Algebra and Discrete Mathematics
Number 4. **(2009).** pp. 120 – 134

SURVEY ARTICLE

The structure of infinite dimensional linear groups satisfying certain finiteness conditions

José M. Muñoz-Escolano, Javier Otal and Nikolaj N. Semko

Communicated by I. Ya. Subbotin

Dedicated to Leonid with all our affection in his 60th birthday

Abstract. We review some recent results on the structure of infinite dimensional linear groups satisfying some finiteness conditions on certain families of subgroups. This direction of research is due to Leonid A. Kurdachenko, who developed the main steps of the theory jointly with mathematicians from several countries.

Introduction

Let V be a vector space over a field F. We recall that *a linear group* G is a subgroup of the group $GL(V, F)$ of all F–automorphisms of V under composition of maps. The theory of linear groups is one of the topics which has played a very important role in algebra and other branches of mathematics. This importance occurs because of the well-known isomorphism between the group of all invertible $n \times n$ matrices with entries in F, $n \in \mathbb{N}$, and linear groups, when $n = \dim_F V$ (finite dimensional linear groups), which gives rise to a rich interplay between geometrical and algebraic ideas associated with such groups. Indeed the theory is rich in many interesting and important results (see, for example, [5] and [36], though the bibliography on the topic is very wide).

Supported by Proyecto MTM2007-60994 of Dirección General de Investigación del Ministerio de Educación (Spain).

2000 Mathematics Subject Classification: *20F22, 20H20.*

Key words and phrases: *Linear group. Minimal, maximal, weak minimal and weak maximal conditions. Central and augmentation dimensions.*

The study of the subgroups of $GL(V, F)$ in the case when V is infinite dimensional over F has been much more limited and normally requires some additional restrictions. The circumstances here are similar to those present in the early development of Infinite Group Theory. One approach there consisted in the application of finiteness conditions to the study of infinite groups. One such restriction that has enjoyed considerable attention in linear groups is the notion of a *finitary linear group*. In the late 1980's, R.E. Phillips, J.I. Hall and others studied infinite dimensional linear groups under finiteness conditions, namely finitary linear groups (see [34, 14, 32, 35, 15, 16])). Here G is called *finitary* if, for each element $g \in G$, the subspace $C_V(g)$ has finite codimension in V; the reader is referred to the above papers to see the type of results that have been obtained. This is a good example of the effectiveness of finiteness conditions in the study of infinite dimensional linear groups. Actually, a finitary linear group can be viewed as the linear analogue of an FC–group (group with finite conjugacy classes); this association suggested that it was reasonable to start a systematic investigation of these "infinite dimensional linear groups" analogous to the fruitful study of finiteness conditions in infinite group theory.

If G is a subgroup of $GL(V, F)$, then it is clear that G acts trivially pointwise on the subspace $C_V(G)$, and hence G properly acts on the factor-space $V/C_V(G)$. Leonid Kurdachenko realized this and, trying to extend the concept of a finitary linear group, defined *the central dimension of* G as the F–dimension of $V/C_V(G)$ (denoted by $\mathrm{centdim}_F G$). Thus G *is finitary if and only if* $\mathrm{centdim}_F\langle g\rangle$ *is finite for every* $g \in G$. At this point, it is worth remarking that the above notion of central dimension of a linear group does not define a class of groups, since it heavily relies in the way in which the linear group G is embedded in a *particular* general linear group. In fact, given an abstract group G, it is easy to construct embeddings of G in the same general linear group such that G has infinite or finite central dimension depending on the embedding.

Example 1. Given a prime p, let $A = \langle a_1\rangle \times \langle a_2\rangle \times \ \dots$ and $B = \langle b_1\rangle \times \langle b_2\rangle \times \ \dots$ be two dos copies of the elementary abelian p–group. Then B acts as A as follows:

$$\begin{cases} a_1 \cdot b_j = a_1 & j \geq 1 \\ a_{j+1} \cdot b_j = a_{j+1}a_1 & j+1 \geq 2 \\ a_k \cdot b_j = a_k & k \neq j+1 \end{cases}$$

We think of A as a vector space V over the prime field F_p of p elements and of B as a subgroup G of the general linear group $GL(V, F)$. We have that $C_V(G) = \langle a_1\rangle$ and hence G has infinite central dimension.

