = 0$. Then $1ab = 0$. Hence $1ba = b1a = ba$ is central. Conversely, assume that R is a semiprime central reversible ring. Let a, b, $c \in R$ with $abc = 0$. We may suppose that a, b and c are nonzero. For any $r \in R$, $abcr = 0$ implies $crab$ is central, and $(crab)^2 = 0$. By assumption $crab = 0$. For any $s \in R$, $crab = 0$ implies $bscra$ is central and $(bscra)^2 = 0$. By similar reason $bscra = 0$. Hence $(bac)^2 = bacbac = 0$. For any $t \in R$, $bactbac$ is central and $(bactbac)^2 = 0$. Then $bactbacRbactbac = 0$. Hence $bactbac = 0$ for all $t \in R$. Being R semiprime we have $bac = 0$. □

The next example provides that there exists a central reversible ring which is not a central symmetric ring.

Example 2.8. Let $Q_8 = \{1, x_{-1}, x_i, x_{-i}, x_j, x_{-j}, x_k, x_{-k}\}$ be the quaternion group and consider the group ring $R = \mathbb{Z}_2Q_8$. The elements of $\mathbb{Z}_2Q_8$ as $\mathbb{Z}_2$-linear combinations of $\{x_g : g \in Q_8\}$. By Courter's result in [10, Corollary 2.3], R is reversible and so central reversible. But R is not symmetric as in the [18, Example 7] by taking $a = 1+x_j$, $b = 1+x_i$ and $c = 1+x_i+x_j+x_k$. Then $abc = 0$ but $bac \neq 0$. In fact $bac = x_i + x_j + x_k + x_{-i} + x_{-k}$ and it is easily checked that $x_i(bac) \neq (bac)x_i$. Hence R is not central symmetric.

Now we show that the class of central symmetric rings is closed under finite direct sums.

Proposition 2.9. *Let I be a finite index set and $\{R_i\}_{i\in I}$ a class of rings. Then R_i is central symmetric for all $i \in I$ if and only if $\bigoplus_{i\in I} R_i$ is central symmetric.*

Proof. Let R_i be central symmetric for all $i \in I$ and $(a_i)_{i\in I}, (b_i)_{i\in I}, (c_i)_{i\in I} \in \bigoplus_{i\in I} R_i$ with $(a_i)(b_i)(c_i) = 0$. Then $a_ib_ic_i = 0$ and by hypothesis $b_ia_ic_i$ is central in R_i for all $i \in I$. Hence $(b_i)(a_i)(c_i)$ is central in $\bigoplus_{i\in I} R_i$. Therefore $\bigoplus_{i\in I} R_i$ is central symmetric. The sufficiency is clear since a subring of a central symmetric ring is central symmetric. □

The following result is a direct consequence of Proposition 2.9.

Corollary 2.10. *Let R be a ring. Then eR and $(1 - e)R$ are central symmetric for some idempotent element e in R if and only if R is central symmetric.*

Note that the homomorphic image of a central symmetric ring need not be central symmetric. Consider the following example.

Example 2.11. Let $\mathbb{Z}_2$ denote the field of integers modulo 2 and $\mathbb{Z}_2(y)$ rational functions field of polynomial ring $\mathbb{Z}_2[y]$ and $R = \mathbb{Z}_2(y)[x]$ the ring of polynomials in x over $\mathbb{Z}_2(y)$ subject to the relation $xy + yx = 1$. It is well known that R is a principal ideal domain and so is a non-commutative domain (see [14, p.30], [11, Note 3.9], [21, Example 5.3]). Let $I = x^2R$. Then I is a maximal ideal of R. Consider the ring $S = R/I$. We write $\bar{x}$ and $\bar{y}$ for the images of x and y respectively under the natural epimorphism from R onto S. Let a, b, $c \in R$ with $abc = 0$. Since R is a domain, at least one of a, b and c is zero. Therefore $bac = 0$ and so bac is central and R is central symmetric. For $\bar{x}, \bar{y} \in S$, we have $\bar{x}^2 = \bar{0}$ and $\bar{x}\bar{y} + \bar{y}\bar{x} = \bar{1}$. Multiplying the last equality from the right by $\bar{x}$ and using $\bar{x}^2 = \bar{0}$, we have $\bar{x}\bar{y}\bar{x} = \bar{x}$. If S were central symmetric, $(\bar{x}\bar{y} - \bar{1})\bar{x} = \bar{0}$ would imply $\bar{x}(\bar{x}\bar{y} - \bar{1}) = -\bar{x}$ is central in S and so $\bar{x}$ is central in S. This is a contradiction since $\bar{x}$ is not central.

