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RESEARCH ARTICLE

Projectivity and flatness over the graded ring of semi-coinvariants

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Abstract. Let k be a field, C a bialgebra with bijective antipode, A a right C-comodule algebra, G any subgroup of the monoid of grouplike elements of C. We give necessary and sufficient conditions for the projectivity and flatness over the graded ring of semi-coinvariants of A. When A and C are commutative and G is any subgroup of the monoid of grouplike elements of the coring $A \otimes C$, we prove similar results for the graded ring of conormalizing elements of A.

Introduction

In the theory of Hopf-Galois extension, it is often important to know whether certain modules over the ring of coinvariants are projective or flat. These properties reflect the notions of principal bundles and homogeneous spaces in a noncommutative setting. In [4], with S. Caenepeel, we gave sufficient conditions for the projectivity over the subring of coinvariants of an H-comodule algebra, where H is a Hopf algebra. In [6], we gave necessary and sufficient conditions for the projectivity and flatness over the endomorphism ring of a finitely generated module. In [9], necessary and sufficient conditions for the projectivity and flatness over the endomorphism ring of a finitely generated comodule over coring have been studied. In [10], these results have been extended to the colour endomorphism ring of a finitely generated G-graded comodule over a G-graded coring, where G is an abelian group with a bicharacter. To establish all these results the methods and techniques are inspired from [7]. In the present paper, C is a bialgebra, A is a C-comodule algebra and G is any subgroup of the monoid of grouplike elements of C. We consider the G-graded ring $\mathcal{S}(A)$ of semi-coinvariants of A which is a subring of A containing the subalgebra of coinvariants of A. We adapt to the graded set-up the methods and techniques of [7] and [9] to give necessary and sufficient conditions for the projectivity and flatness over the graded ring $\mathcal{S}(A)$.

In an appendix, when A and C are commutative and G is any subgroup of the monoid of grouplike elements of the coring $A \otimes C$, we give necessary and sufficient conditions for the projectivity and flatness over the graded ring $\mathcal{N}(A)$ of conormalizing elements of A which is a subring of A.

Throughout we will be working over a field k. All algebras and coalgebras are over k. Background information on comodules over coalgebras and comodules over corings can be found in [1], [2], [3] and [11]. Except where otherwise stated, all unlabelled tensor products and Hom are tensor products and Hom over k. We denote by $\mathcal{M}$ the category of vector spaces.

1. Preliminary results

We will use the following well-known results of graded ring theory [12]. Let G be a group, B a G-graded ring and $\mathcal{M}_{gr-B}$, the category of right G-graded B-modules.

— Let N be a right G-graded B-module. For every x in G, $N(x)$ is the graded B-module obtained from N by a shift of the gradation by x. As vector spaces, N and $N(x)$ coincide, and the actions of B on N and $N(x)$ are the same, but the gradations are related by $N(x)_y = N_{xy}$ for all $y \in G$.

— An object of $\mathcal{M}_{gr-B}$ is projective (resp. flat) in $\mathcal{M}_{gr-B}$ if and only if it is projective (resp. flat) in $\mathcal{M}_B$, the category of right B-modules.

— An object of $\mathcal{M}_{gr-B}$ is free in $\mathcal{M}_{gr-B}$ if it has a B-basis consisting of homogeneous elements or equivalently, if it is isomorphic to some $\oplus_{i\in I} N(x_i)$, where $(x_i, i \in I)$ is a family of elements of G.

— Any object of $\mathcal{M}_{gr-B}$ is a quotient of a free object in $\mathcal{M}_{gr-B}$, and any projective object in $\mathcal{M}_{gr-B}$ is isomorphic to a direct summand of a free object.

— An object of $\mathcal{M}_{gr-B}$ is flat in $\mathcal{M}_{gr-B}$ if and only if it is the inductive limit of finitely generated free objects in $\mathcal{M}_{gr-B}$.

We will recall some preliminaries on corings and comodules over corings. Let A be a k-algebra. An A-coring $\mathcal{C}$ is an (A, A)-bimodule together with two (A, A)-bimodule maps $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \otimes_A \mathcal{C}$ and $\epsilon_{\mathcal{C}} : \mathcal{C} \to A$ such that the usual coassociativity and counit properties hold. Let $\mathcal{C}$ be an A-coring. A right $\mathcal{C}$-comodule is a right A-module M together with a right A-linear map $\rho_{M,\mathcal{C}} : M \to M \otimes_A \mathcal{C}$ such that

$$(id_M \otimes_A \epsilon_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = id_M, \text{ and } (id_M \otimes_A \Delta_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}} = (\rho_{M,\mathcal{C}} \otimes_A id_{\mathcal{C}}) \circ \rho_{M,\mathcal{C}}.$$

We will use Sweedler-Heyneman notation but we will omit the symbol $\sum$:

$$\Delta_{\mathcal{C}}(c) = c_1 \otimes_A c_2 \qquad \rho_{(M,\mathcal{C})}(m) = m_0 \otimes_A m_1.$$

The algebra A is an A-coring called the trivial A coring. Any k-coalgebra is a k-coring. A morphism of right $\mathcal{C}$-comodules $f : M \to N$ is a right A-linear map such that $\rho_{N,\mathcal{C}} \circ f = (f \otimes_A id_M) \circ \rho_{M,\mathcal{C}}$; or equivalently, a right A-linear map such that $f(m)_0 \otimes_A f(m)_1 = f(m_0) \otimes_A m_1$. We denote the set of comodule morphisms between M and N by $\mathrm{Hom}^{\mathcal{C}}(M, N)$, by $\mathcal{M}^{\mathcal{C}}$ the category formed by right $\mathcal{C}$-comodules and morphisms of right $\mathcal{C}$-comodules. By [2], the category $\mathcal{M}^{\mathcal{C}}$ has direct sum.

We write ${}^*\mathcal{C} = {}_A\mathrm{Hom}({}_A\mathcal{C}, {}_AA)$, the left dual ring of $\mathcal{C}$. Then ${}^*\mathcal{C}$ is an associative ring with unit $\epsilon_{\mathcal{C}}$ (see [2, 17.8]): its multiplication is defined by

$$f\#g = f \circ (id_{\mathcal{C}} \otimes_A g) \circ \Delta_{\mathcal{C}}, \quad \text{or equivalently} \quad f\#g(c) = f(c_1 g(c_2))$$

for all left A-linear maps f, g: $\mathcal{C} \to A$ and $c \in \mathcal{C}$. We will denote by ${}_{^*\mathcal{C}}\mathcal{M}$ the category of left ${}^*\mathcal{C}$-modules. Any right $\mathcal{C}$-comodule M is a left ${}^*\mathcal{C}$-module: the action is defined by $f.m = m_0 f(m_1)$ (see [2, 19.1]).