Example 2. As above, let $A = \langle a_1\rangle \times \langle a_2\rangle \times \ldots$ and $B = \langle b_1\rangle \times \langle b_2\rangle \times \ldots$ be copies of the elementary abelian p–group. In this case B acts as A as follows:

$$\begin{cases} a_1 \cdot b_j = a_1 a_{j+1} & j \geq 1 \\ a_k \cdot b_j = a_k & k \geq 2,\ j \geq 1 \end{cases}$$

As above, we think of A as a vector space V on F_p and of B as a subgroup G of $GL(V, F)$. Since $C_V(G) = \langle a_2\rangle \times \langle a_3\rangle\ \ldots$, G has finite central dimension.

Consequently, we may not speak of *the class of groups of finite central dimension*, and so we are focusing on *specific and particular* linear groups that have an structure as abstract groups and not conversely. Even more, the effect of the embedding can be very different of the idea we can intuitively think. For example, a finite (linear) group can have infinite central dimension as the following example shows.

Example 3. Let $A = \langle a_1\rangle \times \langle a_2\rangle \times \langle a_3\rangle \times\ \ldots$ be an elementary abelian p–group, p a prime and $G = \langle g\rangle$ be a cyclic group of order 2. We define an action of G on A as follows:

$$\begin{cases} a_{2j} \cdot g = a_{2j-1} & j \geq 1 \\ a_{2j-1} \cdot g = a_{2j} & j \geq 1 \end{cases}$$

Think of A as a vector space V over the field F_p of p elements and of G as a subgroup of $GL(V, F_p)$. In this example $C_V(G) = \langle a_1 a_2\rangle \times \langle a_3 a_4\rangle \times \ldots$ and so G has infinite central dimension.

Suppose that G has finite central dimension. If $C = C_G(C_V(G))$, then C is a normal subgroup of G and G/C is isomorphic to a subgroup of $GL(n, F)$, where $n = \mathrm{centdim}_F G$. Since C stabilizes the series

$$\{0\} \leq C_V(G) \leq V,$$

we have that C is abelian. Moreover, if the characteristic of F is zero, then C is a torsion-free abelian group, if the characteristic of F is the prime p, then C is an elementary abelian p–group (see [13, Corollary to Theorem 3.8] and [12, section 43] for these assertions). Therefore, the structure of the given group G can be determined by the structure of its factor-group G/C, which is an ordinary finite dimensional linear group. In other words, finite dimensional central linear groups behave as finite dimensional linear groups (or, more precisely, their structure can be deduced from that ordinary case), so the efforts must be addressed to infinite dimensional central linear groups.

In the recent years, Leonid Kurdachenko gave rise to a fundamental idea to study these groups and started an intensive research following it.

Let $\mathcal{L}_{icd}(G)$ be the set of all proper subgroups of G of infinite central dimension. In order to study infinite dimensional linear groups G that are close to finite dimensional, Kurdachenko suggested to start making $\mathcal{L}_{icd}(G)$ *very small* in some sense. That is, by imposing some restrictions to $\mathcal{L}_{icd}(G)$. Given the big success that the study of infinite groups with finiteness conditions had enjoyed, it seemed reasonable to study linear groups with finiteness conditions in a similar way as the above.

In this paper we shall discuss some alternative approaches to the study of a infinite dimensional central linear groups, based on the use of different finiteness conditions, namely the different chain conditions. The results quoted here present a survey of recent results in this direction of research. As we mentioned above, this research was developed by Leonid Kurdachenko and other mathematicians (as the authors of the present work). We should mention that the next results are only a part of the work of Kurdachenko in this topic. Actually, he was able to establish the structure of infinite dimensional linear groups under other finiteness conditions. For example, in [8, 3, 4, 2], an study of linear groups such that every $\mathcal{L}_{icd}(G)$–subgroup has a finite (types of) ranks can be found, while the description of linear groups with boundedly finite G–orbits or related conditions can be seen in [10, 11, 9].

1. Minimal and maximal conditions on subgroups of infinite central dimension

In the theory of infinite groups with finiteness conditions the first important problems were concerned with the maximal and minimal conditions, which were first studied in ring and module theory. Soluble groups with the maximal condition were considered by K.A. Hirsch, and S.N. Cernikov began the investigation of groups with the minimal condition. Connected with these problems was the celebrated problem of O.Yu. Schmidt concerning groups all of whose proper subgroups are finite. The investigations which resulted from these problems determined the further development of the theory of groups with finiteness conditions. If G is a group and $\mathcal{M}$ is a family of subgroups of G, we say that $\mathcal{M}$ satisfies *the minimal* (respectively *the maximal*) condition if given a descending chain (respectively an ascending chain) $\{H_n \mid n \in \mathbb{N}\}$ of subgroups belonging to $\mathcal{M}$ there exists some $m \in \mathbb{N}$ such that $H_n = H_{n+1}$ for all $n \geq m$. The former problems deal with the case when the family $\mathcal{M}$ is composed by all subgroups of G (G satisfies Min or G satisfies Max, respectively).