Our next endeavor is to find conditions when the homomorphic image of a ring is central symmetric. Recall that a ring R is called *unit-central* [15], if all unit elements are central in R. It is proven that every idempotent of a unit-central ring is central.

Lemma 2.12. *Let R be a unit-central ring. If I is a nil ideal of R, then R and R/I are central symmetric.*

Proof. Let $a \in R$ with $a^n = 0$ for some positive integer n. Then $(1 + a)(1 - a + a^2 - a^3 + ... + (-1)^{n-1}a^{n-1}) = 1$. Hence $1 + a$ and therefore a is central. Let a, b and $c \in R$ with $\bar{a}\bar{b}\bar{c} = \bar{0}$ in R/I. Then $abc \in I$. Hence abc is nilpotent. So $1 + abc$, and therefore abc is central. Now $(\bar{c}\bar{a}\bar{b})^2 = \bar{0}$ and $(\bar{b}\bar{c}\bar{a})^2 = \bar{0}$ imply $(cab)^2 \in I$ and $(bca)^2 \in I$, and therefore cab and bca are central. $\bar{a}\bar{b}\bar{c}\bar{r} = \bar{0}$ for all $r \in R$ implies $(\bar{c}\bar{r}\bar{a}\bar{b})^2 = \bar{0}$. So $(crab)^2 \in I$ and $(crab)^2$ is nilpotent and $crab$ is central for all $r \in R$. Similarly, $(\bar{b}\bar{s}\bar{c}\bar{a})^2 = \bar{0}$ implies $bsca$ is central for all $s \in R$. Let $\bar{s}$, $\bar{r} \in R/I$. Then $(\bar{b}\bar{s}\bar{c}\bar{r}\bar{a})^2 = \bar{0}$ since $\bar{c}\bar{r}\bar{a}\bar{b}$ is central nilpotent. Hence $bscra$ is central for s, $r \in R$. Now $(\bar{b}\bar{a}\bar{c})^4 = \bar{0}$ since $\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}$ is central nilpotent. Hence bac is central. Thus $\bar{b}\bar{a}\bar{c}$ is central. □

The next example shows that for a ring R and an ideal I, if R/I is central symmetric, then R need not be central symmetric.

Example 2.13. Let F be a field and R the ring of all 2×2 upper triangular matrices over F and e_{ij} matrix units with 1 at the entry (i, j) and zeros elsewhere. Let $I = e_{12}R$. Then I is an ideal of R and R/I is a commutative ring, therefore central symmetric. Consider $A = e_{22}$,

$B = e_{11} + e_{12}$ and $C = A + B$. Then $ABC = 0$ but $BAC = e_{12}$ is not central since $e_{11}e_{12} = e_{12}$ but $e_{12}e_{11} = 0$. Hence R is not central symmetric.

Let R be a ring with an ideal I. Then I is said to be *prime* if $aRb \subseteq I$ implies a or b is in I, while I is called *completely prime* if $ab \in I$ implies that a or b is in I. Completely prime ideals are prime ideals, but the converse is not true. For example, for any positive integer n, the zero ideal in the ring of all $n \times n$ matrices over a field is a prime ideal, but it is not completely prime. For a ring R and an ideal I, we show that if R/I is a central symmetric ring with a completely prime reduced ideal I, then R is a symmetric ring.

Lemma 2.14. *Let R be a ring. If R/I is a central symmetric ring with a completely prime reduced ideal I, then R is symmetric and so central symmetric.*

Proof. Let $a, b, c \in R$ with $abc = 0$ and $\bar{a}$ will be the image of a under natural epimorphism from R onto R/I. Then $\bar{a}\bar{b}\bar{c} = \bar{0}$. Since R/I is central symmetric and $\bar{a}\bar{b}\bar{c}\bar{r} = \bar{0}$ for any $r \in R$,