A grouplike element of $\mathcal{C}$ is an element $X \in \mathcal{C}$ such that $\Delta_{\mathcal{C}}(X) = X \otimes_A X$ and $\epsilon_{\mathcal{C}}(X) = 1_A$. We know from [1] that if $\mathcal{C}$ contains a grouplike element X, then A becomes a right $\mathcal{C}$-comodule: the coaction is defined by $\rho_{A,X}(a) = Xa$. So we have $a_0 \otimes_A a_1 = a_0 a_1 = Xa$. The algebra A equipped with this structure of a right $\mathcal{C}$-module will be denoted A^X.

Lemma 1.1. *Assume that $\mathcal{C}$ contains a grouplike element X. Then A^X is a cyclic left ${}^*\mathcal{C}$-module under the action defined by $f.a = f(Xa)$ for all $f \in {}^*\mathcal{C}$ and $a \in A$.*

Proof. We already noticed that A^X is a left ${}^*\mathcal{C}$-module with the given ${}^*\mathcal{C}$-action. By [2], there is a ring anti-morphism $i : A \to {}^*\mathcal{C}$ defined by $i(a)(c) = a\epsilon_{\mathcal{C}}(c)$; $a \in A$, $c \in \mathcal{C}$. Now for every $a \in A^X$, we have $i(a).1_A = i(a)(X) = a\epsilon_{\mathcal{C}}(X) = a$. □

Let M be a right $\mathcal{C}$-comodule. We define $M_X = \{m \in M | \rho_{M,\mathcal{C}}(m) = m \otimes_A X\}$. We have $A_X = \{a \in A | \rho_{A,\mathcal{C}}(a) = a \otimes_A X = aX\}$. An element $m \in M_X$ is called a X-coinvariant element in ([2], section 28.4) and will be called a conormal element in this paper.

Lemma 1.2. *For every right $\mathcal{C}$-comodule M, $M_X = \mathrm{Hom}^{\mathcal{C}}(A^X, M)$.*

Proof. (See [2], section 28.4). □

Assume that $\mathcal{C}$ is projective as a left A-module. By [2, 18.14], $\mathcal{M}^{\mathcal{C}}$ is a Grothendieck category and by [2, 19.3], it is a full subcategory of ${}_{*\mathcal{C}}\mathcal{M}$; i.e.,

$$\mathrm{Hom}^{\mathcal{C}}(M,N) = {}_{*\mathcal{C}}\,\mathrm{Hom}(M,N) \quad \text{for any} \quad M,N \in \mathcal{M}^{\mathcal{C}}.$$

As a consequence, an object of $\mathcal{M}^{\mathcal{C}}$ that is projective in ${}_{*\mathcal{C}}\mathcal{M}$ is projective in $\mathcal{M}^{\mathcal{C}}$. From now on all comodules are right comodules.

Let C be a bialgebra with comultiplication Δ_C and counit ϵ_C. We will write $\Delta_C(c) = c_1 \otimes c_2$ for all $c \in C$. If M is a C-comodule, we write $\rho_{M,C}(m) = m_{(0)} \otimes m_{(1)}$ for every $m \in M$. Let A be an algebra. We say that A is a right C-comodule algebra if A is a C-comodule and the unit and the multiplication are right C-colinear; i.e.,

$$\rho_{A,C}(aa') = (aa')_{(0)} \otimes (aa')_{(1)} = a_{(0)}a'_{(0)} \otimes a_{(1)}a'_{(1)} \text{ and } \rho_{A,C}(1_A) = 1_A \otimes 1_C.$$

By [2] or [3], $\mathcal{C} = A \otimes C$ is an A-coring with A-multiplications

$$a'(a \otimes c)a'' = a'aa''_{(0)} \otimes ca''_{(1)}$$

and comultiplication $id_A \otimes \Delta_C$. The category $\mathcal{M}^{\mathcal{C}}$ is isomorphic to the category $\mathcal{M}_A^C$ of relative right-right (A, C) Hopf modules, that is the category of right A-modules M which are also C-comodules such that $\rho_{M,C}(ma) = m_{(0)}a_{(0)} \otimes m_{(1)}a_{(1)}$.

Note that for $M \in \mathcal{M}^{\mathcal{C}}$ we have $m_0 \otimes_A m_1 = m_{(0)} \otimes_A (1_A \otimes m_{(1)})$. The morphisms of $\mathcal{M}^{\mathcal{C}}$ are just the A-linear maps which are also C-colinear maps. We will use the notation $\mathcal{M}^{\mathcal{C}}$ instead of $\mathcal{M}_A^C$. The left dual ${}^*\mathcal{C}$ of $\mathcal{C}$ is anti-isomorphic to the Koppinen smash product ${}_{\#}(C, A)$; i.e., the vector space $\mathrm{Hom}(C, A)$ endowed with the product $f\#g(c) = f(c_2)_{(0)}g(c_1 f(c_2)_{(1)})$ and unit $\iota \circ \epsilon_C$, where ι is the unit of A. Every grouplike element x of C induces a grouplike element $1_A \otimes x$ of $\mathcal{C}$. So the coring $\mathcal{C}$ contains $1_A \otimes 1_C$ as a grouplike element, therefore A is an object of $\mathcal{M}^{\mathcal{C}}$.

2. Main results

We keep the notations and conventions of the preceding paragraph, A is an algebra, C is a bialgebra, $\mathcal{C} = A \otimes C$ and $\mathcal{M}^{\mathcal{C}}$ is the category of $\mathcal{C}$-comodules.

Let us denote by G any subgroup of the monoid of grouplike elements of C and by kG the group algebra of G. Let $x \in G$, and let M be a right $\mathcal{C}$-comodule. Set $M_{1\otimes x} = M_x$. So

$$M_x = \{m \in M \quad | \rho_{M,\mathcal{C}}(m) = m_0 \otimes_A (1_A \otimes x) = m \otimes x = \rho_{M,C}(m)\}.$$