Let $G \leq GL(V, F)$ be a linear group. We say that G *satisfies the minimal condition for subgroups of infinite central dimension*, or G satisfies Min–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the minimal condition. Similarly, we say that G *satisfies the maximal condition for subgroups of infinite central dimension*, or G satisfies Max–icd, if the set $\mathcal{L}_{icd}(G)$ satisfies the maximal condition. Linear groups satisfying Min–icd were studied by Dixon, Evans and Kurdachenko in [6], while Kurdachenko and Subbotin gave a complete description of the structure of linear groups satisfying Max–icd in [30]. Both conditions above are trivially satisfied if $\mathcal{L}_{icd}(G) = \emptyset$, the analogous problem for linear groups of infinite central dimension to O.Yu. Schmidt's foundational problem of Infinite Group Theory.

Theorem 1 ([6])**.** *Suppose that $G \leq GL(V, F)$ is a (locally soluble)–by–finite subgroup of infinite central dimension whose proper subgroups have finite central dimension. Then $G \cong C_{p^\infty}$, where $p \neq char\, F$.*

The next results connect linear groups satisfying Min–icd or Max–icd, finitary linear groups and soluble groups.

Proposition 1 ([6, 30])**.** *Suppose $G \leq GL(V, F)$ and $centdim_F G$ infinite.*

- *If G satisfies Min–icd, then G is finitary or G satisfies Min.*
- *If G satisfies Max–icd, then G is finitary or G is finitely generated.*

Proposition 2 ([6])**.** *Let G be a subgroup of $GL(V, F)$ satisfying Min–icd,*

- *If G is locally soluble, then G is soluble.*
- *If G is locally finite, then G is soluble–by–finite.*

The structure of virtually soluble linear groups with Min–icd is the main result of [6] (and extends a finite dimensional result of Malcev).

Theorem 2. *[6] Let $G \leq GL(V, F)$ be a (locally soluble)–by–finite subgroup. Suppose that G has infinite central dimension and satisfies Min–icd. If G is not Chernikov, then $char\, F = p > 0$ and G has a normal series $H \leq D \leq G$ such that:*

(a) G/D is finite and $D = H \leftthreetimes Q$ where Q is a non-trivial divisible Chernikov p'–subgroup of infinite central dimension;

(b) H is a nilpotent bounded p–subgroup of finite central dimension satisfying the minimal condition on Q–invariant subgroups.

In particular, G is nilpotent–by–abelian–by–finite satisfying Min–n, the minimal condition on normal subgroups.

The description of linear groups with Max–icd is more complicated than the above. We mention here a weak version of the structure theorems, and refer to [30] for a detailed version.

The case when G is infinitely generated (G not finitely generated) is studied first. In this case, G is finitary by Proposition 1.

Theorem 3 ([30]). *Suppose that $G \leq GL(V, F)$ is soluble, has infinite central dimension and satisfies Max–icd. If G/G' is infinitely generated, then G has a normal series $1 \leq H \leq N \leq L \leq G$ such that*

(1) *L has finite index and infinite central dimension;*

(2) *L/H is abelian such that N/H is finitely generated, $centdim_F N$ is finite and $L/N \cong C_{q^\infty}$ where $q \neq char\ F$.*

(3) *H is torsion-free nilpotent when $char\ F = 0$ and H is nilpotent bounded p–subgroup when $char\ F = p > 0$.*

In particular, G is nilpotent–by–abelian–by–finite.

Theorem 4 ([30]). *Suppose that $G \leq GL(V, F)$ is soluble, has infinite central dimension and satisfies Max–icd. If G is infinitely generated, then G has a normal subgroup S of infinite central dimension such that G/S is finitely generated abelian–by–finite and S/S' is infinitely generated.*

Note that Theorem 3 can be applied to find out the structure of S.

We need the following notion. Let $G \leq GL(V, F)$. Then

$$FD(G) = \{x \in G \mid \langle x \rangle \text{ has finite central dimension}\}$$

is a normal subgroup of G ([6]), called the *finitary radical* of G.

We quote now the results for finitely generated groups.

