Partitions of groups and matroids into independent subsets

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ABSTRACT. Can the set $\mathbb{R} \setminus \{0\}$ be covered by countably many linearly (algebraically) independent subsets over the field $\mathbb{Q}$? We use a matroid approach to show that an answer is "Yes" under the Continuum Hypothesis, and "No" under its negation.

In this paper we discuss partitions of groups and other algebraic objects into independent subsets. The notion of an independent set can be defined for any hull operator.

By a *hull operator* on a set X we understand any function $\langle \cdot \rangle : \mathcal{P}_X \to \mathcal{P}_X$ defined on the power-set $\mathcal{P}_X$ of X such that $A \subset \langle A \rangle \subset \langle B \rangle$ for any subsets $A \subset B$ of X.

A subset $A \subset X$ is called *independent* (with respect to the hull operator $\langle \cdot \rangle$) if $a \notin \langle A \setminus \{a\} \rangle$ for any point $a \in A$.

We shall say that a hull operator $\langle \cdot \rangle : \mathcal{P}_X \to \mathcal{P}_X$ on a set X

- is *idempotent* if $\langle \langle A \rangle \rangle = \langle A \rangle$ for each subset $A \subset X$;

- has *finite supports* if for each $A \subset X$ and $a \in \langle A \rangle$ there is a finite subset $F \subset A$ with $a \in \langle F \rangle$;

- has *the MacLane-Steinitz exchange property* if for any subset $A \subset X$ and points $x, y \in X \setminus \langle A \rangle$ the inclusion $x \in \langle A \cup \{y\} \rangle$ is equivalent to $y \in \langle A \cup \{x\} \rangle$;

By a *matroid* we understand a pair $(X, \langle \cdot \rangle)$ consisting of a set X and an idempotent hull operator $\langle \cdot \rangle : \mathcal{P}_X \to \mathcal{P}_X$ that has finite supports and the MacLane-Steinitz exchange property. For finite X our definition of matroid argees with the classical one, see [2, 1.4.4] or [4, p. 270].

Theorem 1. *Let κ be an infinite cardinal, X be a set of cardinality $|X| = \kappa^+$ and $\langle \cdot \rangle : \mathcal{P}_X \to \mathcal{P}_X$ be a hull operator with finite supports and the MacLane-Steinitz exchange property. If $|\langle F \rangle \setminus \langle \varnothing \rangle| \leq \kappa$ for each finite subset $F \subset X$, then the set $X \setminus \langle \varnothing \rangle$ can be covered by κ many independent subsets.*

Proof. If $|X \setminus \langle \varnothing \rangle| \leq \kappa$, then $X \setminus \langle \varnothing \rangle$ can be decomposed into $\leq \kappa$ many singletons (which are independent sets). So, we assume that $|X \setminus \langle \varnothing \rangle| = \kappa^+$.

The finite support property of the hull operator and Zorn's Lemma imply the existence of a maximal independent subset $A \subset X$. We claim that $X = \langle A \rangle$. Assuming the opposite, we can find a point $x \in X \setminus \langle A \rangle$ and consider the set $A \cup \{x\}$ which is dependent as A is maximal independent. Consequently, there is a point $a \in A \cup \{x\}$ such that $a \in \langle (A \cup \{x\}) \setminus \{a\} \rangle$. This point a is not equal to x as $x \notin \langle A \rangle$. Let $B = A \setminus \{a\}$. Since A is independent, $a \notin \langle A \setminus \{a\} \rangle = \langle B \rangle$. Since the hull operator $\langle \cdot \rangle$ has the MacLane-Steinitz exchange property and $x, a \notin \langle B \rangle$, the inclusion $a \in \langle B \cup \{x\} \rangle$ implies $x \in \langle B \cup \{a\} \rangle = \langle A \rangle$, which contradicts the choice of x. Therefore $X = \langle A \rangle$.