When $x = 1_C$, $M_{1_C} = M^{coC}$ is the subspace of C-coinvariants of M and $A_{1_C} = A^{coC}$ is the subring of C-coinvariants of A. An element $m \in M_x$ will be called a semi-coinvariant element. We set $\mathcal{S}(M) = \oplus_{x\in G} M_x$, so $\mathcal{S}(A) = \oplus_{x\in G} A_x$. It is easy to see that $\mathcal{S}(A)$ is a G-graded algebra called the subalgebra of semi-coinvariants of A and $\mathcal{S}(M)$ is a right G-graded $\mathcal{S}(A)$-module called the submodule of semi-coinvariants of M. When C is a Hopf algebra and $G = G(C)$, the algebra $\mathcal{S}(A)$ is called the semi-invariant subalgebra of A in [13]. We will denote by $\mathcal{M}_{gr-\mathcal{S}(A)}$, the category of right G-graded $\mathcal{S}(A)$-modules. The morphisms of this category are the graded morphisms of degree 1_C. Recall that $\mathcal{M}_{gr-\mathcal{S}(A)} = \mathcal{M}^{kG}_{\mathcal{S}(A)}$, the category of relative $(\mathcal{S}(A), kG)$-Hopf modules. For any object $N \in \mathcal{M}_{gr-\mathcal{S}(A)}$, $N \otimes_{\mathcal{S}(A)} A$ is an object of $\mathcal{M}^{\mathcal{C}}$: the A-module structure is the obvious one, while the C-coaction comes from both N and A; i.e., $\rho_{N,C}(n \otimes_{\mathcal{S}(A)} a) = n_x \otimes_{\mathcal{S}(A)} a_{(0)} \otimes xa_{(1)}$ for every $n \in N_x, x \in G$, $a \in A$, where $\rho_{N,kG}(n) = n_x \otimes x$. To each $x \in G$, we associate the functor $(-)_x : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}; \quad M \mapsto M_x$. We also have the semi-coinvariant functor

$$\mathcal{S}(-) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{gr-\mathcal{S}(A)}, \quad M \mapsto \mathcal{S}(M) = \oplus_x M_x$$

and an induction functor

$$F(-) = - \otimes_{\mathcal{S}(A)} A : \mathcal{M}_{gr-\mathcal{S}(A)} \to \mathcal{M}^{\mathcal{C}}; \quad N \mapsto F(N) = N \otimes_{\mathcal{S}(A)} A.$$

It is easy to show that $(F(-), \mathcal{S}(-))$ is an adjoint pair of functors; in other words: for any $M \in \mathcal{M}^{\mathcal{C}}$ and $N \in \mathcal{M}_{gr-\mathcal{S}(A)}$, $\mathrm{Hom}^{\mathcal{C}}(N \otimes_{\mathcal{S}(A)} A, M) \cong \mathrm{Hom}_{gr-\mathcal{S}(A)}(N, \mathcal{S}(M))$. The unit and counit of the pair $(F(-), \mathcal{S}(-))$ are the following: for $N \in \mathcal{M}_{gr-\mathcal{S}(A)}$ and $M \in \mathcal{M}^{\mathcal{C}}$:

$$u_N : N \to \mathcal{S}(N \otimes_{\mathcal{S}(A)} A), \quad u_N(n) = n \otimes_{\mathcal{S}(A)} 1$$

$$c_M : \mathcal{S}(M) \otimes_{\mathcal{S}(A)} A \to M, \quad c_M(m \otimes_{\mathcal{S}(A)} a) = ma.$$

The adjointness property means that we have

$$\mathcal{S}(c_M) \circ u_{\mathcal{S}(M)} = id_{\mathcal{S}(M)}, \quad c_{F(N)} \circ F(u_N) = id_{F(N)} \qquad (\star).$$

Let $x \in G$, and let M be a $\mathcal{C}$-comodule. We can define (see [13, page 346], where C is a Hopf algebra and $G = G(C)$) a new $\mathcal{C}$-comodule M^x, the underlying A-module of which is the same as that of M, while the C-coaction is new and is given by

$$\rho_{M,x}(m) = m_{(0)} \otimes xm_{(1)} = m_{(0)} \otimes_A (1_A \otimes xm_{(1)}) = m_{(0)} \otimes_A (1_A \otimes x)(1_A \otimes m_{(1)}).$$

We call M^x the twisted $\mathcal{C}$-comodule obtained from M and x. Note that M^{1_C} is exactly M with its original $\mathcal{C}$-comodule structure. Note also that

A^x is A with the $\mathcal{C}$-coaction defined by the grouplike element $1 \otimes x$ of $\mathcal{C}$, that is, $A^x = A^{1_A \otimes x}$. So A^{1_C} is exactly the $\mathcal{C}$-comodule A.

By Lemma 1.1, A^x is a cyclic left ${}^*\mathcal{C}$-module, so [6] or [9] gives necessary and sufficient conditions for the projectivity and flatness over the endomorphism ring $\mathrm{Hom}^{\mathcal{C}}(A^x, A^x) = {}_{^*\mathcal{C}}\mathrm{Hom}(A^x, A^x)$.

We have $(M^x)^y = M^{xy}$, $(M^x)_y = M_{x^{-1}y}$ and $A^x \otimes M = M^x$, for all $x, y \in G$. To each element $x \in G$, we associate an equivalent functor

$$(-)^x : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}; \quad M \mapsto M^x,$$

which has inverse $(-)^{x^{-1}}$. Lemma 1.2 implies that the functor $(-)_x$ is isomorphic to $\mathrm{Hom}^{\mathcal{C}}(A^x, -)$.

Let us recall that over any ring A, a left module Λ is called finitely presented if there is an exact sequence $A^m \to A^n \to \Lambda \to 0$ for some natural integers m and n. If A is left noetherian, every finitely generated left A-module is finitely presented.

Lemma 2.1. *The functor $\mathcal{S}(-)$ commutes with direct sums; it commutes with direct limits if ${}^*\mathcal{C}$ is left noetherian.*

Proof. Let $\{M_i\}_{i \in I}$ be a family of objects in $\mathcal{M}^{\mathcal{C}}$. By Lemma 1.1, every A^x is a cyclic ${}^*\mathcal{C}$-module. So the functor $\mathrm{Hom}^{\mathcal{C}}(A^x, -) = {}_{^*\mathcal{C}}\mathrm{Hom}(A^x, -)$ commutes with direct sums in $\mathcal{M}^{\mathcal{C}}$. We have

$$\begin{aligned} \mathcal{S}(\oplus_i M_i) &= \oplus_x \mathrm{Hom}^{\mathcal{C}}(A^x, \oplus_i M_i) \\ &= \oplus_x \oplus_i \mathrm{Hom}^{\mathcal{C}}(A^x, M_i) \\ &= \oplus_i \oplus_x \mathrm{Hom}^{\mathcal{C}}(A^x, M_i) \\ &= \oplus_i \mathcal{S}(M_i) \end{aligned}$$

and we get the first assertion. Assume that ${}^*\mathcal{C}$ is left noetherian, and let $\{M_i\}_{i \in I}$ be a directed family of objects in $\mathcal{M}^{\mathcal{C}}$. Then every A^x is a finitely presented left ${}^*\mathcal{C}$-module since A^x is a finitely generated left ${}^*\mathcal{C}$-module and ${}^*\mathcal{C}$ is left noetherian. So the functor $\mathrm{Hom}^{\mathcal{C}}(A^x, -) = {}_{^*\mathcal{C}}\mathrm{Hom}(A^x, -)$ commutes with direct limits in $\mathcal{M}^{\mathcal{C}}$, and

$$\begin{aligned} \mathcal{S}(\varinjlim M_i) &= \oplus_x \mathrm{Hom}^{\mathcal{C}}(A^x, \varinjlim M_i) \\ &= \oplus_x \varinjlim \mathrm{Hom}^{\mathcal{C}}(A^x, M_i) \\ &= \varinjlim \oplus_x \mathrm{Hom}^{\mathcal{C}}(A^x, M_i) \\ &= \varinjlim \mathcal{S}(M_i) \end{aligned}$$