$$\bar{b}\bar{a}\bar{c}\bar{r} \text{ is central in } R/I \qquad (*)$$

also, $\bar{r}\bar{a}\bar{b}\bar{c} = \bar{0}$ for any $r \in R$ implies

$$\bar{b}\bar{r}\bar{a}\bar{c} \text{ is central in } R/I \qquad (**)$$

By using (*) and (**) we prove $(\bar{b}\bar{a}\bar{c})^4 = \bar{0}$. By (*), we have $(\bar{b}\bar{a}\bar{c})^4 = (\bar{b}\bar{a}\bar{c}\bar{b})\bar{a}\bar{c}\bar{b}\bar{a}\bar{c}\bar{b}\bar{a}\bar{c} = \bar{a}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{c}\bar{b})\bar{c}\bar{b}\bar{a}\bar{c}$. Hence we have $\bar{a}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{c}\bar{b})\bar{c}\bar{b}\bar{a}\bar{c} = \bar{a}\bar{c}(\bar{b}\bar{a}\bar{b}\bar{a}\bar{c})\bar{b}\bar{c}\bar{b}\bar{a}\bar{c} = \bar{a}\bar{c}\bar{b}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{b}\bar{a}\bar{c})\bar{c}$. Thus $\bar{a}\bar{c}\bar{b}\bar{c}\bar{b}\bar{a}(\bar{b}\bar{a}\bar{b}\bar{a}\bar{c})\bar{c} = \bar{a}\bar{c}(\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{c})\bar{c} = \bar{a}(\bar{b}\bar{c}\bar{b}\bar{a}\bar{b}\bar{a}\bar{b}\bar{a}\bar{c})\bar{c}\bar{c} = \bar{0}$. It follows $(bac)^4 \in I$ and $bac \in I$ and so one of a, b and c belongs to I since I is completely prime. So $cab \in I$ and $(cab)^2 = 0$ implies $cab = 0$ since I is reduced. Similarly, $(bsca)^2 = 0$ implies $bsca = 0$, and $(arbsc)^2 = 0$ implies $arbsc = 0$, and $(cuarbs)^2 = 0$ implies $cuarbs = 0$. Hence $bscuar = 0$ for all r, s, $u \in R$. This implies $(bac)^2 = bacbac = 0$. Thus $bac = 0$. This completes the proof. $\square$

Symmetric rings are generalized by Ouyang and Chen to weak symmetric rings in [19]. A ring R is said to be *weak symmetric*, if for all $a, b, c \in R$, if abc is nilpotent, then acb is nilpotent. We now give an example to show that there exists a weak symmetric ring which is not a central symmetric ring.

Example 2.15. Consider the ring $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} \end{bmatrix}$, and the elements

$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ of R. Then $ABC = 0$. But $BAC = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not central in R. Hence R is not central symmetric. However R is weak symmetric by [19, Proposition 2.3].

A module M has the *summand intersection property* if the intersection of two direct summands is again a direct summand of M. A ring R is said to have the *summand intersection property* if the right R-module R has the summand intersection property. A module M has the *summand sum property* if the sum of two direct summands is a direct summand of M and a ring R is said to have the *summand sum property* if the right R-module R has the summand sum property. A ring R is said to be *abelian* if every idempotent is central. In this case we have the following.

Proposition 2.16. *Let R be a central symmetric ring. Then we have the following.*

(1) *R is an abelian ring.*

(2) *R has the summand intersection property.*

(3) *R has the summand sum property.*

Proof. (1) Let e be an idempotent of R and $x \in R$. Since $e(xe - exe) = 0$ and $(ex - exe)e = 0$, being R central symmetric, $(xe - exe)e$ and $e(ex - exe)$ are central. Then $(xe - exe)e = e(xe - exe) = 0$ and $e(ex - exe) = (ex - exe)e = 0$. Hence we have $ex = xe$ for all $x \in R$. Therefore R is abelian.

(2) Let e and f be idempotents of R. By (1), e and f are central, we have $eR \cap fR = efR = feR$ and $(ef)^2 = ef$. This completes the proof.

(3) Let eR and fR be right ideals of R with $e^2 = e$, $f^2 = f \in R$. Then $e + f - ef$ is an idempotent of R by (1). Since R is abelian, it is easy to check that $eR + fR = (e + f - ef)R$. So $eR + fR$ is a direct summand of R. □

The converse of Proposition 2.16 (1) does not hold in general, that is, every abelian ring need not be central symmetric, as the following example shows.

Example 2.17. Consider the ring

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid \mathrm{a, b, d, c} \in \mathbb{Z}, \mathrm{a} \equiv \mathrm{d} (\mathrm{mod} 2), \mathrm{b} \equiv \mathrm{c} \equiv 0 (\mathrm{mod} 2) \right\}$$

with the usual matrix operations. Since 0 and the identity matrices are the only idempotents of R, R is an abelian ring. Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -4 \\ 0 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \in R$. Then $ABC = 0$ but BAC is not central.

The next result shows that the converse statement of Proposition 2.16 (1) holds for right principally projective rings.