Claim 1. $|\langle B \rangle \setminus \langle \varnothing \rangle| \leq \max\{|B|, \kappa\}$ *for any subset $B \subset X$.*

Proof. Let $[B]^{<\omega}$ denote the family of finite subsets of B. Since the hull operator $\langle \cdot \rangle$ has finite supports, $\langle B \rangle = \bigcup_{F \in [B]^{<\omega}} \langle F \rangle$. By our assumption, $|\langle F \rangle \setminus \langle \varnothing \rangle| \leq \kappa$ for each finite subset $F \subset X$. Consequently,

$$|\langle B \rangle \setminus \langle \emptyset \rangle| = \left| \bigcup_{F \in [B]^{<\omega}} \langle F \rangle \setminus \langle \varnothing \rangle \right| \leq$$
$$\sum_{F \in [B]^{<\omega}} |\langle F \rangle \setminus \langle \varnothing \rangle| \leq \max\{\kappa, |[B]^{<\omega}|\} \leq \max\{\kappa, |B|\}. \qquad \square$$

By Claim 1, $\kappa^+ = |X \setminus \langle \varnothing \rangle| = |\langle A \rangle \setminus \langle \varnothing \rangle| \leq \max\{\kappa, |A|\}$ and thus $|A| = \kappa^+$. Fix an injective enumeration $A = \{a_\alpha : \alpha < \kappa^+\}$.

For every ordinal $\alpha < \kappa^+$, put

$$A_\alpha = \{a_\gamma : \gamma < \alpha\} \quad \text{and} \quad X_\alpha = \langle A_{\alpha+1} \rangle \setminus \langle A_\alpha \rangle.$$

Then $X \setminus \langle \varnothing \rangle = \bigcup_{\alpha < \kappa^+} X_\alpha$ and for every $\alpha < \kappa^+$

$$|X_\alpha| = |\langle A_{\alpha+1} \rangle \setminus \langle A_\alpha \rangle| \leq |\langle A_{\alpha+1} \rangle \setminus \langle \varnothing \rangle| \leqslant \kappa$$

according to Claim 1. For every $\alpha < \kappa^+$ fix an injection $\chi_\alpha : X_\alpha \to \kappa$ and define a function $\chi : X \setminus \langle \varnothing \rangle \to \kappa$ letting $\chi | X_\alpha = \chi_\alpha$ for all $\alpha < \kappa^+$. For every $\lambda < \kappa$ consider the set $Z_\lambda = \chi^{-1}(\lambda)$. We claim that $\{Z_\lambda : \lambda < \kappa\}$ is a desired partition into κ many independent subsets.

Assuming that some set Z_λ is dependent, we can find a finite dependent subset $F \subset Z_\lambda$. We can assume that F is a minimal dependent subset. Since F is dependent, there is an element $a \in F$ such that $a \in \langle F \setminus \{a\} \rangle$. We claim that $x \in \langle F \setminus \{x\} \rangle$ for any element $x \in F$. This is clear if $x = a$. So, we assume that $x \neq a$. Consider the set $E = F \setminus \{x, a\}$. Since F is minimal dependent, the subsets $E \cup \{a\}$ and $E \cup \{x\}$ are independent and thus $a, x \notin \langle E \rangle$. Since $a \in \langle F \setminus \{a\} \rangle = \langle E \cup \{x\} \rangle$ the MacLane-Steinitz exchange property of the hull operator $\langle \cdot \rangle$ guarantees that $x \in \langle E \cup \{a\} \rangle = \langle F \setminus \{x\} \rangle$.

Since χ is injective on each subset X_α, there exist a numeration $F = \{f_1, \ldots, f_n\}$ and ordinals $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \kappa^+$ such that $f_1 \in Y_{\alpha_1}, \ldots, f_n \in Y_{\alpha_n}$. Then $f_n \notin \langle \{f_1, \ldots, f_{n-1}\} \rangle = \langle F \setminus \{f_n\} \rangle$, which is a desired contradiction. $\square$

Corollary 1. *Let κ be an infinite cardinal, $X \setminus \{0\}$ be a vector space over a field F such that $|F| \leqslant \kappa$ and $\dim X = \kappa^+$. Then $X \setminus \{0\}$ can be partitioned in κ linearly independent subsets.*