□

Lemma 2.2. *Let M be a $\mathcal{C}$-comodule. Then*

(1) $\mathcal{S}(M)(x) = \mathcal{S}(M^{x^{-1}})$ for every $x \in G$

(2) The k-linear map $f : \mathcal{S}(A^x) \otimes_{\mathcal{S}(A)} A \to A^x$; $u \otimes_{\mathcal{S}(A)} a \mapsto ua$ is an isomorphism in $\mathcal{M}^{\mathcal{C}}$ for all $u \in \mathcal{S}(A^x)$ and $a \in A$.

Proof. (1) We have $\mathcal{S}(M)(x) = \oplus_{y\in G} M_{xy}$ and $\mathcal{S}(M^{x^{-1}}) = \oplus_{y\in G}(M^{x^{-1}})_y$. On the other hand, $m \in M_{xy}$ if and only if $\rho_{M,C}(m) = m \otimes xy$ if and only if $m_{(0)} \otimes m_{(1)} = m \otimes xy$ if and only if $m_{(0)} \otimes x^{-1}m_{(1)} = m \otimes y$ if and only if $\rho_{M^{x^{-1}},C}(m) = m \otimes y$ if and only if $m \in (M^{x^{-1}})_y$.

(2) Assume that u is homogeneous of degree y. Note that $u \otimes_{\mathcal{S}(A)} a = 1 \otimes_{\mathcal{S}(A)} ua$ for every $a \in A$. Then f is an A-linear isomorphism: its inverse is defined by $a \mapsto 1 \otimes_{\mathcal{S}(A)} a$. Now we have $\rho_{A^x,C}(u) = u \otimes y$; i.e., $u_{(0)} \otimes xu_{(1)} = u \otimes y$; i.e., $u_{(0)} \otimes u_{(1)} = u \otimes x^{-1}y$. It follows that

$$\begin{aligned}(ua)_{(0)} \otimes x(ua)_{(1)} &= ua_{(0)} \otimes xx^{-1}ya_{(1)}\\ &= ua_{(0)} \otimes ya_{(1)}\\ &= f((u \otimes_{\mathcal{S}(A)} a)_{(0)}) \otimes ((u \otimes_{\mathcal{S}(A)} a)_{(1)}\end{aligned}$$

So f is C-colinear. □

Let A be projective in $\mathcal{M}^{\mathcal{C}}$. Then each A^x is also projective in $\mathcal{M}^{\mathcal{C}}$. Therefore Lemma 1.2 implies that the functor $(-)_x$ is exact for every $x \in G$. It follows that the functor $\mathcal{S}(-)$ is exact. We refer the reader to [13, Proposition 1.3] for necessary and sufficient conditions for A to be projective in $\mathcal{M}^{\mathcal{C}}$ if C is a Hopf algebra.

In the remainder of this section, $(x_i, i \in I)$ is a family of elements of G.

Lemma 2.3. *For every index set I,*

(1) $c_{\oplus_{i\in I} A^{x_i^{-1}}}$ is an isomorphism;

(2) $u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}$ is an isomorphism;

(3) if A is projective in $\mathcal{M}^{\mathcal{C}}$, then u is a natural isomorphism; in other words, the induction functor $F = (-) \otimes_{\mathcal{S}(A)} A$ is fully faithful.

Proof. (1) It is straightforward to check that the canonical isomorphism

$$\oplus_{i\in I}\mathcal{S}(A)(x_i) \otimes_{\mathcal{S}(A)} A \simeq \oplus_{i\in I} A^{x_i^{-1}} \text{ is just } c_{\oplus_{i\in I} A^{x_i^{-1}}} \circ (\kappa \otimes id_A),$$

where κ is the isomorphism $\oplus_{i\in I}\mathcal{S}(A)(x_i) \cong \mathcal{S}(\oplus_{i\in I} A^{x_i^{-1}})$, (see Lemmas 2.1 and 2.2). So $c_{\oplus_{i\in I} A^{x_i^{-1}}}$ is an isomorphism.

(2) Putting $M = \oplus_{i\in I} A^{x_i^{-1}}$ in $(\star)$, we find

$$\mathcal{S}(c_{\oplus_{i\in I} A^{x_i^{-1}}}) \circ u_{\mathcal{S}(\oplus_{i\in I} A^{x_i^{-1}})} = id_{\mathcal{S}(\oplus_{i\in I} A^{x_i^{-1}})}.$$

From Lemmas 2.1 and 2.2, we get

$$\mathcal{S}(c_{\oplus_{i\in I} A^{x_i^{-1}}}) \circ u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)} = id_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}.$$

From (1), $\mathcal{S}(c_{\oplus_{i\in I}A^{x_i^{-1}}})$ is an isomorphism, hence $u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}$ is an isomorphism.

(3) Take a free resolution $\oplus_{j\in J}\mathcal{S}(A)(x_j) \to \oplus_{i\in I}\mathcal{S}(A)(x_i) \to N \to 0$ of a right graded $\mathcal{S}(A)$-module N. Since u is natural, we have a commutative diagram

$$\begin{array}{ccccccc}
\oplus_{j\in J}\mathcal{S}(A)(x_j) & \longrightarrow & \oplus_{i\in I}\mathcal{S}(A)(x_i) & \longrightarrow & N & \longrightarrow & 0 \\
\downarrow{\scriptstyle u_{\oplus_{j\in J}\mathcal{S}(A)(x_j)}} & & \downarrow{\scriptstyle u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}} & & \downarrow{\scriptstyle u_N} & & \\
\mathcal{S}(\oplus_{j\in J}A^{x_j^{-1}}) & \longrightarrow & \mathcal{S}(\oplus_{i\in I}A^{x_i^{-1}}) & \longrightarrow & \mathcal{S}(N\otimes_{\mathcal{S}(A)} A) & \longrightarrow & 0
\end{array}$$

The top row is exact. The bottom row is exact, since the sequence $\oplus_{j\in J}A^{x_j^{-1}} \to \oplus_{i\in I}A^{x_i^{-1}} \to N\otimes_{\mathcal{S}(A)} A \to 0$ is exact in $\mathcal{M}^{\mathcal{C}}$ (because $-\otimes_{\mathcal{S}(A)} A$ is right exact) and $\mathcal{S}(-)$ is an exact functor. By (2), $u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}$ and $u_{\oplus_{j\in J}\mathcal{S}(A)(x_j)}$ are isomorphisms. It follows from the five lemma that u_N is an isomorphism. $\square$

We can now give equivalent conditions for projectivity and flatness of $P \in \mathcal{M}_{gr-\mathcal{S}(A)}$.