Theorem 5 ([30]). *Suppose that $G \leq GL(V, F)$ is finitely generated soluble, has infinite central dimension and satisfies Max–icd. If the central dimension of $FD(G)$ is finite, then G has a nilpotent normal subgroup $U \leq FD(G)$ of finite central dimension such that*

(1) *G/U is polycyclic;*

(2) *U is torsion-free when $char\ F = 0$; and U is a bounded p–group when $char\ F = p > 0$.*

(3) *U satisfies Max–$\langle g \rangle$ for every $g \in G \setminus FD(G)$.*

In particular, G is nilpotent–by–polycyclic.

Theorem 6 ([30]). *Suppose that $G \leq GL(V,F)$ is finitely generated soluble, has infinite central dimension and satisfies Max–icd. If the central dimension of $FD(G)$ is infinite, then G has a normal subgroup $L \leq FD(G)$ of infinite central dimension such that*

(1) *G/L is abelian–by–finite and L/L' is infinitely generated;*

(2) *L satisfies Max–$\langle g \rangle$ for every $g \in G \setminus FD(G)$.*

The complete structure of G is obtained applying Theorem 3 to L.

2. Weak minimal and weak maximal conditions on infinite central dimensional subgroups

The weak minimal and weak maximal conditions, introduced simultaneously in 1968 by R. Baer [1] and D. I. Zaitsev [37], are the natural generalization of the ordinary minimal and maximal conditions. Let G be a group and let $\mathcal{M}$ be a family of subgroups of G. We say that $\mathcal{M}$ satisfies *the weak minimal* (respectively *the weak maximal*) condition if given a descending chain (respectively an ascending chain) $\{H_n \mid n \in \mathbb{N}\}$ of $\mathcal{M}$–subgroups, there exists some $m \in \mathbb{N}$ such that the indices $|H_n : H_{n+1}|$ (respectively $|H_{n+1} : H_n|$) are finite for all $n \geq m$. Groups with the weak minimal and maximal conditions for several families of subgroups have been studied in many instances. For example, the weak chain conditions on normal subgroups ([19, 21, 22]), subnormal subgroups ([20]), non-normal subgroups ([23]) and non-subnormal subgroups ([28, 29]). See also [31, Section 5.1] and [17] to obtain more information on this topic.

We say that a group $G \leq GL(V,F)$ satisfies the weak minimal (respectively maximal) condition for subgroups of infinite central dimension (or briefly Wmin–icd (respectively Wmax–icd)) if $\mathcal{L}_{icd}(G)$ satisfies the weak minimal (respectively maximal) condition. Periodic linear groups satisfying the conditions Wmin–icd and Wmax–icd were characterized by the authors of this paper in [33]. We first mention an extension of a well-known Zassehhaus' result of finite dimensional linear groups.

Proposition 3 ([33]). *Let G be a periodic subgroup of $GL(V,F)$ satisfying either Wmin–icd or Wmax–icd. Then*

- *If G is locally finite, then G is finitary or G is Chernikov.*
- *If G is locally soluble, then G is soluble.*

In particular, periodic non-Chernikov locally soluble linear groups satisfying Wmin–icd or Wmax–icd are soluble finitary linear groups.

The main result of [33] is the following theorem.

Theorem 7 ([33]). *Let G be a periodic locally soluble subgroup of $GL(V,F)$ of infinite central dimension satisfying either Wmin–icd or Wmax–icd. The following hold:*

(1) If char $F = 0$, then G is Chernikov; and

(2) If char $F = p > 0$, then either G is Chernikov or G has a series of normal subgroups $H \le D \le G$ satisfying the following conditions:

(2a) H is a nilpotent bounded p–subgroup whose central dimension is finite;

(2b) D has finite index and it is a semidirect product $D = H \leftthreetimes Q$, where Q is a Chernikov divisible p'–group whose central dimension is infinite; and

(2c) if K is a Prüfer q–subgroup of Q and $\mathrm{centdim}_F K$ is finite, then H has a finite K–composition series.

From this and Theorem 2, we deduce the following consequences.

Corollary 1 ([33]). *Let $G \le GL(V,F)$ be a periodic locally soluble linear group of infinite central dimension. Then the following conditions are equivalent.*

(1) G satisfies Wmin–icd,

(2) G satisfies Wmax–icd,

(3) G satisfies Min–icd.

Moreover, when G is a periodic locally nilpotent group and one of these conditions holds then G is Chernikov.