Proposition 2.18. *Let R be a right principally projective ring. If R is abelian, then it is central symmetric.*

Proof. Let $a, b, c \in R$ with $abc = 0$. By hypothesis, there exists an idempotent e of R such that $r_R(a) = eR$. Hence $ae = 0$ and $bc = ebc$. It follows that $bca = ebca = bcae = 0$. Thus R is symmetric and so central symmetric. □

Recall that a ring R is called *directly finite* whenever $a, b \in R$, $ab = 1$ implies $ba = 1$. Then we have the following.

Corollary 2.19. *Every central symmetric ring is directly finite.*

Proof. It is clear from Proposition 2.16 since every abelian ring is directly finite. □

Recall that a ring R is called *semicommutative* if for any a, $b \in R$, $ab = 0$ implies $aRb = 0$. A ring R is called *central semicommutative* [3], if for any $a, b \in R$, $ab = 0$ implies arb is a central element of R for each $r \in R$, while R is *weakly semicommutative* [17], if for any $a, b \in R$, $ab = 0$ implies arb is a nilpotent element for each $r \in R$. In the next proposition, we prove that all central symmetric rings are central semicommutative and weakly semicommutative.

Proposition 2.20. *Let R be a central symmetric ring. Then the followings hold.*

(1) *R is central semicommutative.*

(2) *R is weakly semicommutative.*

Proof. (1) Let $a, b \in R$ with $ab = 0$. For any $r \in R$, $rab = 0$ implies arb is central in R. Hence R is central semicommutative.
(2) Let $a, b \in R$ with $ab = 0$. Since R is central symmetric, ba is central in R. Hence for each $r \in R$, $(arb)^2 = arbarb = ar^2bab = 0$. Therefore R is weakly semicommutative. □

It is well known that a ring is a domain if and only if it is prime and symmetric. In addition to this fact, we have the following proposition when we deal with central case.

Proposition 2.21. *Let R be a ring. Then R is a domain if and only if R is a prime and central symmetric ring.*

Proof. First assume R is a domain. It is clear that R is prime and symmetric and so central symmetric. Conversely, assume R is a prime and central symmetric ring. Let $a, b \in R$ with $ab = 0$. Then $rab = 0$ and $abr = 0$ for all $r \in R$. Being R central symmetric arb and bar are contained in the center of R. Hence we have $(arb)R(arb) = 0$ for any $r \in R$. Since R is prime, $arb = 0$ for any $r \in R$ and so $aRb = 0$. This implies that $a = 0$ or $b = 0$. Therefore R is a domain. □

Let $P(R)$ denote the prime radical and $N(R)$ the set of all nilpotent elements of the ring R. The ring R is called 2-*primal* if $P(R) = N(R)$ (see namely [12] and [13]). In this direction we obtain the following result.

Theorem 2.22. *If R is a central symmetric ring, then it is 2-primal. The converse holds for semiprime rings.*

Proof. To complete the proof it is enough to show that $N(R) \leq P(R)$ since $P(R)$ is a nil ideal. Let $a \in N(R)$. We first assume that $a^2 = 0$. By hypothesis ara is central for any $r \in R$. Commuting ara with sa for any $s \in R$ we have $arasa = 0$. It follows that $a \in P$ for any prime ideal P and so $a \in P(R)$. Assume now $a^3 = 0$. Then a^2ra and ara^2 are central. Commuting a^2ra by sa for any $s \in R$ we have $a^2rasa = 0$. By hypothesis $arasata$ is central for any $t \in R$. Again commute $arasata$ with az for any $z \in R$ and use the centrality of ara^2 for all $r \in R$ to obtain $(az)(arasata) = (arasata)(az) = aras(ata^2)z = (ata^2)arasz = 0$. Since z, t, r and s are arbitrary in R, $a \in P(R)$. By induction on the index of nilpotency we may conclude that $P(R)$ consists of all nilpotent elements of R. Hence R is 2-primal. Conversely, let R be a semiprime and 2-primal ring. Then R is symmetric and so central symmetric. □

Corollary 2.23. *Let R be a central symmetric ring. Then the ring $R/P(R)$ is central symmetric.*

Recall that a ring R is said to be *von Neumann regular* if for every $a \in R$ there exists $b \in R$ with $a = aba$. A ring R is called *strongly regular* if for any $a \in R$ there exists $b \in R$ such that $a = a^2b$. Now we give some relations between symmetric, central symmetric, regular, strongly regular and abelian rings. Also the following theorem provides some conditions for the converses of Proposition 2.3 and Proposition 2.16(1).