Theorem 2.4. *For $P \in \mathcal{M}_{gr-\mathcal{S}(A)}$, we consider the following statements.*

(1) $P\otimes_{\mathcal{S}(A)} A$ is projective in $\mathcal{M}^{\mathcal{C}}$ and u_P is injective;

(2) P is projective as a right graded $\mathcal{S}(A)$-module;

(3) $P\otimes_{\mathcal{S}(A)} A$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\oplus_{i\in I}A^{x_i^{-1}}$, and u_P is bijective;

(4) there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $\oplus_{i\in I}A^{x_i^{-1}}$, and $P \cong \mathcal{S}(Q)$ in $\mathcal{M}_{gr-\mathcal{S}(A)}$;

(5) $P\otimes_{\mathcal{S}(A)} A$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\oplus_{i\in I}A^{x_i^{-1}}$.

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).

If A is projective in $\mathcal{M}^{\mathcal{C}}$, then (5) $\Rightarrow$ (3) $\Rightarrow$ (1).

Proof. (2) $\Rightarrow$ (3). If P is projective as a right graded $\mathcal{S}(A)$-module, then we can find an index set I and $P' \in \mathcal{M}_{gr-\mathcal{S}(A)}$ such that $\oplus_{i\in I}\mathcal{S}(A)(x_i) \cong P\oplus P'$. Then obviously $\oplus_{i\in I}A^{x_i^{-1}} \cong \oplus_{i\in I}\mathcal{S}(A)(x_i)\otimes_{\mathcal{S}(A)}A \cong (P\otimes_{\mathcal{S}(A)}A)\oplus(P'\otimes_{\mathcal{S}(A)} A)$. Since u is a natural transformation, we have a commutative diagram:

$$\begin{array}{ccc}
\oplus_{i\in I}\mathcal{S}(A)(x_i) & \xrightarrow{\cong} & P\oplus P' \\
\downarrow{\scriptstyle u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}} & & \downarrow{\scriptstyle u_P\oplus u_{P'}} \\
\mathcal{S}(\oplus_{i\in I}A^{x_i^{-1}}) & \xrightarrow[\cong]{} & \mathcal{S}(P\otimes_{\mathcal{S}(A)} A)\oplus\mathcal{S}(P'\otimes_{\mathcal{S}(A)} A)
\end{array}$$

From the fact that $u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}$ is an isomorphism, it follows that u_P (and $u_{P'}$) are isomorphisms.

(3) $\Rightarrow$ (4). Take $Q = P \otimes_{\mathcal{S}(A)} A$.

(4) $\Rightarrow$ (2). Let $f : \oplus_{i\in I} A^{x_i^{-1}} \to Q$ be a split epimorphism in $\mathcal{M}^{\mathcal{C}}$. Then the map $\mathcal{S}(f) : S(\oplus_{i\in I} A^{x_i^{-1}}) \cong \oplus_{i\in I}\mathcal{S}(A)(x_i) \to \mathcal{S}(Q) \cong P$ is split surjective in $\mathcal{M}_{gr-\mathcal{S}(A)}$, hence P is projective as a right graded $\mathcal{S}(A)$-module.

(4) $\Rightarrow$ (5). We already proved that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). Since (5) is contained in (3), we get (4) $\Rightarrow$ (5).

(1) $\Rightarrow$ (2). Take an epimorphism $f : \oplus_{i\in I}\mathcal{S}(A)(x_i) \to P$ in $\mathcal{M}_{gr-\mathcal{S}(A)}$. Then

$$F(f) =: \oplus_{i\in I}\mathcal{S}(A)(x_i) \otimes_{\mathcal{S}(A)} A \cong \oplus_{i\in I} A^{x_i^{-1}} \to P \otimes_{\mathcal{S}(A)} A$$

is surjective, and splits in $\mathcal{M}^{\mathcal{C}}$ since $P \otimes_{\mathcal{S}(A)} A$ is projective in $\mathcal{M}^{\mathcal{C}}$. Consider the commutative diagram

$$\begin{array}{ccccc}
\oplus_{i\in I}\mathcal{S}(A)(x_i) & \xrightarrow{f} & P & \longrightarrow & 0 \\
\Big\downarrow{\scriptstyle u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}} & & \Big\downarrow{\scriptstyle u_P} & & \\
\mathcal{S}(\oplus_{i\in I} A^{x_i^{-1}}) & \xrightarrow[\mathcal{S}F(f)]{} & \mathcal{S}(P \otimes_{\mathcal{S}(A)} A) & \longrightarrow & 0
\end{array}$$

The bottom row is split exact, since any functor, in particular $\mathcal{S}(-)$ preserves split exact sequences. By Lemma 2.3(2), $u_{\oplus_{i\in I}\mathcal{S}(A)(x_i)}$ is an isomorphism. A diagram chasing argument tells us that u_P is surjective. By assumption, u_P is injective, so u_P is bijective. We deduce that the top row is isomorphic to the bottom row, and therefore splits. Thus $P \in \mathcal{M}_{gr-\mathcal{S}(A)}$ is projective.

(5) $\Rightarrow$ (3). Under the assumption that A is projective in $\mathcal{M}^{\mathcal{C}}$, (5) $\Rightarrow$ (3) follows from Lemma 2.3(3).