Corollary 2 ([33]). *Let $G \le GL(V,F)$ be a periodic locally soluble linear group of infinite central dimension. If G satisfies Max–icd, then G satisfies Min–icd.*

For non-periodic groups the situation is more complicated. Locally nilpotent linear groups satisfying Wmin–icd and Wmax–icd were studied in [25] and [27]. The next result shows that for nilpotent linear groups the weak chain conditions on subgroups of infinite central dimensional and the weak chain conditions on subgroups are equivalent.

Theorem 8 ([25]). *Let G be a nilpotent subgroup of $GL(V,F)$ of infinite central dimension satisfying either Wmin–icd or Wmax–icd. Then G is minimax.*

Other results of [25] were deduced in characteristic prime only. In the next result, $t(G)$ is *the torsion subgroup* of the locally nilpotent group G, $G^{\mathfrak{F}}$ *the finite residual* of G (the intersection of all subgroups of G of finite index), and $G^{\mathfrak{N}}$ *the nilpotent residual* of G (the intersection of all normal subgroups H of G such that G/H is nilpotent).

Theorem 9 ([25])**.** *Let $G \leq GL(V, F)$ be a locally nilpotent linear group of infinite central dimension satisfying either Wmin–icd or Wmax–icd. Suppose that char $F = p > 0$. Then*

- *$G/t(G)$ is minimax*
- *$G/G^{\mathfrak{F}}$ is nilpotent and minimax*
- *$G/G^{\mathfrak{N}}$ is minimax*

In particular, if $t(G)$ has infinite central dimension, then G is minimax.

If G satisfies the weak minimal condition something more can be said.

Theorem 10 ([27])**.** *Let G be a subgroup of $GL(V, F)$ of infinite central dimension satisfying Wmin–icd.*

(i) If G is locally nilpotent, then G is either minimax or finitary.

(ii) If G is hypercentral and char $F = p > 0$, then G is minimax.

Similar results for the condition Wmax–icd are false. Indeed in [27, Section 4], an example of a hypercentral linear group satisfying Wmax–icd, which is neither minimax nor finitary was constructed.

In [26], soluble linear groups satisfying Wmin–icd or Wmax–icd were studied. The main result of that paper shows that their structure is rather like the structure of finite dimensional soluble groups. Recall that $x \in G \leq GL(V, F)$ is *unipotent* if there exists some $n \in \mathbb{N}$ such that $V(x-1)^n = 0$. A subgroup H of G is called *unipotent* if every element of H is unipotent and *boundedly unipotent* if n is independent of x.

Theorem 11 ([26])**.** *Let G be a soluble subgroup of $GL(V, F)$ of infinite central dimension satisfying Wmin–icd or Wmax–icd. Then either G is minimax or G satisfies the following conditions:*

(i) G has a normal boundedly unipotent subgroup L of finite central dimension such that G/L is minimax;

(ii) L is a torsion-free nilpotent subgroup when char $F = 0$;

(iii) L is a bounded nilpotent p–subgroup when char $F = p > 0$.

As in the locally nilpotent case we have

Theorem 12 ([26]). *If $G \leq GL(V, F)$ is soluble, has infinite central dimensional and satisfies Wmin–icd, then G is either minimax or finitary.*

We mention that non-minimax soluble finitary linear groups satisfying Min–icd (hence Wmin–icd) can be constructed ([6, Section 5]). Also, it is easy to construct non-finitary minimax soluble linear groups. On the other hand, analogous results to above for the weak maximal condition are not true. In fact, the example mentioned above ([27, Section 4]) is a metabelian linear group satisfying Wmax–icd which is neither minimax nor finitary. Summing up, our next result states the equivalence of weak chain conditions in linear groups as the last result of this section.

Corollary 3. *Let $G \leq GL(V, F)$. If $centdim_F G$ is infinite and G is*

- *either locally nilpotent non-finitary; or*
- *soluble non-finitary; or*
- *hypercentral and char $F = p > 0$.*

Then the following statements are equivalent

- *G satisfies Wmin–icd,*
- *G satisfies Wmin,*
- *G satisfies Wmax,*
- *G is minimax*

Moreover, if G is nilpotent, then the conditions are also equivalent to the weak maximal condition on subgroups of infinite central dimension.

3. Augmentation dimension: antifinitary linear groups

If $G \leq GL(V, F)$, G acts trivially on $V/V(\omega FG)$, and hence G properly acts on the subspace $V(\omega FG)$. As in [7], the *augmentation dimension* of G is defined to be the F–dimension of $V(\omega FG)$ and is denoted by $augdim_F G$. The study of finite augmentation dimensional linear groups can be reduced to the study of finite dimensional linear groups, just as it happens for finite central dimensional linear groups. Once more, this concept was introduced by Kurdachenko, when considering linear groups whose infinite augmentation dimensional subgroups satisfy the minimal condition ([7]). Later on, linear groups with some rank restrictions on the same subgroups were studied ([4, 2]).