Theorem 2.24. *Let R be a ring. Then the following conditions are equivalent.*

(1) *R is strongly regular.*

(2) *R is von Neumann regular and symmetric.*

(3) *R is von Neumann regular and central symmetric.*

(4) *R is von Neumann regular and abelian.*

Proof. (1) $\Rightarrow$ (2) The first assertion is clear. Let $a, b, c \in R$ with $abc = 0$. Since $(bca)^2 = 0$ and $bac = (bac)^2 r$ for some $r \in R$, we have $bca = 0$. Then R is symmetric.

(2) $\Rightarrow$ (3) Obvious. (3) $\Rightarrow$ (4) By Proposition 2.16.

(4) $\Rightarrow$ (1) Let $a \in R$. By hypothesis, there exists $b \in R$ such that $a = aba$. Since ab is an idempotent, ab is central. Hence $a = a^2 b$ and therefore R is strongly regular. $\square$

Let S denote a multiplicatively closed subset of a ring R consisting of central regular elements. Let $S^{-1}R$ be the localization of R at S. Then we have the following.

Proposition 2.25. *A ring R is central symmetric if and only if $S^{-1}R$ is central symmetric.*

Proof. Let R be a central symmetric ring and $a/s_1, b/s_2, c/s_3 \in S^{-1}R$, where $a, b, c \in R$, $s_1, s_2, s_3 \in S$ with $(a/s_1)(b/s_2)(c/s_3) = 0$. Since $abc/s_1 s_2 s_3 = 0$, we have $abc = 0$. So bac is central in R, then $(b/s_2)(a/s_1)(c/s_3)$ is also central in $S^{-1}R$. Therefore $S^{-1}R$ is central symmetric. Conversely, assume that $S^{-1}R$ is a central symmetric ring. Since R may be embedded in $S^{-1}R$, the rest is clear. $\square$

Corollary 2.26. *Let R be a ring. Then $R[x]$ is central symmetric if and only if $R[x, x^{-1}]$ is central symmetric.*

Proof. Consider the subset $S = \{1, x, x^2, x^3, \dots\}$ of $R[x]$ consisting of central regular elements. Then it follows from Proposition 2.25. $\square$

Let R be a ring and $f(x) = \sum_{i=0}^{n} a_i x^i$, $g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Rege and Chhawchharia [20] introduce the notion of an Armendariz ring, that is, a ring R is called *Armendariz*, $f(x)g(x) = 0$ implies $a_i b_j = 0$ for all i and j. The name of the ring was given due to Armendariz [6] who proved that reduced rings satisfied this condition. The interest of this notion lies in its natural and useful role in understanding the relation between the annihilators of the ring R and the annihilators of the polynomial ring $R[x]$. So far, Armendariz rings are generalized in different ways. A ring R is called *nil-Armendariz* [5], if $f(x)g(x)$ has nilpotent coefficients, then a_ib_j is nilpotent for $0 \leq i \leq n$, $0 \leq j \leq s$. According to Harmanci et al. [1], a ring R is called *central Armendariz*, $f(x)g(x) = 0$ implies that a_ib_j is a central element of R for all i and j. In [4, Theorem 5], Anderson and Camillo proved that for a ring R and $n \geq 2$ a natural number, $R[x]/(x^n)$ is Armendariz if and only if R is reduced. For central symmetric rings, we obtain the following result.

Theorem 2.27. *Let R be a right principally projective ring and $n \geq 2$ a natural number. Then R is central symmetric if and only if $R[x]/(x^n)$ is central Armendariz.*

Proof. Suppose R is a central symmetric ring. By Theorem 2.6(2), R is a reduced ring. From [4, Theorem 5], $R[x]/(x^n)$ is Armendariz and so central Armendariz. Conversely, assume that $R[x]/(x^n)$ is central Armendariz. By hypothesis and [1, Theorem 2.5], $R[x]/(x^n)$ is Armendariz. It follows from [4, Theorem 5] that R is reduced and so central symmetric. □

We wind up the paper with some observations.

Theorem 2.28. *If R is a central symmetric ring, then R is nil-Armendariz.*

Proof. If R is central symmetric, then it is 2-primal by Theorem 2.22 and so $N(R)$ is an ideal of R. Proposition 2.1 in [5] states that a ring in which the set of all nilpotent elements forms an ideal is nil-Armendariz. □

Corollary 2.29. *If R is a central symmetric ring, then $R[x]/(x^n)$ is nil-Armendariz, where $n \geq 2$ is a natural number and (x^n) is the ideal generated by x^n.*

Proof. If R is central symmetric, then it is nil-Armendariz by Theorem 2.28. From [5, Proposition 4.1], $R[x]/(x^n)$ is nil-Armendariz. □

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