(3) $\Rightarrow$ (1). By (3), $P \otimes_{\mathcal{S}(A)} A$ is a direct summand of some $\oplus_{i\in I} A^{x_i^{-1}}$. If A is projective in $\mathcal{M}^{\mathcal{C}}$, then $\oplus_{i\in I} A^{x_i^{-1}}$ is projective in $\mathcal{M}^{\mathcal{C}}$. So $P \otimes_{\mathcal{S}(A)} A$ being a direct summand of a projective object of $\mathcal{M}^{\mathcal{C}}$ is projective in $\mathcal{M}^{\mathcal{C}}$. □

Theorem 2.5. *Assume that ${}^*\mathcal{C}$ is left noetherian. For $P \in \mathcal{M}_{gr-\mathcal{S}(A)}$, the following assertions are equivalent.*

(1) P is flat as a right graded $\mathcal{S}(A)$-module;

(2) $P \otimes_{\mathcal{S}(A)} A = \varinjlim Q_i$, where $Q_i \cong \oplus_{j\leq n_i} A^{x_{ij}^{-1}}$ in $\mathcal{M}^{\mathcal{C}}$ for some positive integer n_i, and u_P is bijective;

(3) $P \otimes_{\mathcal{S}(A)} A = \varinjlim Q_i$*, where* $Q_i \in \mathcal{M}^{\mathcal{C}}$ *is a direct summand of some* $\oplus_{j \in I_i} A^{x_{ij}^{-1}}$ *in* $\mathcal{M}^{\mathcal{C}}$*, and* u_P *is bijective;*

(4) there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$*, such that* $Q_i \cong \oplus_{j \le n_i} A^{x_{ij}^{-1}}$ *for some positive integer* n_i *and* $\mathcal{S}(Q) \cong P$ *in* $\mathcal{M}_{gr-\mathcal{S}(A)}$*;*

(5) there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$*, such that* Q_i *is a direct summand of some* $\oplus_{j \in I_i} A^{x_{ij}^{-1}}$ *in* $\mathcal{M}^{\mathcal{C}}$*, and* $\mathcal{S}(Q) \cong P$ *in* $\mathcal{M}_{gr-\mathcal{S}(A)}$*.*

If A *is projective in* $\mathcal{M}^{\mathcal{C}}$*, these conditions are also equivalent to conditions (2) and (3) without the assumption that* u_P *is bijective.*

Proof. (1) $\Rightarrow$ (2). $P = \varinjlim N_i$, with $N_i = \oplus_{j \le n_i} \mathcal{S}(A)(x_{ij})$. Take $Q_i = \oplus_{j \le n_i} A^{x_{ij}^{-1}}$, then

$$\varinjlim Q_i \cong \varinjlim (N_i \otimes_{\mathcal{S}(A)} A) \cong (\varinjlim N_i) \otimes_{\mathcal{S}(A)} A \cong P \otimes_{\mathcal{S}(A)} A.$$

Consider the following commutative diagram:

$$\begin{array}{ccc}
P = \varinjlim N_i & \xrightarrow{\lim(u_{N_i})} & \varinjlim \mathcal{S}(N_i \otimes_{\mathcal{S}(A)} A) \\
\downarrow{\scriptstyle u_P} & & \downarrow{\scriptstyle f} \\
\mathcal{S}((\varinjlim N_i) \otimes_{\mathcal{S}(A)} A) & \xrightarrow[\cong]{} & \mathcal{S}(\varinjlim (N_i \otimes_{\mathcal{S}(A)} A))
\end{array}$$

By Lemma 2.3(2), the u_{N_i} are isomorphisms. By Lemma 2.1, the natural homomorphism f is an isomorphism. Hence u_P is an isomorphism.

(2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) are obvious.

(2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5). Put $Q = P \otimes_{\mathcal{S}(A)} A$. Then $u_P : P \to \mathcal{S}(P \otimes_{\mathcal{S}(A)} A)$ is the required isomorphism.

(5) $\Rightarrow$ (1). We have a split exact sequence $0 \to N_i \to P_i = \oplus_{j \in I_i} A^{x_{ij}^{-1}} \to Q_i \to 0$ in $\mathcal{M}^{\mathcal{C}}$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F\mathcal{S}(N_i) & \longrightarrow & F\mathcal{S}(P_i) & \longrightarrow & F\mathcal{S}(Q_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c_{N_i}} & & \downarrow{\scriptstyle c_{P_i}} & & \downarrow{\scriptstyle c_{Q_i}} & & \\
0 & \longrightarrow & N_i & \longrightarrow & P_i & \longrightarrow & Q_i & \longrightarrow & \mathbf{0}
\end{array}$$

We know from Lemma 2.3(1) that c_{P_i} is an isomorphism. Both rows in the diagram are split exact, so it follows that c_{N_i} and c_{Q_i} are also isomorphisms. Next consider the commutative diagram:

$$\begin{array}{ccc}
(\varinjlim \mathcal{S}(Q_i)) \otimes_{\mathcal{S}(A)} A & \xrightarrow{f \otimes id_A} & \mathcal{S}(Q) \otimes_{\mathcal{S}(A)} A \\
\uparrow{\scriptstyle h} & & \downarrow{\scriptstyle c_Q} \\
\varinjlim (\mathcal{S}(Q_i) \otimes_{\mathcal{S}(A)} A) & \xrightarrow{\lim c_{Q_i}} & \mathbf{Q}
\end{array}$$

where h is the natural homomorphism and f is the isomorphism $\varinjlim \mathcal{S}(Q_i) \cong \mathcal{S}(\varinjlim(Q_i))$ (see Lemma 2.1). h is an isomorphism, because the functor $(-) \otimes_{\mathcal{S}(A)} A$ preserves inductive limits. $limc_{Q_i}$ is an isomorphism, because every c_{Q_i} is an isomorphism. It follows that c_Q is an isomorphism, hence $\mathcal{S}(c_Q)$ is an isomorphism. From ($\star$), we get $\mathcal{S}(c_Q) \circ u_{\mathcal{S}(Q)} = id_{\mathcal{S}(Q)}$. It follows that $u_{\mathcal{S}(Q)}$ is also an isomorphism. Since $\mathcal{S}(Q) \cong P$, u_P is an isomorphism. Consider the isomorphisms

$$P \cong \mathcal{S}(P \otimes_{\mathcal{S}(A)} A) \cong \mathcal{S}(\mathcal{S}(Q) \otimes_{\mathcal{S}(A)} A) \cong \mathcal{S}(Q) \cong \varinjlim \mathcal{S}(Q_i);$$

where the first isomorphism is u_P, the third is $\mathcal{S}(c_Q)$ and the last one is f. By Lemmas 2.1 and 2.2, each $\mathcal{S}(P_i) \cong \oplus_{j \in I} \mathcal{S}(A)(x_{ij})$ is projective as a right graded $\mathcal{S}(A)$-module, hence each $\mathcal{S}(Q_i)$ is also projective as a right graded $\mathcal{S}(A)$-module, and we conclude that $P \in \mathcal{M}_{gr-\mathcal{S}(A)}$ is flat. The final statement is an immediate consequence of Lemma 2.3(3). □

3. Appendix

The notations introduced in the preceding sections will be retained throughout. We want to extend the results of section 2 to the more general type of grouplike elements of $\mathcal{C} = A \otimes C$. Assume that A and C are commutative. Then the coring $\mathcal{C} = A \otimes C$ becomes a commutative associative algebra with identity element $1_A \otimes 1_C$: the multiplication in $\mathcal{C}$ is given by $(a \otimes c)(a' \otimes c') = aa' \otimes cc'$. Denote by $G(\mathcal{C})$ the set of grouplike elements of $\mathcal{C}$. Let $a_i \in A$ and $c_i \in C$. Then $\sum(a_i \otimes c_i)$ is an element of $G(\mathcal{C})$ if and only if

$$\sum(a_i \otimes c_{i1} \otimes c_{i2}) = \sum(a_i a_{j(0)} \otimes c_i a_{j(1)} \otimes c_j) \quad \text{and} \quad \sum a_i \epsilon_C(c_i) = 1_A.$$

The product of two grouplike elements of $\mathcal{C}$ is a grouplike element.