Though the concept of augmentation dimension resembles to be opposite of the concept of central dimension, there is an apparently strong relationship between them. In fact $\dim_F V/C_V(g)) = \dim_F V(g-1)$ for $g \in GL(V,F)$. More generally, if $G \leq GL(V,F)$ is finitely generated, then it is easy to see that the finiteness of one of the dimensions implies the finiteness of the other. However, in the general case, this does not hold as the following specific examples show.

Example 4. Let V and G as in Example 1. Since $C_V(G) = \langle a_1 \rangle = V(\omega(F_pG))$, we have that G has infinite central dimension and finite augmentation dimension.

Example 5. Now, let V and G as in Example 2. In this case $C_V(G) = \langle a_2 \rangle \times \langle a_3 \rangle \times = V(\omega F_p G)$, and so G has finite central dimension and infinite augmentation dimension.

In [18], some conditions over G such that the finiteness of $\mathrm{centdim}_F G$ implies the finiteness of $\mathrm{augdim}_F G$ are established. Given a prime p, it is said that an abstract group G has *finite factor p-rank r* (respectively, *finite factor 0-rank r*) if whenever U and V are normal subgroups of G such that V/U is an abelian p-group (respectively, torsion-free abelian) and H is an intermediate subgroup $U \leq H \leq V$ such that H/U is finitely generated, then the minimal number of elements required to generated H/U is less or equal to r, and r is the least such integer with this property.

Theorem 13 ([18])**.** *Let G be a locally soluble subgroup of $GL(V,F)$ such that char $F = p \geq 0$. Suppose that G has finite factor p–rank. If centdim$_F G$ is finite, then so is augdim$_F G$.*

Theorem 14 ([18])**.** *Let $G \leq GL(V,F)$ be periodic and let char $F = p > 0$. Suppose that G has an ascending series of normal subgroups such that every factor is either a p–group or a p'–group, and there is an integer r such that every finitely generated p–subgroup of G can be generated by r elements. If centdim$_F G$ is finite, then so is augdim$_F G$.*

As we mentioned above, finitary linear groups can be defined as those linear groups whose finitely generated subgroups have finite augmentation dimension. Therefore the following linear groups are in the antipodes of finitary linear groups. We say that a group $G \leq GL(V,F)$ is an *antifinitary linear group* if each proper infinitely generated subgroup of G has finite augmentation dimension. These groups were carefully studied in [24]. We remark that the study has to be focused on non-finitary antifinitary linear groups because of the next result.

Theorem 15 ([7]). *Let G be a (locally soluble)–by–finite linear group of infinite augmentation dimension such that every proper subgroup has finite augmentation dimension. Then $G \cong C_{p^\infty}$ where $p \neq char\, F$.*

In particular, this happens if G is (locally soluble)–by–finite finitary and antifinitary linear group of infinite central dimension. For $\mathcal{L}_{iac}(G) = \emptyset$ and it suffices to apply Theorem 15. Another interesting result about locally finite antifinitary linear groups is the following

Proposition 4 ([24]). *Let $G \leq GL(V, F)$ be a locally finite antifinitary linear group. If G is not finitary, then G is Chernikov.*

The study of antifinitary linear groups splits in a natural way in two cases depending on whether or not the group considered is finitely generated. We recall that a group G is said to be *generalized radical* if G has an ascending series whose factors are locally nilpotent or locally finite. For infinitely generated groups, we were able to establish that

Proposition 5 ([24]). *Let $G \leq GL(V, F)$ be an infinitely generated locally generalized radical antifinitary linear group. If G is not finitary, then G is locally finite.*

Thus, Proposition 4 and Proposition 5 at once give that an infinitely generated antifinitary linear group G that is locally generalized radical is Chernikov. Even more, we have the following detailed description.

Theorem 16 ([24]). *Let G be an infinitely generated locally generalized radical antifinitary subgroup of $GL(V, F)$ such that $G \neq FD(G)$.*

(1) If $G/FD(G)$ is infinitely generated, then G is a Prüfer p–group for some prime p.