Corollary 2. *Let κ be an infinite cardinal, $X \setminus \{0\}$ be an extension of a field F such that $|F| \leqslant \kappa$ and $|X| = \kappa^+$. Then $X \setminus \{0\}$ can be partitioned in κ subsets algebraically independent over F.*

Corollary 3. *Let κ be an infinite cardinal, K_{κ^+} be a complete graph with κ^+ many vertices. Then there exists an edge coloring of K_{κ^+} in κ colors with no monochrome cycles.*

Now we consider independent subsets in groups. We define a subset A of a group G to be *independent* if A contains no point $a \in A$ with $a \in \langle A \setminus \{a\} \rangle$ where $\langle A \setminus \{a\} \rangle$ is the subgroup of G generated by the set $A \setminus \{a\}$. Thus A is independent with respect to the hull operator $\langle \cdot \rangle : \mathcal{P}_G \to \mathcal{P}_G$ assigning to each subset $B \subset G$ the subgroup $\langle B \rangle$ generated by B. This hull operator is idempotent and has finite supports, but in general fails to have the MacLane-Steinitz exchange property. Because of that, Theorem 1 is not applicable to the following open problem.

Problem 1. *Let G be an (abelian) group G of cardinality $|G| = \aleph_1$ with neutral element $\{e\}$. Can $G \setminus \{e\}$ be covered by countably many independent subsets?*

Let us prove that for groups of cardinality $> \aleph_1$ this problem has a negative solution.

Theorem 2. *Let κ be an infinite cardinal and G be a group of cardinality $|G| > \kappa^+$. Then, for every coloring $\chi : G \setminus \{e\} \to \kappa$, there exists a dependent monochrome subset A of G of cardinality $|A| = 4$.*

We shall need a simple combinatorial lemma.

Lemma 1. *Let $\lambda \geq 1$ be a cardinal and κ be an infinite cardinal. Let X, Y be sets of cardinality $|X| = \kappa^+$ and $|Y| > (\kappa^+)^\lambda$. For any κ-coloring $\chi : X \times Y \to \kappa$ there are subsets $A \subset X$ and $Z \subset Y$ such that $|A| = \lambda$, $|Z| > (\kappa^+)^\lambda$ and the set $A \times Z$ is monochrome.*

Proof. Since $|X| > \kappa$, for every $y \in Y$, there exist a subset $A(y) \subset X$ of cardinality $|A(y)| = \lambda$ such that $A(y) \times \{y\}$ is monochrome and hence $\chi(A(y) \times \{y\}) = \{\widetilde{\chi}(y)\}$ for some color $\widetilde{\chi}(y)$. Now consider the function $f : Y \to [X]^\lambda \times \kappa$ defined by

$$f(y) = (A(y), \widetilde{\chi}(y)).$$

Since $|[X]^\lambda \times \kappa| \leq (\kappa^+)^\lambda < |Y|$ there exists a pair $(A, \alpha) \in [X]^\lambda \times \kappa$ such that the preimage $Z = f^{-1}(A, \alpha)$ has cardinality $|Z| > (\kappa^+)^\lambda$. Then $A \times Z$ is a required monochrome rectangle. $\qquad\square$

Proof of Theorem 2. Since $|G| > \kappa^+$, we can choose two sets $X, Y \subset G \setminus \{e\}$ of cardinality $|X| = \kappa^+$ and $|Y| > \kappa^+$ such that $Y \cap X^{-1} = \emptyset$. Given any κ-coloring $\chi : G \setminus \{e\} \to \kappa$, consider the coloring $\widetilde{\chi} : X \times Y \to \kappa$ defined by $\widetilde{\chi}(x, y) = \chi(xy)$. By Lemma 1, there are a 2-element set $A = \{a, b\} \subset X$ and a subset $Z \subset X$ of cardinality $|Z| > \kappa^+$ such that the set $A \times Z$ is monochrome. Choose any point $x \in Z$ and any point $y \in Z \setminus \{a^{-1}bx, b^{-1}ax\}$. Then the set $B = \{ax, bx, ay, by\} \subset G$ has cardinality $|B| = 4$ and is monochrome with respect to the coloring χ. Since $ax = ay(by)^{-1}bx$, the set B is dependent. $\qquad\square$

Next, we consider covers of abelian groups by linearly independent subsets. Following [1, §16] or [3, §4.2], we call a subset A of an abelian group G *linearly independent* if for any pairwise distinct points $a_1, \ldots, a_n \in A$ and integer numbers $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$ the equality

$$\lambda_1 a_1 + \cdots + \lambda_n a_n = 0$$

implies $\lambda_1 a_1 = \cdots = \lambda_n a_n = 0$.