Let $X = \sum(a_i \otimes c_i)$ be an element of $G(\mathcal{C})$, and let M be a $\mathcal{C}$-comodule. We have $\rho_{M,\mathcal{C}}(m) = m_0 \otimes_A m_1 = m_{(0)} \otimes_A (1_A \otimes m_{(1)})$. For every $X \in G(\mathcal{C})$, we can define a new $\mathcal{C}$-comodule M^X, the underlying A-module of which is the same as that of M, while the C-coaction is new and is given by $\rho_{M,X}(m) = m_0 \otimes_A m_1 = m_{(0)} \otimes_A X(1_A \otimes m_{(1)})$; i.e.; $\rho_{M,X}(m) = \sum m_{(0)} a_i \otimes c_i m_{(1)}$, where $X = \sum(a_i \otimes c_i) \in G(\mathcal{C})$. We call M^X the twisted $\mathcal{C}$-comodule obtained from M and X. Note that $M^{1_A \otimes 1_C}$ is exactly M with its original $\mathcal{C}$-comodule structure. For every $a \in A^X$, we have $\rho_{A,X}(a) = a_{(0)} \otimes_A X(1_A \otimes a_{(1)})$. So A^X is exactly the one we have defined in section 1. We have $(M^X)^Y = M^{XY}$ and $A^X \otimes_A M = M^X$, for all $X, Y \in G(\mathcal{C})$.

From now on we assume that G is any subgroup of the monoid $G(\mathcal{C})$. We have $(M^X)_Y = M_{X^{-1}Y}$ for every $X \in G$. We set $\mathcal{N}(M) = \oplus_{X \in G} M_X$, so $\mathcal{N}(A) = \oplus_{X\in G} A_X$. Then $\mathcal{N}(A)$ is a commutative G-graded algebra called the subalgebra of conormalizing elements of A and $\mathcal{N}(M)$ is a right G-graded $\mathcal{N}(A)$-module called the submodule of conormalizing elements of M. We will denote by $\mathcal{M}_{gr-\mathcal{N}(A)}$ the category of G-graded $\mathcal{N}(A)$-modules. The morphisms of this category are the graded morphisms of degree $1_A \otimes 1_C$. Recall that $\mathcal{M}_{gr-\mathcal{N}(A)}$ is the category $\mathcal{M}^{kG}_{\mathcal{N}(A)}$ of relative right-right $(\mathcal{N}(A), kG)$-Hopf modules. For any object $N \in \mathcal{M}_{gr-\mathcal{N}(A)}$, $N \otimes_{\mathcal{N}(A)} A$ is an object of $\mathcal{M}^{\mathcal{C}}$: the A-module structure is the obvious one and the C-coaction comes from both N and A; i.e., $\rho_{N,\mathcal{C}}(n \otimes_{\mathcal{N}(A)} a) = n_X \otimes_{\mathcal{N}(A)} a_{(0)} \otimes_A X(1_A \otimes a_{(1)})$ for every $n \in N_X; X \in G$, $a \in A$, where $\rho_{N,kG}(n) = n_X \otimes X$. We have an induction functor,

$$G = - \otimes_{\mathcal{N}(A)} A : \mathcal{M}_{gr-\mathcal{N}(A)} \to \mathcal{M}^{\mathcal{C}}; \quad N \mapsto N \otimes_{\mathcal{N}(A)} A.$$

To each element $X \in G$, we associate an equivalent functor

$$(-)^X : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{C}}; \quad M \mapsto M^X,$$

which has inverse $(-)^{X^{-1}}$. We also associate to each $X \in G$ a functor

$$(-)_X : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{gr-\mathcal{N}(A)}; \quad M \mapsto M_X.$$

We define the conormalizing functor

$$\mathcal{N}(-) : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}_{gr-\mathcal{N}(A)}, \quad M \mapsto \mathcal{N}(M) = \oplus_X M_X.$$

Lemma 3.1. $(-\otimes_{\mathcal{N}(A)} A, \quad \mathcal{N}(-))$ *is an adjoint pair of functors; in other words, for any* $M \in \mathcal{M}^{\mathcal{C}}$ *and* $N \in \mathcal{M}_{gr-\mathcal{N}(A)}$, $\mathrm{Hom}^{\mathcal{C}}(N \otimes_{\mathcal{N}(A)} A, M) \cong \mathrm{Hom}_{gr-\mathcal{N}(A)}(N, \mathcal{N}(M))$.

Proof. Let N be an object of $\mathcal{M}_{gr-\mathcal{N}(A)}$, M an object of $\mathcal{M}^{\mathcal{C}}$ and $f \in \mathrm{Hom}^{\mathcal{C}}(N \otimes_{\mathcal{N}(A)} A, M)$. Let n be a homogeneous element of N of degree X, then $n \otimes_{\mathcal{N}(A)} 1_A$ is an element of $(N \otimes_{\mathcal{N}(A)} A)_X$ and $f(n \otimes_{\mathcal{N}(A)} 1_A) \in M_X$. Let us define k-linear maps

$$\phi : \mathrm{Hom}^{\mathcal{C}}(N \otimes_{\mathcal{N}(A)} A, M) \to \mathrm{Hom}(N, \mathcal{N}(M))$$

by $\phi(f)(n) = f(n \otimes_{\mathcal{N}(A)} 1_A)$ and

$$\psi : \mathrm{Hom}_{gr-\mathcal{N}(A)}(N, \mathcal{N}(M)) \to \mathrm{Hom}(N \otimes_{\mathcal{N}(A)} A, M)$$

by $\psi(g)(n \otimes_{\mathcal{N}(A)} a) = g(n)a$. It is easy to show that

$$\phi(f) \in \mathrm{Hom}_{gr-\mathcal{N}(A)}(N, \mathcal{N}(M)), \quad \psi(g) \in \mathrm{Hom}^{\mathcal{C}}(N \otimes_{\mathcal{N}(A)} A, M)$$

and that ϕ is a bijection with inverse ψ. □

Let us denote by F' the functor $- \otimes_{\mathcal{N}(A)} A$. The unit and counit of the adjunction pair $(F', \quad \mathcal{N}(-))$ are the following: for $N \in \mathcal{M}_{gr-\mathcal{N}(A)}$ and $M \in \mathcal{M}^{\mathcal{C}}$:

$$u_N : N \to \mathcal{N}(N \otimes_{\mathcal{N}(A)} A), \quad u_N(n) = n \otimes_{\mathcal{N}(A)} 1_A$$

$$c_M : \mathcal{N}(M) \otimes_{\mathcal{N}(A)} A \to M, \quad c_M(m \otimes_{\mathcal{N}(A)} a) = ma.$$