(2) If $G/FD(G)$ is finitely generated, then $G = K\langle g\rangle$ satisfying the following conditions:

(2a) K is a divisible abelian Chernikov normal q–subgroup of G, for some prime q

(2b) g is a p–element, where p is a prime such that $p = |G/FD(G)|$;

(2c) K is G–divisibly irreducible, i.e. K has no proper G–invariant subgroups;

(2d) if $q = p$, then K has finite special rank equal to $p^{m-1}(p-1)$ where $p^m = |\langle g\rangle/C_{\langle g\rangle}(K)|$; and

(2e) if $q \neq p$, then K has finite special rank $o(q, p^m)$ where as above $p^m = |\langle g\rangle/C_{\langle g\rangle}(K)|$ and $o(q, p^m)$ is the order of q modulo p^m.

On the other hand, finitely generated antifinitary linear groups can be characterized as follows.

Theorem 17 ([24]). *Let G be a finitely generated radical antifinitary subgroup of $GL(V,F)$. If G has infinite augmentation dimension, then the following conditions holds:*

(1) $augdim_F FD(G)$ is finite;

(2) G has a normal subgroup U such that G/U is polycyclic;

(3) U is boundedly unipotent and, in particular, U is nilpotent;

(4) U is torsion-free if char $F=0$ and is a bounded p–subgroup if char $F=p>0$; and

(5) if

$$\langle 1\rangle = Z_0 \leq Z_1 \leq \cdots \leq Z_m = U$$

is an upper central series of U, then Z_1/Z_0, ..., Z_m/Z_{m-1} are finitely generated $\mathbb{Z}\langle g\rangle$–modules for each element $g \in G \setminus FD(G)$. In particular U satisfies the maximal condition on $\langle g\rangle$–invariant subgroups (Max–$\langle g\rangle$) for every $g \in G \setminus FD(G)$.

References

[1] R. Baer, *Polyminimaxgruppen*, Math. Ann. **175**, 1968, pp.1-43.

[2] O. Yu. Dashkova, *Infinite-dimensional linear groups with restrictions on subgroups that are not soluble A_3–groups*, Algebra and Logic **46(5)**, 2007, pp.297-302.

[3] O. Yu. Dashkova, M. R. Dixon, L. A. Kurdachenko, *Linear groups with rank restrictions on the subgroups of infinite central dimension*, J. Pure Appl. Alg. **208(3)**, 2006, pp.785-795.

[4] O. Yu. Dashkova, M. R. Dixon, L. A. Kurdachenko, *Infinite dimensional linear groups with the restriction on the subgroups of infinite ranks*, Izvestiua Gomel University **3(36)**, 2006, pp.109-123.

[5] J. D. Dixon, *The structure of linear groups*, Van Nostrand, New York 1971.

[6] M. R. Dixon, M. J. Evans, L. A. Kurdachenko, *Linear groups with the minimal condition on subgroups of infinite central dimension*, J. Algebra **277**, 2004, pp.172-186.

[7] M. R. Dixon, M. J. Evans, L. A. Kurdachenko, *Linear groups with the minimal condition on some infinite dimensional subgroups*, Ukrainian Math. J. **57(11)**, 2005, pp.1726-1740.

[8] M. R. Dixon, L. A. Kurdachenko, *Linear groups with infinite central dimension*, in Groups St Andrews 2005, Vol. 1 (C. M. Campbell et al., eds.), London Math. Soc. Lecture Note Ser. **399**. CUP, Cambridge 2007, pp.306-312.

[9] M. R. Dixon, L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, *Trends in infinite dimensional linear groups*, in Groups St Andrews 2009, to appear.

[10] M. R. Dixon, L. A. Kurdachenko, J. Otal, *Linear groups with bounded action*, Alg. Colloquium, to appear.

[11] M. R. Dixon, L. A. Kurdachenko, J. Otal, *Linear groups with finite dimensional orbits*, to appear.

[12] L. Fuchs, *Infinite abelian groups, Vol. 1.* Academic Press, New York 1970.

[13] P. Hall, *Nilpotent groups*, Queen Mary College Notes, London 1969.

[14] J.I. Hall, *Finitary linear transformations groups and elements of finite local degree*, Arch. Math. **50**, 1988, pp. 313–318.

[15] J.I. Hall, *Locally finite simple groups of finitary linear transformations*, In *Finite and locally finite groups, Istambul 1994*, NATO ASI Series Vol. 471, Kluver, Dordrecht 1995, pp. 147–188.

[16] J. I. Hall, *Periodic simple groups of finitary linear transformations*, Ann. of Math. **163(2)**, 2006, pp. 445-498.

[17] L. S. Kazarin, L. A. Kurdachenko, *Conditions for finiteness and factorization in infinite groups*, Russian Math. Surveys **47**, 1992, pp. 81-126.