Observe that a subset A of an abelian group G is linearly independent if and only if it is independent with respect to the hull operator $[\,\cdot\,] : \mathcal{P}_G \to \mathcal{P}_G$ assigning to each subset $B \subset G$ the subset

$$[B] = \{0\} \cup \{x \in G : \exists n \in \mathbb{N} \ \ nx \in \langle B \rangle \setminus \{0\}\}.$$

The hull operator $[\,\cdot\,]$ is idempotent and has finite supports and the MacLane-Steinitz exchange property. So, the pair $(G, [\,\cdot\,])$ is a matroid.

Since $\langle A \rangle \subset [A]$ for any $A \subset X$, each independent subset of G is linearly independent. On the other hand, the subset $\{2, 3\}$ of the group of integers $\mathbb{Z}$ is independent but not linearly independent.

For an abelian group G and a natural number $n \in \mathbb{N}$ consider the subgroup

$$G[n] = \{x \in G : nx = 0\}.$$

The union $G[\mathbb{N}] = \bigcup_{n \in \mathbb{N}} G[n]$ is called the *torsion part* of G.

Theorem 3. *Let κ be an infinite cardinal and G be an abelian group. The set $G \setminus \{0\}$ can be covered by κ many linearly independent subsets if and only if $|G| \le \kappa^+$ and either $|G[\mathbb{N}]| \le \kappa$ or $G = G[p]$ for some prime number p.*

Proof. First we prove the "if" part. Assume that $|G| \le \kappa^+$ and either $|G[\mathbb{N}]| \le \kappa$ or $G[p] = G$ for some prime number p.

If $|G| \le \kappa$, then $G \setminus \{0\}$ decomposes into $|G \setminus \{0\}| \le \kappa$ many singletons, which are linearly independent. So, we assume that $|G| = \kappa^+$.

Consider the hull operator $[\,\cdot\,] : \mathcal{P}_G \to \mathcal{P}_G$ turning G into a matroid whose independent sets coincide with linearly indepenent subsets of G. Theorem 1 will imply that $G \setminus \{0\}$ can be covered by κ many linearly independent subsets as soon as we prove that for each finite subset $F \subset G$ the hull $[F]$ has cardinality $|[F]| \le \kappa$.

First we consider the case $|G[\mathbb{N}]| \le \kappa$. For every $n \in \mathbb{N}$ consider the homomorphism $n : G \to G$, $n : x \mapsto nx$, whose kernel coincides with the subgroup $G[n]$ that has cardinality $|G[n]| \le |G[\mathbb{N}]| \le \kappa$. Taking into account that $|\langle F \rangle| \le \aleph_0$ and

$$[F] = \{0\} \cup \bigcup_{n \in \mathbb{N}} n^{-1}(\langle F \rangle \setminus \{0\}) \subset \bigcup_{n \in \mathbb{N}} n^{-1}(\langle F \rangle),$$

we conclude that

$$|[F]| \le \sum_{n \in \mathbb{N}} |n^{-1}(\langle F \rangle)| \le \sum_{n \in \mathbb{N}} |G[n]| \cdot |\langle F \rangle| \le \kappa.$$

Next, assume that $G = G[p]$ for a prime number p. Then G is a linear space over the p-element field $\mathbb{F}_p$ and by Corollary 1, G can be covered by κ many linearly independent subsets.

To prove the "only if" part, assume that $G \setminus \{0\}$ can be covered by κ many linearly independent subsets. By Theorem 2, $|G| \le \kappa^+$. It remains to prove that $|G[\mathbb{N}]| > \kappa$ implies $G[p] = G$ for a prime number p. For every prime number p, consider the subgroup $G[p^\infty] = \bigcup_{k \in \mathbb{N}} G[p^k]$. By [3, 4.1.1], $G[\mathbb{N}] = \bigoplus_{p \in \Pi} G[p^\infty]$ where Π denotes the set of all prime numbers.