The adjointness property means that we have

$$\mathcal{N}(c_M) \circ u_{\mathcal{N}(M)} = id_{\mathcal{N}(M)}, \quad c_{F'(N)} \circ F'(u_N) = id_{F'(N)} \qquad (\star\star).$$

The proofs of the following results are similar to those of the preceding section and we omit them.

Lemma 3.2. *The functor $\mathcal{N}(-)$ commutes with direct sums; it commutes with direct limits if ${}^*\mathcal{C}$ is left noetherian.*

Let A be projective in $\mathcal{M}^{\mathcal{C}}$. Then each A^X is projective in $\mathcal{M}^{\mathcal{C}}$. So by Lemma 1.2, the functor $(-)_X$ is exact for every $X \in G$. It follows that the functor $\mathcal{N}(-)$ is exact.

Lemma 3.3. *Let M be a $\mathcal{C}$-comodule. Then*
(1) $\mathcal{N}(M)(X) = \mathcal{N}(M^{X^{-1}})$ for every $X \in G$;
(2) The k-linear map $f : \mathcal{N}(A^X) \otimes_{\mathcal{N}(A)} A \to A^X$; $u \otimes_{\mathcal{N}(A)} a \mapsto ua$ is an isomorphism in $\mathcal{M}^{\mathcal{C}}$.

Lemma 3.4. *For every index set I,*
(1) $c_{\oplus_{i\in I} A^{X_i^{-1}}}$ is an isomorphism;
(2) $u_{\oplus_{i\in I}\mathcal{N}(A)(X_i)}$ is an isomorphism;
(3) if A is projective in $\mathcal{M}^{\mathcal{C}}$, then u is a natural isomorphism; in other words, the induction functor $F' = (-) \otimes_{\mathcal{N}(A)} A$ is fully faithful.

Theorem 3.5. *For $P \in \mathcal{M}_{gr-\mathcal{N}(A)}$, we consider the following statements.*
(1) $P \otimes_{\mathcal{N}(A)} A$ is projective in $\mathcal{M}^{\mathcal{C}}$ and u_P is injective;
(2) P is projective as a graded $\mathcal{N}(A)$-module;
(3) $P \otimes_{\mathcal{N}(A)} A$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\oplus_{i\in I} A^{X_i^{-1}}$, and u_P is bijective;
(4) there exists $Q \in \mathcal{M}^{\mathcal{C}}$ such that Q is a direct summand of some $\oplus_{i\in I} A^{X_i^{-1}}$, and $P \cong \mathcal{N}(Q)$ in $\mathcal{M}_{gr-\mathcal{N}(A)}$;
(5) $P \otimes_{\mathcal{N}(A)} A$ is a direct summand in $\mathcal{M}^{\mathcal{C}}$ of some $\oplus_{i\in I} A^{X_i^{-1}}$.
Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5).
If A is projective in $\mathcal{M}^{\mathcal{C}}$, then (5) $\Rightarrow$ (3) $\Rightarrow$ (1).

Theorem 3.6. *Assume that* $^*\mathcal{C}$ *is left noetherian. For* $P \in \mathcal{M}_{gr-\mathcal{N}(A)}$*, the following assertions are equivalent.*

(1) P *is flat as a graded* $\mathcal{N}(A)$*-module;*

(2) $P \otimes_{\mathcal{N}(A)} A = \varinjlim Q_i$*, where* $Q_i \cong \oplus_{j \leq n_i} A^{X_{ij}^{-1}}$ *in* $\mathcal{M}^{\mathcal{C}}$ *for some positive integer* n_i*, and* u_P *is bijective;*

(3) $P \otimes_{\mathcal{N}(A)} A = \varinjlim Q_i$*, where* $Q_i \in \mathcal{M}^{\mathcal{C}}$ *is a direct summand of some* $\oplus_{j \in I_i} A^{X_{ij}^{-1}}$ *in* $\mathcal{M}^{\mathcal{C}}$*, and* u_P *is bijective;*

(4) there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$*, such that* $Q_i \cong \oplus_{j \leq n_i} A^{X_{ij}^{-1}}$ *for some positive integer* n_i *and* $\mathcal{N}(Q) \cong P$ *in* $\mathcal{M}_{gr-\mathcal{N}(A)}$*;*

(5) there exists $Q = \varinjlim Q_i \in \mathcal{M}^{\mathcal{C}}$*, such that* Q_i *is a direct summand of some* $\oplus_{j \in I_i} A^{X_{ij}^{-1}}$ *in* $\mathcal{M}^{\mathcal{C}}$*, and* $\mathcal{N}(Q) \cong P$ *in* $\mathcal{M}_{gr-\mathcal{N}(A)}$*.*

If A *is projective in* $\mathcal{M}^{\mathcal{C}}$*, these conditions are also equivalent to conditions (2) and (3), without the assumption that* u_P *is bijective.*

We conclude the paper by the following remarks:

Remarks 3.7. By [8, Propostion 2.3], if C is a finite-dimensional Hopf algebra, then $G(A \otimes C)$ is a group. If $G(A \otimes C) = \{1 \otimes c; \quad c \in G(C)\}$, then $G(A \otimes C)$ is obviously a group isomorphic to $G(C)$. In this case, the conormal elements and the semi-coinvariant elements are exactly the same. By [5, Proposition 5.1], this can happen in the following situation: k is algebraically closed, A is a finitely generated normal k-algebra and C is the affine coordinate ring of a connected algebraic group $\mathcal{G}$ acting rationally on A. More precisely, in this situation, we have

$$G(A \otimes C) = \{1 \otimes \phi; \quad \phi \in G(C) = \chi(\mathcal{G})\},$$

where $\chi(\mathcal{G})$ is the group of characters of $\mathcal{G}$.

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