[18] V. V. Kirichenko, L. A. Kurdachenko, N. V. Polyakov, *On certain finitary modules*, in *Third International Algebraic Conference in Ukraine*, Nats. Akad. Nauk Ukr., Inst. Mat. Kiev, 2002, pp.283-296.

[19] L. A. Kurdachenko, *Groups that satisfy weak minimality and maximality conditions for normal subgroups*, Siberian Math. J. **20**, 1979, pp.755-761.

[20] L. A. Kurdachenko, *Groups satisfying the weak minimum and maximum conditions for subnormal subgroups*, Math. Notes. **29**, 1981, pp.11-61.

[21] L. A. Kurdachenko, *Locally nilpotent groups with the weak minimal condition for normal subgroups*, Siberian Math. J. **25**, 1984, pp.589-594.

[22] L. A. Kurdachenko, *Locally nilpotent groups with weak minimality and maximality conditions for normal subgroups*, Dokl. Akad. Nauk Ukrain. SSR Ser. A **8**, 1985, pp.9-12.

[23] L. A. Kurdachenko, V. E. Goretskii, *Groups with weak minimality and maximality conditions for non-normal subgroups*, Ukrainian Math. J. **41**, 1989, pp.1474-1477.

[24] L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, *Antifinitary linear groups*, Forum Mat. **20**, 2008, pp.27-44.

[25] L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, *Locally nilpotent linear groups with the weak chain conditions on subgroups of infinite central dimension*, Pub. Mat. **52**, 2008, pp.151-169.

[26] L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, *Soluble linear groups with some restrictions on their subgroups of infinite central dimension*, in Ischia Group Theory 2008, to appear.

[27] L. A. Kurdachenko, J. M. Muñoz-Escolano, J. Otal, N. N. Semko, *Locally nilpotent linear groups with restrictions on their subgroups of infinite central dimension*, Geom. Ded. **138**, 2009, pp.69-81.

[28] L. A. Kurdachenko, H. Smith, *Groups with the weak minimal condition for non-subnormal subgroups*, Ann. mat. pura appl. **173(4)**, 1997, pp.299-312.

[29] L. A. Kurdachenko, H. Smith, *Groups with the weak maximal condition for non-subnormal subgroups*, Ricerche Mat. **47(1)**, 1998, pp.29-49.

[30] L. A. Kurdachenko, I. Ya. Subbotin, *Linear groups with the maximal condition on subgroups of infinite central dimension.* Pub. Mat. **50**, 2006, pp.103-131.

[31] J. C. Lennox, D. J. S. Robinson, *The Theory of Infinite Soluble Groups*, OUP, Oxford, 2004.

[32] U. Meirfrankenfeld, R.E. Phillips, O. Puglisi, *Locally solvable finitary linear groups*, J. London Math. Soc. **47**, 1993, pp. 31–40.

[33] J. M. Muñoz-Escolano, J. Otal and N. N. Semko, *Periodic linear groups with the weak chain conditions on subgroups of infinite central dimension*, Com. Alg. **36(2)**, 2008, pp.749-763.

[34] R.E. Phillips, *The structure of groups of finitary transformations*, J. Algebra **119**, 1988, pp. 400–448.

[35] R. E. Phillips, *Finitary linear groups: a survey*, in *Finite and Locally Finite Groups, Istanbul 1994*, (B. Hartley et al., eds.), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **471**. Kluwer Acad. Publ., Dordrecht 1995, pp.111-146.

[36] B. A. F. Wehrfritz, *Infinite Linear Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete **76**, Springer-Verlag, New York, Heidelberg, Berlin 1973.

[37] D. I. Zaitsev, *The groups satisfying the weak minimal condition*, Ukrain. Math. J. **20**, 1968, pp.472-482.

Contact information

J. M. Muñoz-Escolano Departamento de Matemáticas – IUMA, Universidad de Zaragoza, Pedro Cerbuna 12. 50009 Zaragoza, SPAIN.
E-Mail: jmescola@unizar.es

J. Otal Departamento de Matemáticas – IUMA, Universidad de Zaragoza, Pedro Cerbuna 12. 50009 Zaragoza, SPAIN.
E-Mail: otal@unizar.es

N. N. Semko Department of Mathematics, National State Tax Service Academy of Ukraine, 09200 Irpen, UKRAINE.
E-Mail: n_semko@mail.ru

Received by the editors: 23.08.2009
and in final form 23.08.2009.