Since $|G[\mathbb{N}]| = \kappa^+$, there exists a prime number p such that $|G[p^\infty]| = \kappa^+$ and hence $|G[p^n]| = \kappa^+$ for some $n \in \mathbb{N}$. We can assume that n is the smallest number with this property, i.e., $|G[p^{n-1}]| < \kappa^+$.

We claim that $G = G[p^n]$. Otherwise, we can choose a point $a \in G \setminus G[p^n]$. By our assumption, $G \setminus \{0\}$ can be covered by κ many linearly independent subsets. The coset $a + G[p^n]$ has cardinality κ^+ and hence contains two distinct points $a + x$, $a + y$ that lie in a linearly independent subset of G. On the other hand, the set $\{a+x, a+y\}$ is linearly dependent since $p^n(a + x) - p^n(a + y) = 0$ but $p^n(a + x) = p^n a \neq 0 \neq p^n(a + y)$. This contradiction shows that $G = G[p^n]$, which means that the group G has bounded height and then by Prüfer-Baer Theorem 4.3.5 in [3], G is a direct sum $\oplus_{\alpha < \kappa^+} H_\alpha$ of cyclic p-groups of order $|H_\alpha| \leq p^k$. Each group H_α contains a cyclic subgroup C_α of order p. Then the subgroup $\oplus_{\alpha < \kappa^+} C_\alpha \subset G[p]$ has cardinality κ^+ and hence $p^n = p$ by the choice of n. Therefore, $G = G[p^k] = G[p]$. $\qquad \square$

Theorem 3 implies that for the direct sum $G = \oplus^{\aleph_1} C_4$ $\aleph_1$ many cyclic groups of order 4 the set $G \setminus \{0\}$ cannot be covered by countably many linearly independent sets. The same concerns the group $\oplus^{\aleph_1} C_6 = (\oplus^{\aleph_1} C_2) \oplus (\oplus^{\aleph_1} C_3)$.

Problem 2. *Can the groups $\oplus^{\aleph_1} C_4$ and $\oplus^{\aleph_1} C_6$ with removed zeros be covered by countably many independent sets.*

Corollaries 1, 2 and Theorem 2 imply the following interesting equivalence.

Theorem 4. *The following statements are equivalent:*

1. *For the linear space $\mathbb{R}$ over the field $\mathbb{Q}$ there is a partition of $\mathbb{R} \setminus \{0\}$ into countably many linearly independent subsets over $\mathbb{Q}$.*

2. *For the extension $\mathbb{R}$ of the field $\mathbb{Q}$ there is a partition of $\mathbb{R} \setminus \{0\}$ into countably many algebraically independent subsets over $\mathbb{Q}$.*

3. *For the abelian group $\mathbb{R}$ there is a partition of $\mathbb{R} \setminus \{0\}$ into countably many independent subsets.*

4. *The Continuum Hypothesis is true.*

Remark 1. Let κ, μ be infinite cardinals such that $\kappa^+ < \mu$. It follows from Lemma 1 that for every κ-coloring of the set of edges of the complete graph K_μ, there exists a monochrome cycle of length 4. Moreover, S. Slobodianyuk observed that the proof of Lemma 1 gives a monochrome complete bipartite

subgraph $K_{\lambda,\mu}$ provided that $\lambda \leqslant \kappa^+$ and $(\kappa^+)^\lambda < \mathrm{cf}\,\mu$. In particular, there exists a monochrome cycle of any even length.

On the other hand, for the complete graph K_{2^κ} its set of vertice $V = 2^\kappa$ can be identified with the set of all functions $f : \kappa \to \{0,1\}$, which allows us to define a coloring $\chi : [V]^2 \to \kappa$ by the rule

$$\chi(\{f,g\}) = \min\{\alpha : f(\alpha) \neq g(\alpha)\}.$$

For this coloring, K_{2^κ} has no monochrome cycles of odd length.

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